Analytical 2nd Order Self-force

Some preliminary work..

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MAX-PLANCK-GESELLSCHAFT

Post-Newtonian Self-Force

has it's uses..

2nd order

- **Parameter space**

  e.g. fill in large-r, low e fluxes [C. Munna] ✓ *lower accuracy needed at 2SF*

- **PN/PM/EOB**

  -porting information back and forth [A. Antonelli] ✓ *FULL 5pn (6pn?) [Bini et al 2019/20]*
  -cross validation

- **'Tool'**

  testing aspects of numerical codes
  e.g. regularisation, debugging *…probably*

- **Hybrid codes?**

  part numeric, part PN *e.g. this talk*

What do we actually know at 1st order

-local 1st order field only

e.g. PN solutions to Teukolsky

~~$\psi_4^{\mathrm{PN}}(r,\omega)$ - $r \gg M$~~

$\psi_4^{\mathrm{PN}}(r,\omega)$ – $r_0 \gg M, \quad r \sim r_0,$
$\omega \sim r_0^{-3/2}$

1st order source $\propto \delta(r - r_0)$, —> only need local solutions

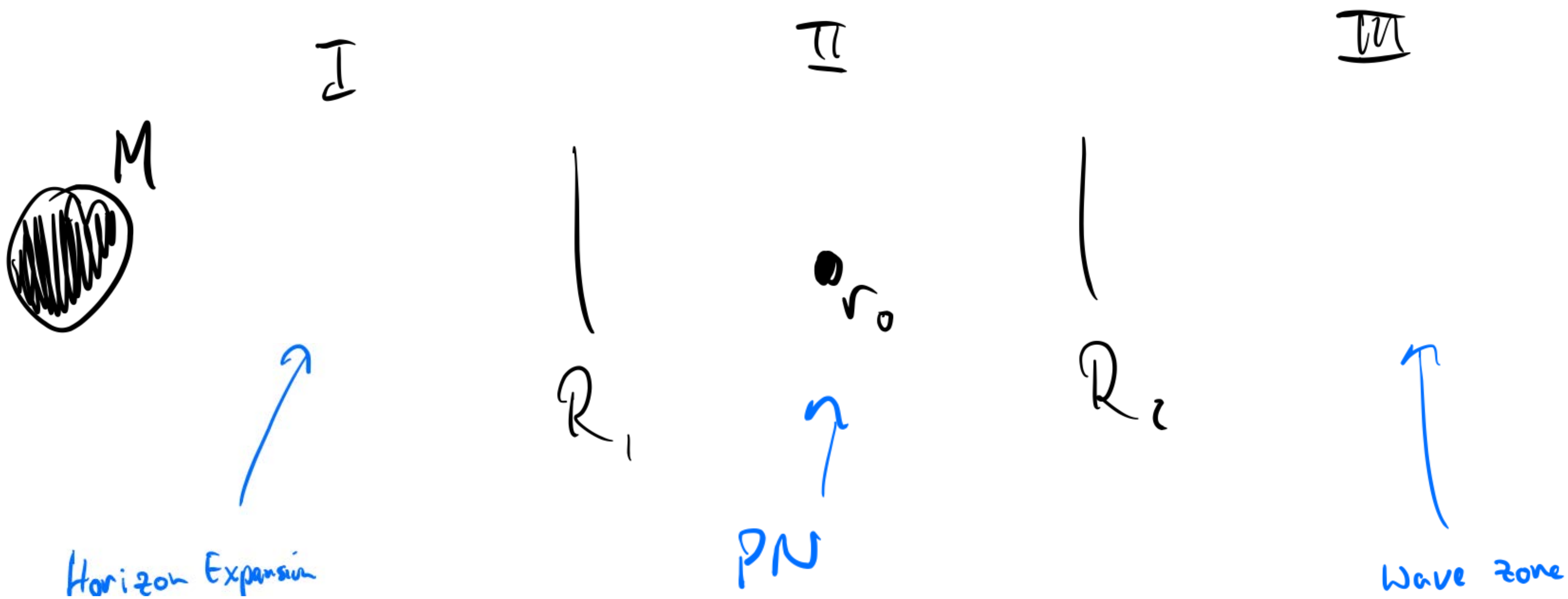
What do we actually know at 1st order

-local 1st order field only

2nd order EFEs

$$\delta G_{\mu\nu}[h^2] = -\,\delta^2 G_{\mu\nu}[h^1, h^1]$$

need solutions *everywhere*



2nd order scalar SF

Flat spacetime

Toy field equations

$$\Box \phi^1 = -4\pi\varrho$$

$$\Box \phi^2 = t^{\alpha\beta}\partial_\alpha\phi^1\partial_\beta\phi^1 \equiv S^2[\phi^1, \phi^1]$$

