On Calculating the Lorenz Gauge Metric Perturbation

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Outline

- Motivation
- More specifically what are we calculating?
- How we calculate the Lorenz gauge metric perturbation
- Roadblocks
- Results and Future Work

Motivation

The Self-force is best understood in the Lorenz gauge.
To second order, the formulation is known ***only*** in the Lorenz gauge.

(Pound, Wardell, Warburton and Miller 2019)

Require Lorenz gauge metric perturbation to first and second order in the small mass ratio.

First Order Metric Perturbation in Lorenz Gauge

Goal: Calculate $h_{\mu\nu}$

- First Order
- Lorenz gauge ($h^{\mathrm{L}}_{\mu\nu}$)
- Vacuum background spacetime (starting with Schwarzschild)
- Homogenous Case
- Analytically (Done numerically by Barack and Lousto/Barack and Sago)

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Resolving roadblocks: A path to a **gauge transformation between Regge-Wheeler and Lorenz gauges** and intuits approach to finding $\mathbf{h}^{\mathbf{L}}_{\mu\nu}$ **in Kerr(?)**

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis

Metric Perturbation: $\tilde{g}_{\mu\nu} = g_{\mu\nu} + h_{\mu\nu}$

Linearised Field Equations in Lorenz Gauge: $\Box \bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\nu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = 0$

Lorenz Gauge Condition: $\bar{h}_{\mu\nu}{}^{;\nu} = 0$

Trace-Reversed Metric: $\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2}g_{\mu\nu}h$

$$h_{\mu\nu}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty} e^{-i\omega t}\left(h^{o,lm}_{\mu\nu}(\omega,r,\theta,\phi) + h^{e,lm}_{\mu\nu}(\omega,r,\theta,\phi)\right)d\omega$$

Barack and Lousto, 2005
Berndtson, 2007
Barack and Sago, 2010
Thompson, 2018

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis: Odd Sector

$$h^{o,lm}_{\mu\nu}(\omega, r, \theta, \phi) = \begin{pmatrix} 0 & 0 & h_0^{lm}(\omega, r)\csc\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\phi} & -h_0^{lm}(\omega, r)\sin\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} \\ * & 0 & h_1^{lm}(\omega, r)\csc\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\phi} & -h_1^{lm}(\omega, r)\sin\theta\frac{\partial Y_{lm}(\theta,\phi)}{\partial\theta} \\ * & * & -h_2^{lm}(\omega, r)X_{lm}(\theta, \phi) & h_2^{lm}(\omega, r)\sin\theta W_{lm}(\theta, \phi) \\ * & * & * & h_2^{lm}(\omega, r)\sin^2\theta X_{lm}(\theta, \phi) \end{pmatrix}$$

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis: Odd Sector

$\omega \neq 0$ and $l \geq 2$ case, Schwarzschild background

Linearised Field Equations in Lorenz Gauge: $\Box \bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\nu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = 0$

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$$h_0^{\mathrm{odd}} = \frac{1}{i\omega}\left(\psi_1^{\mathrm{RW}} + \frac{2\lambda}{3}\psi_2^{\mathrm{RW}}\right)$$

$$h_1^{\mathrm{odd}} = \frac{1}{(i\omega)^2}\left[-\frac{2\lambda}{3}\psi_2^{\mathrm{RW}\prime} + \frac{2}{r}\psi_1^{\mathrm{RW}} - \frac{2\lambda}{3r}\psi_2^{\mathrm{RW}} - \psi_1^{\mathrm{RW}\prime}\right] \qquad \text{(Berndtson, 2007)}$$

$$h_2^{\mathrm{odd}} = \frac{1}{(i\omega)^2}\left[(r-2M)\psi_2^{\mathrm{RW}\prime} + \psi_1^{\mathrm{RW}} + \frac{(3+2\lambda)r - 6M}{3r}\psi_2^{\mathrm{RW}}\right]$$

Can transform between Regge-Wheeler gauge and Lorenz gauge.
(Hopper and Evans, 2012)

