A view to second-order self-force for eccentric orbits

23rd Capra Meeting on Radiation Reaction in General Relativity

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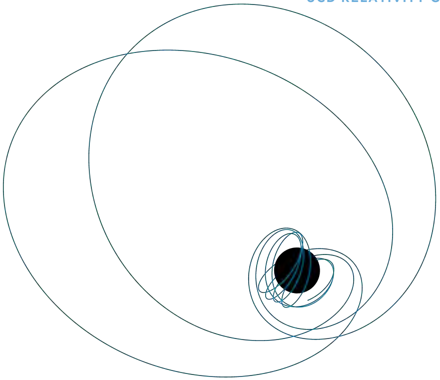
State of the art

- Second-order self-force calculation of gravitational binding energy - Pound, Wardell, Warburton, and Miller (2019).
- Circular orbits in Schwarzschild spacetime (in progress). See Adam's talk...
- Aim: Astrophysically realistic scenario of noncircular orbits in Kerr spacetime.



Worldtube method for eccentric orbits with exponential convergence

Effective source method for the Teukolsky equation (preliminary)

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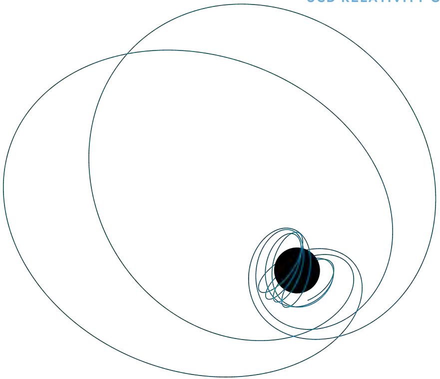

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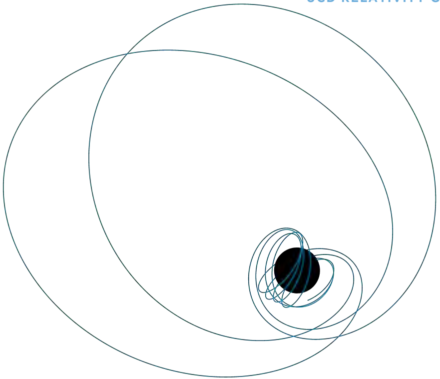

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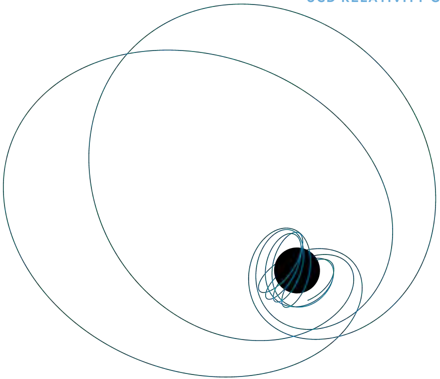

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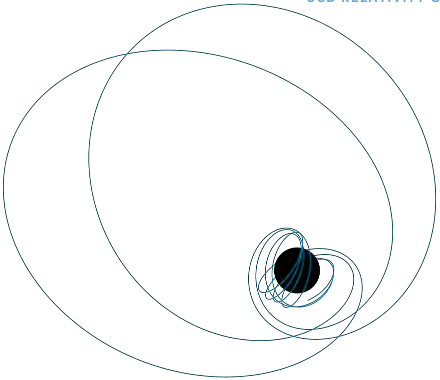

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Field Equations

$$G_{\mu\nu}[\mathrm{g}] = G_{\mu\nu}[g] + \epsilon\,\delta G_{\mu\nu}[h^{(1)}] + \epsilon^2\left(\delta G_{\mu\nu}[h^{(2)}] + \delta^2 G_{\mu\nu}[h^{(1)}, h^{(1)}]\right) + \mathcal{O}(\epsilon^3)$$

- Peform singular-regular split

$$h^{(n)}_{\mu\nu} = h^{S(n)}_{\mu\nu} + h^{R(n)}_{\mu\nu}$$

- Let's consider the first two orders in ϵ.

$$\delta G_{\mu\nu}[h^{(1)}] = 8\pi T^{(1)}_{\mu\nu}\,,$$

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Linearized Einstein operator

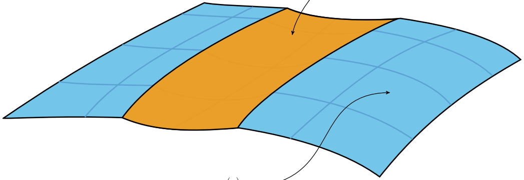
$$\delta G_{\mu\nu}[\dots] = \partial_t^2 - \partial_{r_*}^2 + \dots$$

Non-compact. Sourced by first-order perturbation and diverges at the particle.