$$, \varrho = \frac{\delta(x^i - x_0^i)}{dt/d\tau}$$

$$u^\alpha = (t, x^i) = u^t(1,0,0,\Omega)$$

2nd order scalar SF

Toy field equations-modes

$$\Box_r \phi^1_{lm} = -4\pi\varrho_{lm}$$
$$\Box_r \phi^2_{lm} = (t^{\alpha\beta}\partial_\alpha\phi^1\partial_\beta\phi^1)_{lm} \equiv S^2_{lm}[\phi^1, \phi^1]$$

$$\phi^i = \sum_{lm} e^{-im\Omega t}\phi^i_{lm}(r)Y_{lm}(\theta, \varphi)$$

Homogeneous solutions $\phi^{\pm}_{lm}$

$$\phi^i_{lm} = C^+_{lm}(r)\phi^+_{lm}(r) + C^-_{lm}(r)\phi^-_{lm}(r)$$

With e.g.

$$C^+_{lm} = \frac{1}{W}\int_0^r \phi^-_{lm}(r')S^i_{lm}(r')dr'$$

1st order: $C^{\pm}_{lm}(r) = C^{\pm}_{lm}(r_0)$

2nd order: need $\phi^{\pm}_{lm}(r)$ *'everywhere'*

2nd order scalar SF

Toy field equations-modes

$$\Box_r \phi^1_{lm} = -4\pi \varrho_{lm}$$
$$\Box_r \phi^2_{lm} = (t^{\alpha\beta}\partial_\alpha \phi^1 \partial_\beta \phi^1)_{lm} \equiv S^2_{lm}[\phi^1, \phi^1]$$

$$\phi^i = \sum_{lm} e^{-im\Omega t} \phi^i_{lm}(r) Y_{lm}(\theta, \varphi)$$

To do list:

- Construct $\phi^\pm_{lm}$ 'everywhere'

  Split domain- asymptotic matching

- Local 2nd order source- only needs local 1st order PN

Second order source-local behaviour

Infinite mode coupling

Miller, Pound, Wardell arxiv:1608.06783 (MPW 2016)

$$\phi^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-im\Omega t} \phi^1_{lm}(r) Y_{lm}(\theta, \varphi)$$

$$S^2[\phi^1, \phi^1] = t^{\alpha\beta} \partial_\alpha \phi^1 \partial_\beta \phi^1$$