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis: Even Sector

$$h_{\mu\nu}{}^{e,lm}(\omega, r, \theta, \phi) =$$

$$\begin{pmatrix} \left(1-\frac{2M}{r}\right) H_0^{lm}(\omega,r)Y_{lm} & H_1^{lm}(\omega,r)Y_{lm} & h_0^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & h_0^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\phi} \\ * & \frac{H_2^{m}(\omega,r)Y_{lm}}{\left(1-\frac{2M}{r}\right)} & h_1^{lm}(\omega,r)\frac{\partial Y_{lm}}{\partial\theta} & h_1^{lm}(\omega,r)X_{lm} \\ * & * & \begin{array}{c} r^2\left(K^{lm}(\omega,r)Y_{lm}\right. \\ \left.+G^{lm}(\omega,r)W_{lm}\right) \end{array} & r^2\sin\theta G^{lm}(\omega,r)X_{lm} \\ * & * & * & \begin{array}{c} r^2\sin^2\theta\left(K^{lm}(\omega,r)Y_{lm}\right. \\ \left.-G^{lm}(\omega,r)W_{lm}\right) \end{array} \end{pmatrix}$$

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis: Even Sector

$\omega \neq 0$ and $l \geq 2$ case, Schwarzschild background

$$\text{Linearised Field Equations in Lorenz Gauge:} \quad \Box \bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\nu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = 0$$

$$\text{Lorenz Gauge Condition:} \quad \bar{h}_{\mu\nu}{}^{;\nu} = 0$$

$$H_0^{\text{even}}, H_1^{\text{even}}, H_2^{\text{even}}, h_0^{\text{even}}, h_1^{\text{even}}, K^{\text{even}}, G^{\text{even}} \to \psi_0^{\text{RW}}, \psi_1^{\text{RW}}, \psi_2^{\text{Z}}, M_{2af} \;\; \text{and derivatives}$$

$$\Bigg\{ \underbrace{f(r)^2 \frac{d^2}{dr^2} + \frac{2M}{r^2} f(r) \frac{d}{dr} - f(r) \left(\frac{l(l+1)}{r^2} - \frac{2M}{r^3} \right) + \omega^2}_{\text{Regge-Wheeler operator } \mathcal{L}_0} \Bigg\} M_{2af} = f(r)\psi_0^{\text{RW}}$$

Calculating $h^{\mathrm{L}}_{\mu\nu}$ in a Tensor Spherical Harmonic Basis: Even Sector

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$H_0^{\text{even}}, H_1^{\text{even}}, H_2^{\text{even}}, h_0^{\text{even}}, h_1^{\text{even}}, K^{\text{even}}, G^{\text{even}} \to \psi_0^{\text{RW}}, \psi_1^{\text{RW}}, \psi_2^{\text{Z}}, M_{2af}$ and derivatives

$$\Bigg\{ \underbrace{f(r)^2 \frac{d^2}{dr^2} + \frac{2M}{r^2} f(r) \frac{d}{dr} - f(r)\left(\frac{l(l+1)}{r^2} - \frac{2M}{r^3}\right) + \omega^2}_{\text{Regge-Wheeler operator } \mathcal{L}_0} \Bigg\} M_{2af} = f(r)\psi_0^{\text{RW}}$$

- M_{2af} has been solved numerically (Berndtson, Nolan and Warburton)
- M_{2af} is difficult to solve, non compact source.
- Try using MST method (low frequency expansion) and/or Green's function.
- Is there a way to get around solving M_{2af}?

An Alternate Route to Metric Reconstruction

There may be a route to calculating $h^{\mathrm{L}}_{\mu\nu}$ without the need to solve for M_{2af}.

$$\text{Linearised Field Equations in Lorenz Gauge:} \quad \Box\bar{h}_{\mu\nu} + 2R^{\alpha}{}_{\nu}{}^{\beta}{}_{\nu}\bar{h}_{\alpha\beta} = 0$$

$$\text{Lorenz Gauge Condition:} \quad \bar{h}_{\mu\nu}{}^{;\nu} = 0$$

$$\text{Metric Components:} \quad h_0^{\mathrm{odd}} = \frac{1}{i\omega}\left(\psi_1 + \frac{2\lambda}{3}\psi_2\right)\ldots$$

Einstein field equations in Lorenz gauge and Lorenz gauge conditions take the following form:

$$A_{l\omega}(r)(\mathcal{L}_0 M_{2af} - f(r)\psi_0^{\mathrm{RW}}) + B_{l\omega}(r)\mathcal{L}_0\psi_0^{\mathrm{RW}} + C_{l\omega}(r)\mathcal{L}_1\psi_1^{\mathrm{RW}} + D_{l\omega}(r)\mathcal{L}_2\psi_2^{\mathrm{RW}}$$
$$+ E_{l\omega}(r)\mathcal{L}_{2\mathrm{Z}}\psi_2^{\mathrm{Z}} + \text{derivatives} = 0$$

$$\mathcal{L}_0 M_{2af} = f(r)\psi_0^{\mathrm{RW}}, \quad \mathcal{L}_s\psi_s = 0$$

How does $\mathbf{M_{2af}}$ contribute to the metric perturbation?