Worldtube and effective source

- Define a *worldtube* around the worldline
- Solve field equations for $h^{(n)}_{\mu\nu}$ outside worldtube
- Solve field equations for $h^{\mathcal{R}(n)}_{\mu\nu} \equiv h^{(n)}_{\mu\nu} - h^{\mathcal{P}(n)}_{\mu\nu} \sim h^{R(n)}_{\mu\nu}$ inside worldtube
- At first-order we have

$$\delta G_{\mu\nu}[h^{\mathcal{R}(1)}] = 8\pi T^{(1)}_{\mu\nu} - \delta G_{\mu\nu}[h^{\mathcal{P}(1)}] \equiv S^{\text{eff}(1)}_{\mu\nu}$$



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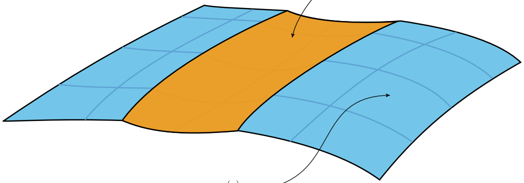
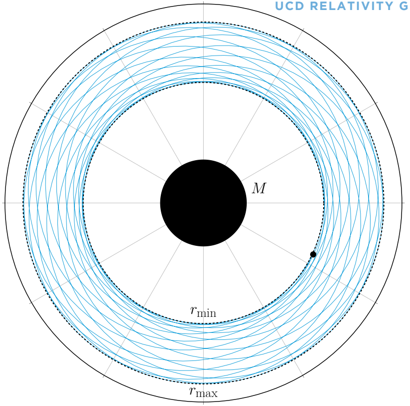
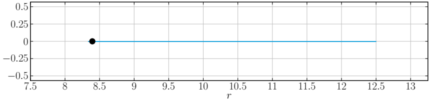


The challenge of eccentric orbits

- Discontinuity in the punctures is now time-dependant
- Need to take Fourier Transform for the new radial harmonic, n.
- At first-order, one can use Extended Homogeneous Solutions (EHS)
- Consider a compact source, spread across libration region





Scalar self-force (SSF) model

- "Toy model" scalar field, ψ

$$\Box_{\ell mn}\psi_{\ell mn}(r) = S_{\ell mn}(r)$$

- Solve with an auxiliary worldtube

Outside the worldtube
$(r \leq r_{\mathbf{min}},\ r \geq r_{\mathbf{max}})$

$$\Box_{\ell mn}\psi_{\ell mn}(r) = 0$$

Inside the worldtube
$(r_{\mathbf{min}} < r < r_{\mathbf{max}})$

$$\Box_{\ell mn}\psi^{R}_{\ell mn}(r) = S^{\text{eff}}_{\ell mn}(r)$$

$$S^{\text{eff}}_{\ell mn}(r) \equiv S_{\ell mn}(r) - \Box_{\ell mn}\psi^{\mathcal{P}}_{\ell mn}(r)$$

- Retarded B.C. s
- Exponential convergence
- Integration over $S^{\text{eff}}_{\ell mn}(r)$

- FT of $S^{\text{eff}}_{\ell m}(t, r)$ not trivial
- Algebraic convergence
- Gibbs phenomenon

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Compact Source

- Solve with an auxiliary worldtube

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A Naive Approach

- FT of effective source is found numerically

$$S^{\text{eff}}_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} S^{\text{eff}}_{\ell m}(t,r)e^{i\omega_{mn}t}dt \implies S^{\text{eff}}_{\ell m}(t,r) = \sum_{n=-\infty}^{\infty} S^{\text{eff}}_{\ell mn}(r)e^{-i\omega_{mn}t}$$

- Computationally difficult (slow)
- In practise this leads to poor convergence
- $S^{\text{eff}}_{\ell mn}(t,r)$ is C^0 differentiable $\longmapsto$ Gibbs ringing
- Acausal homogeneous solutions excited