In practice, only up to some $l_{\max}$

Each mode of the source involves

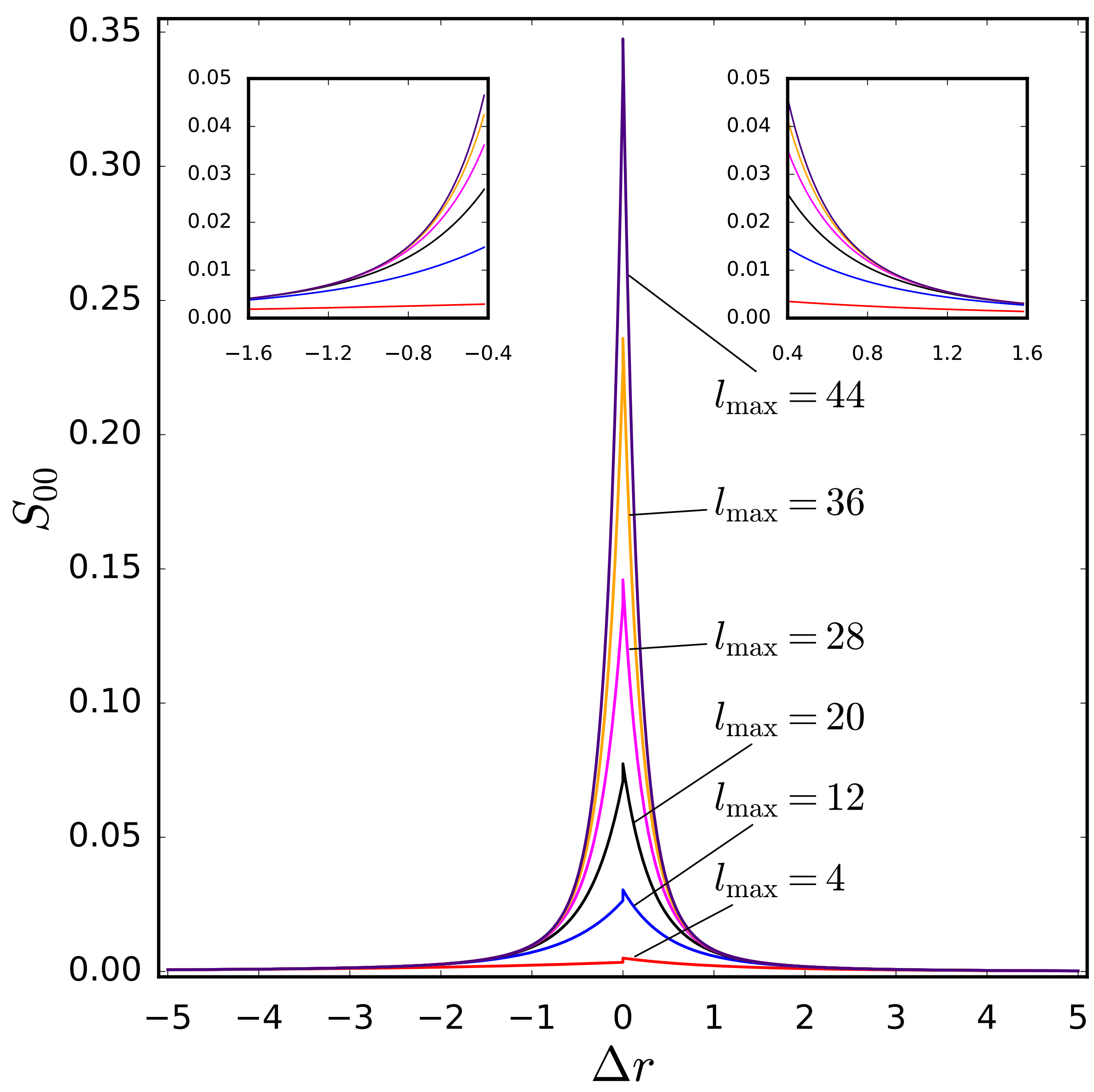
$$S^2_{lm}[\phi^1, \phi^1] = \oint \left[t^{\alpha\beta} \partial_\alpha \sum_{l_1, m_1} e^{-im_1 \Omega t} \phi^1_{l_1 m_1} Y_{l_1 m_1} \partial_\beta \sum_{l_2, m_2} e^{-im_2 \Omega t} \phi^1_{l_2 m_2} Y_{l_2 m_2} \right] Y^*_{lm} \sin\theta' d\theta' d\varphi'$$

NEITHER sum converges well near r_0, so given a finite number of modes, how do you compute S^2_{lm}??

Second order source

Infinite mode coupling

Figure 1 of MPW 2016



Numerical solution: puncture method

$$\phi^1 = \phi^{1,\mathscr{R}} + \phi^{1,\mathscr{P}}$$

$$S^2[\phi^1, \phi^1] = S^2[\phi^{1,\mathscr{R}}, \phi^{1,\mathscr{R}}] + 2S^2[\phi^{1,\mathscr{R}}, \phi^{1,\mathscr{P}}] + S^2[\phi^{1,\mathscr{P}}, \phi^{1,\mathscr{P}}]$$