How Does M_{2af} Contribute to $h^{\mathrm{L}}_{\mu\nu}$?

$$h_{\mu\nu}(t,r,\theta,\phi) = \sum_{l=0}^{\infty}\sum_{m=-l}^{l}\int_{-\infty}^{\infty} e^{-i\omega t}\left(h^{o,lm}_{\mu\nu}(\omega,r,\theta,\phi) + h^{e,lm}_{\mu\nu}(\omega,r,\theta,\phi)\right) d\omega$$

- $l = 0$ Scalar Perturbations
- $l = 1$ Scalar and Vector Perturbations
- $l \geq 2$ Scalar, Vector and Tensor Perturbations (Contains Radiative Modes)
- Metric Perturbation = Radiative Modes + $\delta a + \delta M$+ Gauge

Recall M_{2af} was derived for the case where $\omega \neq 0$ and $l \geq 2$. We shall therefore focus on examining M_{2af}'s contribution to the radiative modes of $h^{\mathrm{L}}_{\mu\nu}$ to start.

Contribution of M_{2af} to Radiative Modes

Wald's Result on perturbations of a Kerr black hole:

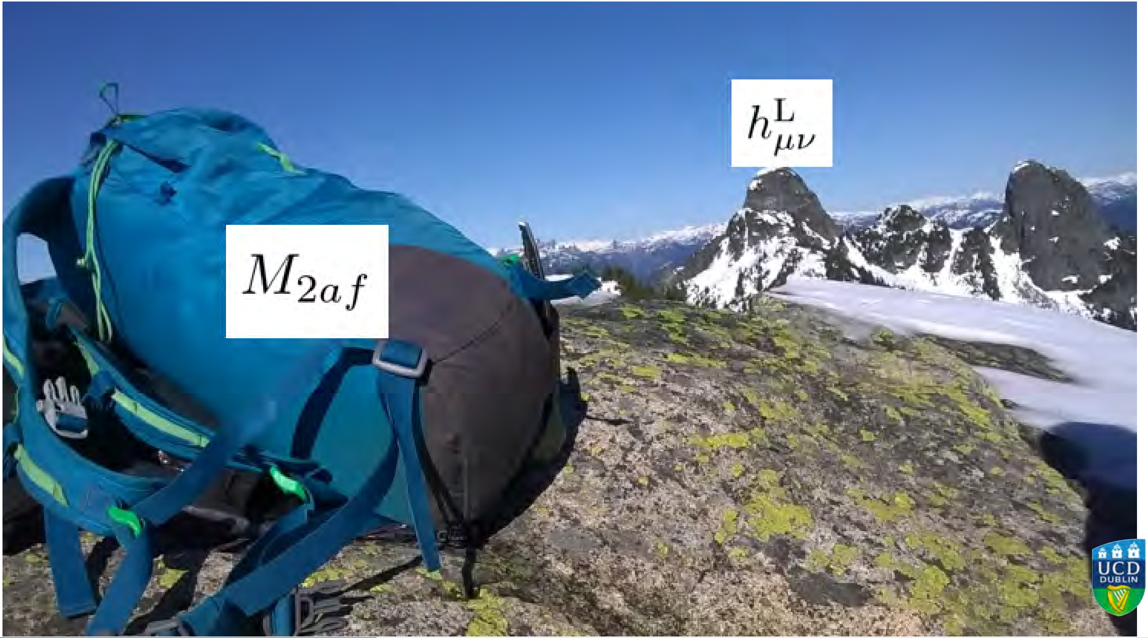
Either of the perturbed Weyl scalars $\psi_0^{\rm P}$ or $\psi_4^{\rm P}$ alone uniquely satisfies the non-trivial part of gravitational perturbations to a Kerr black hole.