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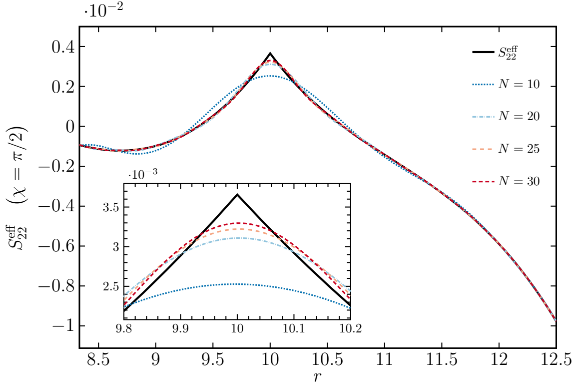
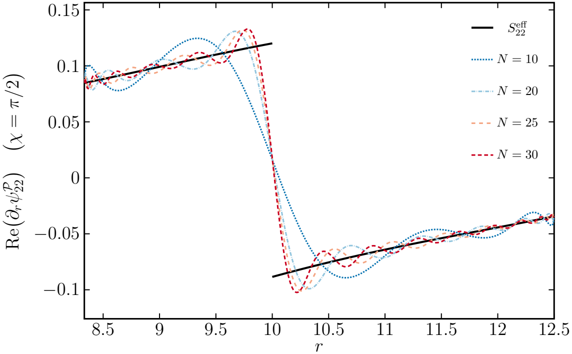
Gibbs Phenomenon

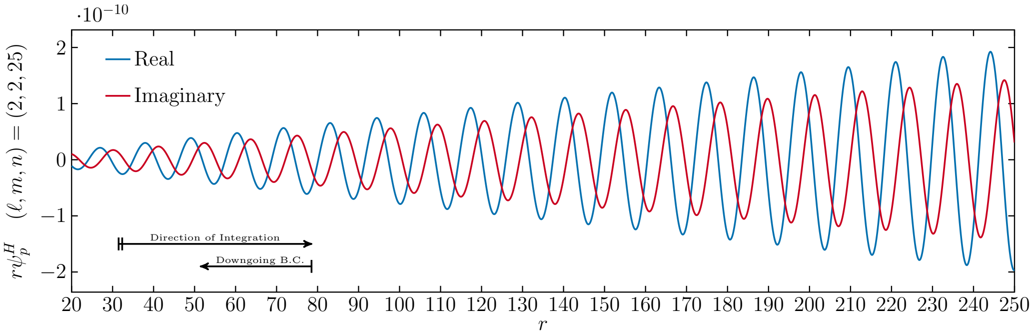
- Similar to EHS by Barack, Ori, and Sago (2008)



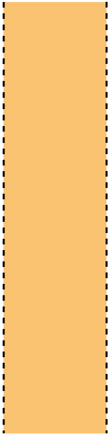


Excited Homogeneous Solutions

- Extended Particular Solutions originally formulated by Hopper and Evans (2013)



Extended Particular Solutions



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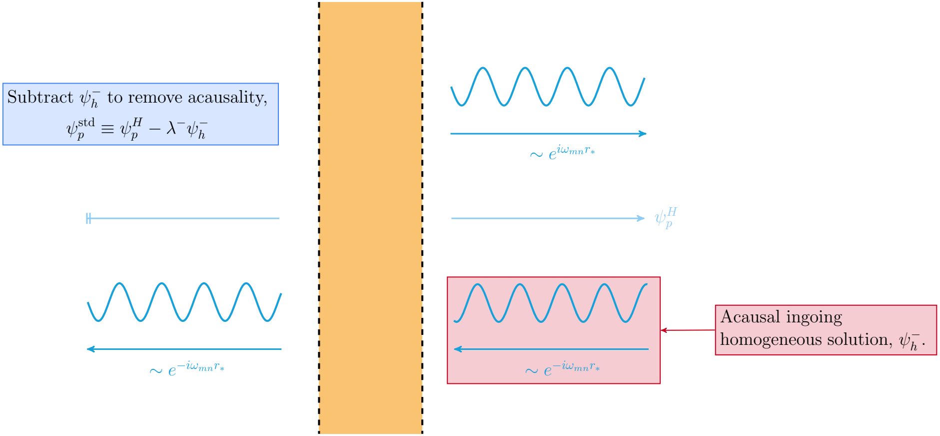