Convergent l sums

Know full $4D$ time domain form— integrate against Y_{lm} directly

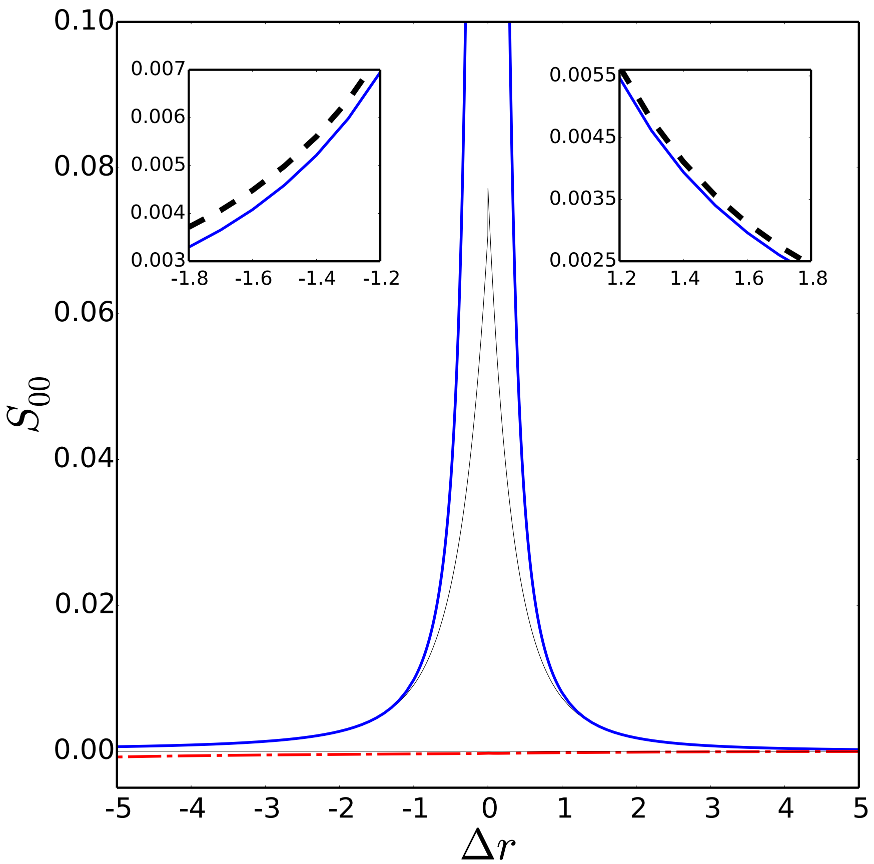


Figure 2 of MPW 2016

each lm mode:

2 ‘mode-coupling’ sums + numerical integral

- *by far the most expensive part of current 2nd order codes*

Numerical solution: PN method

Miller, Pound, Wardell arxiv:1608.06783

$$\phi^1 = \sum_{l=0}^{\infty} \sum_{m=-l}^{l} e^{-im\Omega t} \phi^1_{lm}(r) Y_{lm}(\theta, \varphi)$$

$$S^2[\phi^1, \phi^1] = t^{\alpha\beta} \partial_\alpha \phi^1 \partial_\beta \phi^1$$

In practice, only up to some $l_{\max}$

In PN, know *all* l. e.g.

$$\phi^+_{lm} = (r\omega\eta)^{l+1} \left(1 + \left(\frac{r^2\omega^2}{2(2l-1)} + \frac{l+1}{r} \right) \eta^2 + \dots \right)$$

Numerical solution: PN method

$$S^2_{lm}[\phi^1,\phi^1] = \oint \left[t^{\alpha\beta}\partial_\alpha \sum_{l_1,m_1} e^{-im_1\Omega t}\phi^1_{l_1m_1}Y_{l_1m_1}\partial_\beta \sum_{l_2,m_2} e^{-im_2\Omega t}\phi^1_{l_2m_2}Y_{l_2m_2} \right] Y^*_{lm}\sin\theta' d\theta' d\varphi'$$

$$\oint Y_{l_1m_1}Y_{l_2m_2}Y^*_{lm}d\Omega = C^{lm}_{l_1m_1l_2m_2} \qquad |l-l_1| \le l_2 \le l+l_1$$