$$\delta R_{\mu\nu\rho\sigma} = \frac{1}{2}(\nabla_\rho\nabla_\nu h_{\mu\sigma} + \nabla_\sigma\nabla_\mu h_{\nu\rho} - \nabla_\rho\nabla_\mu h_{\nu\sigma} - \nabla_\sigma\nabla_\nu h_{\mu\rho} + h_\mu{}^\tau R_{\tau\nu\rho\sigma} - h_\nu{}^\tau R_{\tau\mu\rho\sigma})$$

$$\begin{aligned} {}_2\Psi_{\ell m\omega} &= \Psi_0^{\rm P} = \delta C_{1313} = \delta R_{\mu\nu\rho\sigma}\ell^\mu m^\nu \ell^\rho m^\sigma \qquad \text{(Kinnersley tetrad)} \\ {}_{-2}\Psi_{\ell m\omega} &= \Psi_4^{\rm P} = \delta C_{2424} = \delta R_{\mu\nu\rho\sigma} n^\mu \bar{m}^\nu n^\rho \bar{m}^\sigma \end{aligned}$$

Main Result: Using Berndtson's expressions for the metric perturbation components for the case $\omega \neq 0$ with $l \geq 2$, M_{2af} does not contribute to perturbed Weyl scalars.

$\Longrightarrow$ **$\mathbf{M_{2af}}$ does not contribute to the radiative part of $\mathbf{h^L_{\mu\nu}}$.**


$h^{\mathrm{L}}_{\mu\nu}$
M_{2af}

Results: Path to Gauge Transformation

- M_{2af} does not contribute to the radiative part of $h^{\mathrm{L}}_{\mu\nu}$
- In fact, only ψ_2^{RW} contributes to the perturbed Weyl scalars which is to be expected, and the Chandrasekhar transformation is recovered:

$$
{}_2\Psi_{\ell m\omega} = \Psi_0^{\mathrm{P}} = -\frac{iC_\ell}{4r^2}\mathcal{D}^\dagger\mathcal{D}^\dagger\left(\frac{2r\psi_2^{\mathrm{RW}}}{i\omega}\right), \qquad \mathcal{D}^\dagger \equiv \partial_r - \frac{i\omega}{f}
$$

$$
{}_{-2}\Psi_{\ell m\omega} = \Psi_4^{\mathrm{P}} = -\frac{iC_\ell}{16}r^2 f^2\mathcal{D}\mathcal{D}\left(\frac{2r\psi_2^{\mathrm{RW}}}{i\omega}\right), \qquad \mathcal{D} \equiv \partial_r + \frac{i\omega}{f}
$$

- Can construct radiative part of metric purely from ψ_2^{RW}, without M_{2af}, and can use the MST method to transform between Regge-Wheeler and Lorenz gauge for homogeneous perturbations.

Results: Path to $h^{\mathrm{L}}_{\mu\nu}$ in Kerr?

- Homogenous Lorenz gauge metric perturbation components with $\omega \neq 0$ and $l \geq 2$ can be written in terms of Regge-Wheeler master functions (ψ_2^{RW} only).
- Can the metric perturbation be written in terms of a Hertz potential, Φ and completion pieces?

$$h^{\mathrm{L}}_{\mu\nu} = \underbrace{\text{``}\nabla_\mu \nabla_\nu \Phi^{\mathrm{L}}\text{''}}_{\text{(Radiative Modes)}} + \delta a + \delta M + \text{Gauge}$$

$$\text{Hertz Potential} \begin{cases} \text{Gravitational Case for Schwarzschild and Kerr in Radiation guage} \\ \text{EM case for Kerr in Lorenz gauge (Dolan)} \\ \text{Gravitational case for Schwarzschild in Lorenz gauge?} \\ \text{Gravitational case in Kerr in Lorenz gauge? (NP/GHP formalism)} \end{cases}$$

Future Work

- Contribution of M_{2af} to δM or δa? (We expect M_{2af} to be pure gauge).
- Construct full Lorenz gauge solution including completion pieces of the metric perturbation: *Which fields contribute to δM, δa and gauge?*
- Explore transformation between Regge-Wheeler and Lorenz gauges using MST (we now have this for static and non-static homogeneous perturbations with $l \geq 2$)
- Using NP/GHP formalism: $h^{\mathrm{L}}_{\mu\nu}$ in Schwarzschild $\rightarrow$ $h^{\mathrm{L}}_{\mu\nu}$ in Kerr