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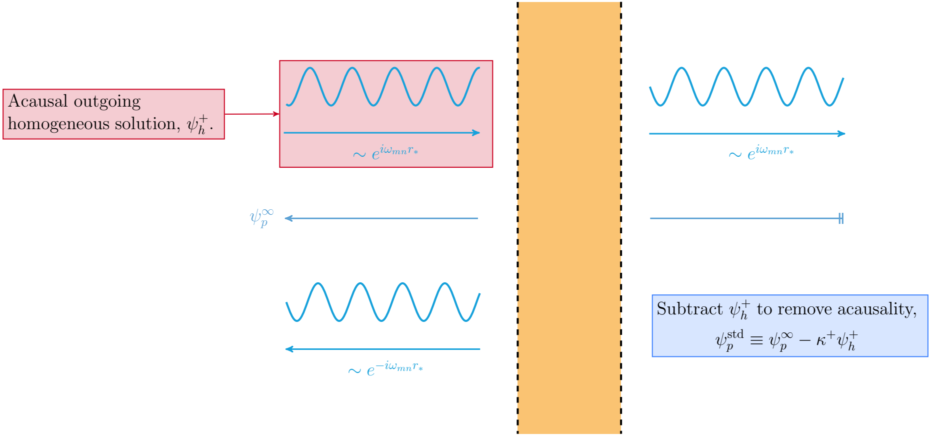

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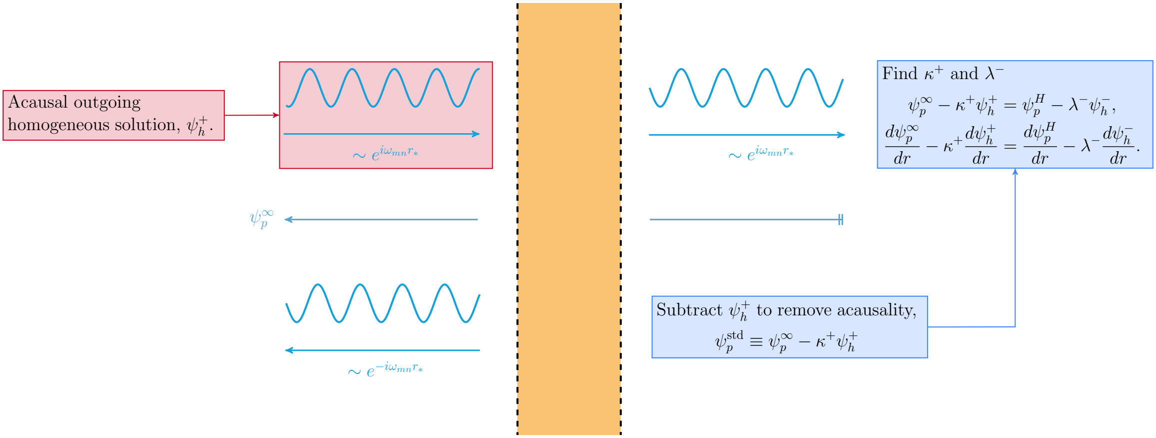

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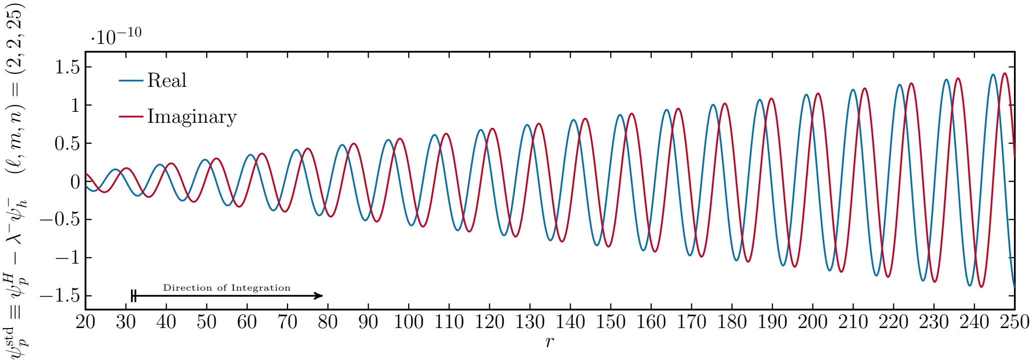
Extended Particular Solutions II



Extended Effective Sources (EES)