- **Evaluate S^2_{lm} as a function of l_1**

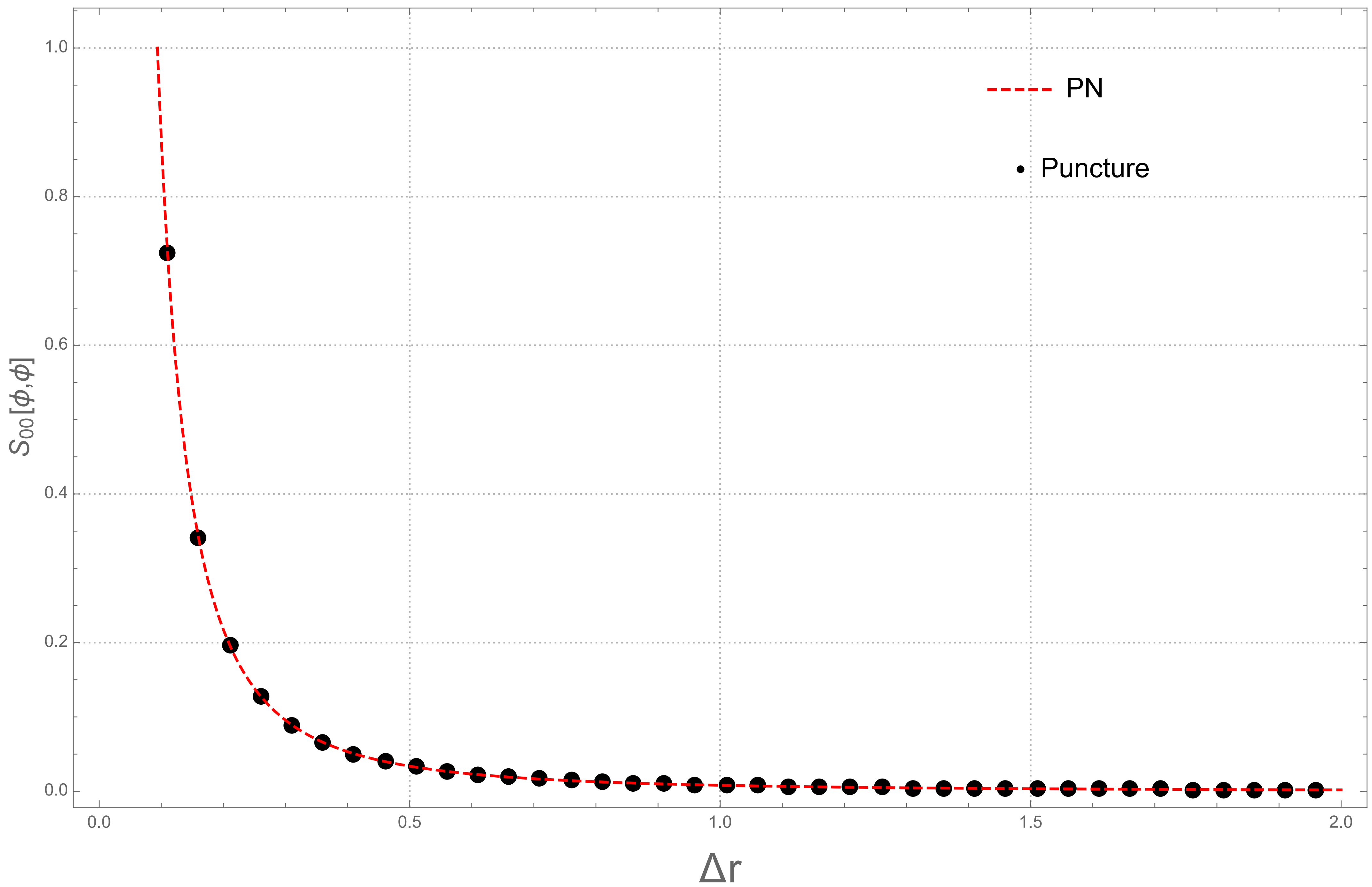
$$S^{2,l_1}_{00} = \frac{2\sqrt{\pi}(l_1+1)}{r_0^4(u^t)^2}\epsilon^{2l_1+4}\eta^8 + \ldots \qquad \epsilon = r_0/r,\ \eta \text{ - PN order counting}$$