- Form two smooth functions: $\hat{S}^{\text{eff},+}_{\ell m}(t,r)$ and $\hat{S}^{\text{eff},-}_{\ell m}(t,r)$
- In the FD

$$\hat{S}^{\text{eff},\pm}_{\ell mn}(r) = \frac{1}{T_r}\int_0^{T_r} \hat{S}^{\text{eff},\pm}_{\ell m}(t,r)e^{i\omega_{mn}t}dt.$$

- We postulate, as $N \longrightarrow \infty$ in the partial sums

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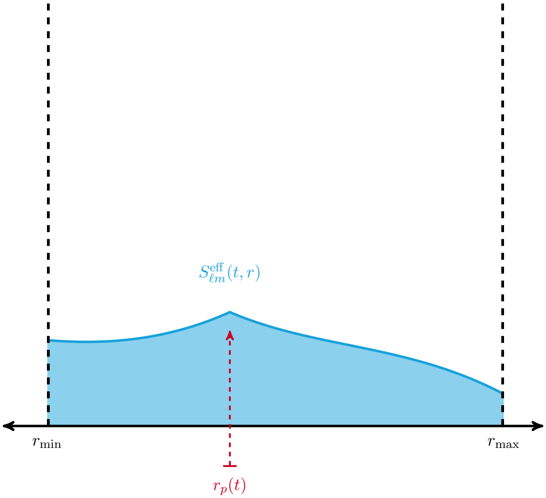

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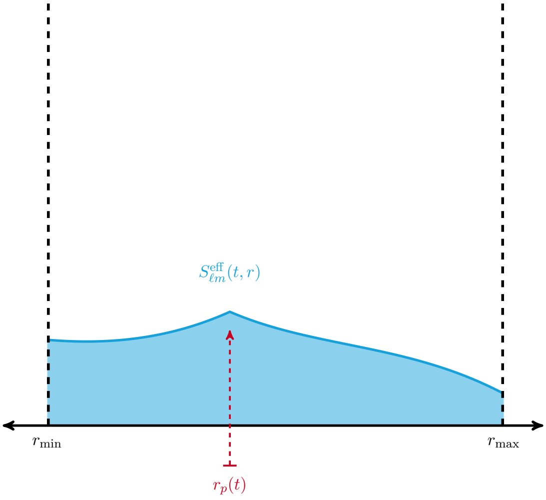

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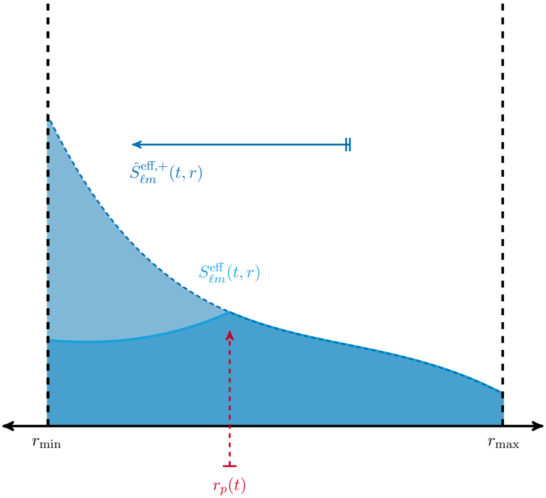

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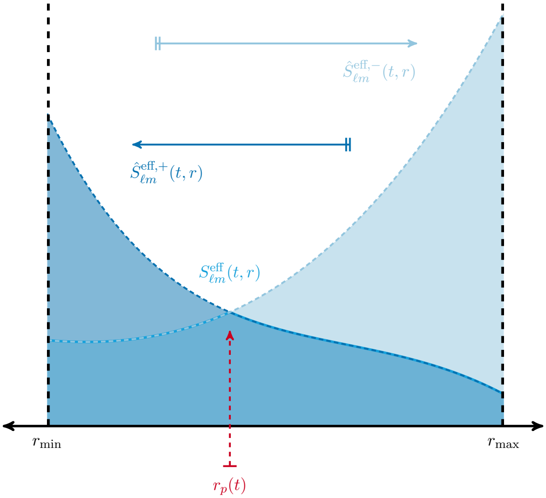