- **sum to ∞, at each PN order**

$$S^2_{00} = \frac{2\sqrt{\pi}\epsilon^4}{r_0^4(u^t)^2\left(\epsilon^2-1\right)^2}\eta^8 + \frac{2\sqrt{\pi}\Omega^2\epsilon^4}{r_0^2(u^t)^2\left(\epsilon^2-1\right)^2}\eta^{10} - \left(\frac{\sqrt{\pi}\Omega^4\left(\epsilon^4-62\epsilon^2+25\right)}{16(u^t)^2\left(\epsilon^2-1\right)^2} + \frac{\sqrt{\pi}\Omega^4\left(\epsilon^2+25\right)\tanh^{-1}(\epsilon)}{16(u^t)^2\epsilon}\right)\eta^{12}\ldots$$

Numerical solution: PN method

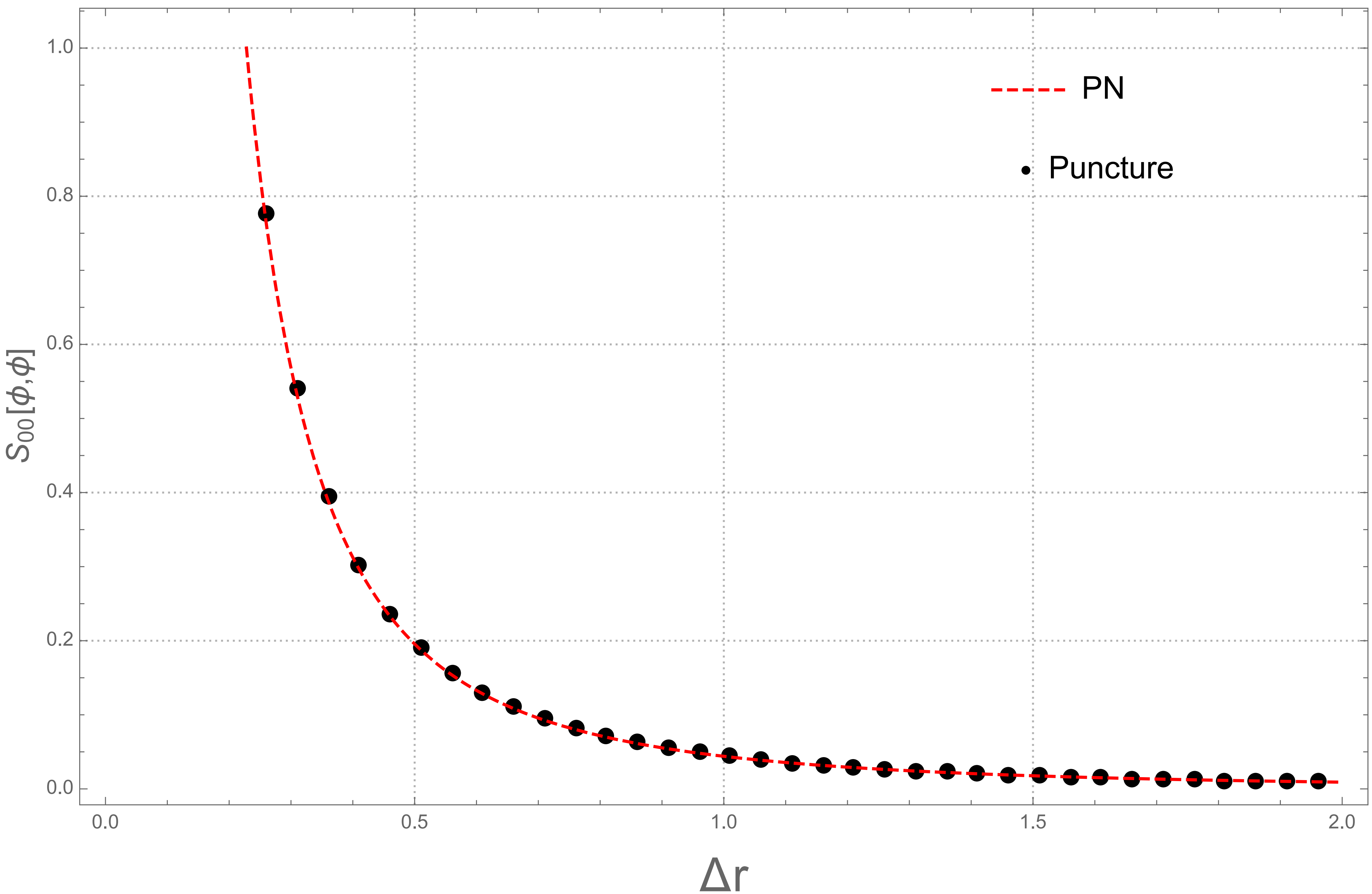
$$r_0 = 10$$



Acknowledge: Barry Wardell

Numerical solution: PN method

$$r_0 = 4$$



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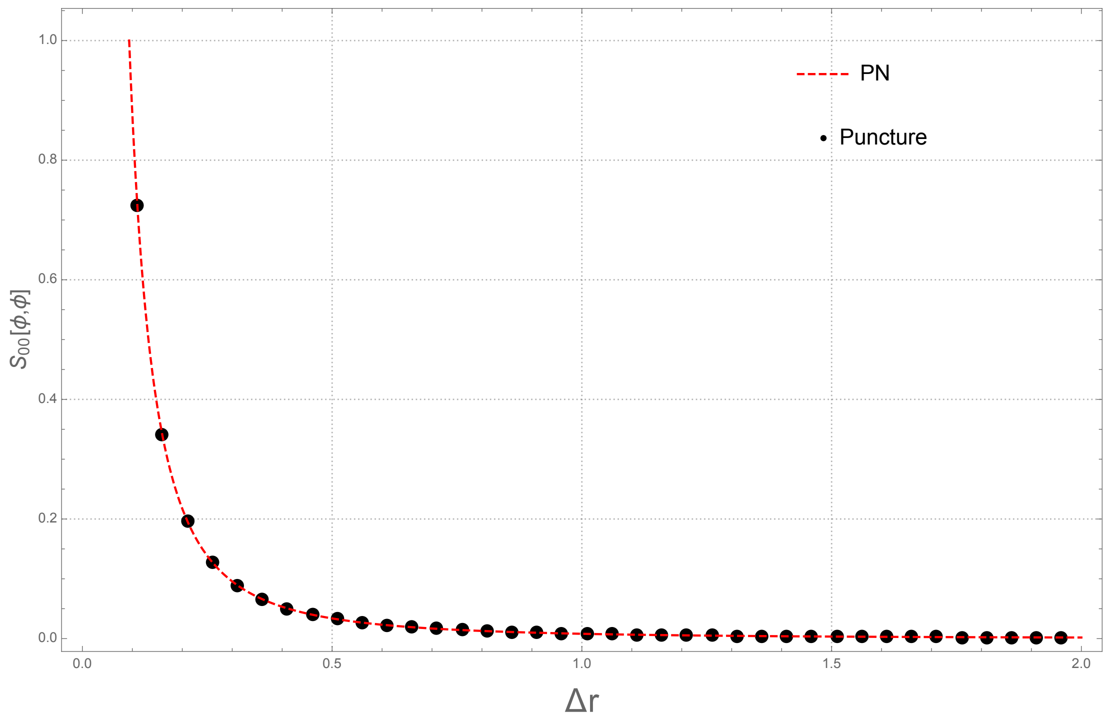
Numerical solution: PN method



Infinite mode coupling:
not a problem for PN

Idea:

Compute $S^{2,\mathrm{PN}}$ using high order 1SF

Use it as the source for the full 2nd order numerical code

Future

Toy models

Local S^2_{lm} in curved space -PN convergence?

Implement asymptotic matching $\rightarrow \phi^2_{lm}$ (flat + Schwarzschild)

GSF

Computing S^2_{lm} for Teukolsky?

- *radiation gauge PN solutions as a testing ground for HRG? [Spiers, Upton talks yesterday]*

Lorenz gauge MST? [Durkan talk monday]

- *Will PN be accurate enough to completely replace the source!*

…Next Capra

Schwarzschild scalar source

$$r_0 = 10M, \quad 2\text{PN}$$