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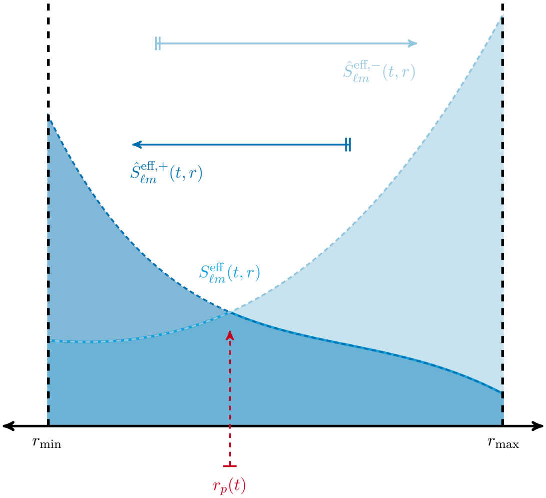
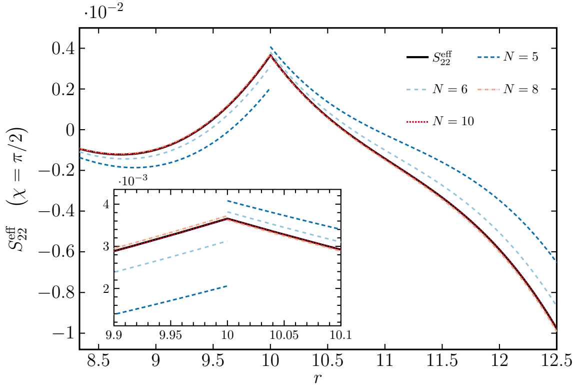
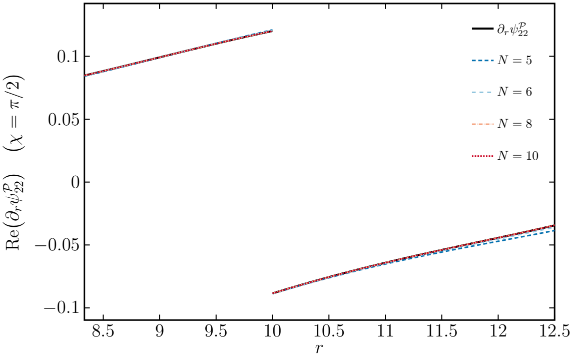


Extended Effective Sources (EES)

- By extension, for any t and r,

$$S^{\text{eff}}_{\ell m}(t,r) = \hat{S}^{\text{eff},+}_{\ell m}(t,r)\Theta[r - r_p(t)] + \hat{S}^{\text{eff},-}_{\ell m}(t,r)\Theta[r_p(t) - r].$$





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- We define the FD EPS (with EES) to be $\hat{\psi}_p^{\pm}$
- We then define

$$\tilde{\psi}^{+} \equiv \hat{\psi}_p^{+} - \kappa^{+}\psi_h^{+},$$
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- The FD EPS can then be transferred to the TD

$$\tilde{\psi}_{\ell m}^{\pm}(r,t) \equiv \sum_{n=-\infty}^{\infty} \tilde{\psi}_{\ell mn}^{\pm}(r)e^{-i\omega_{mn}t}$$

- The solution for the regular field is then given by the weak solution,

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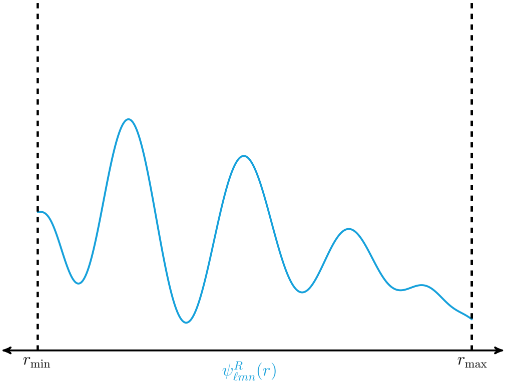

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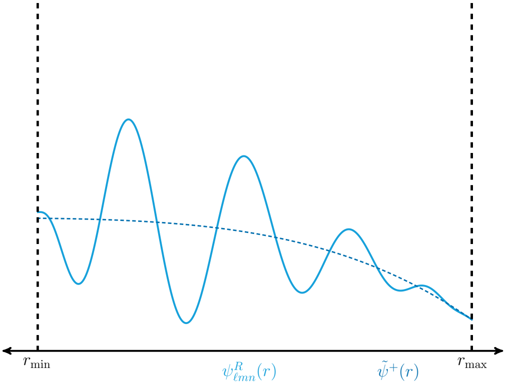

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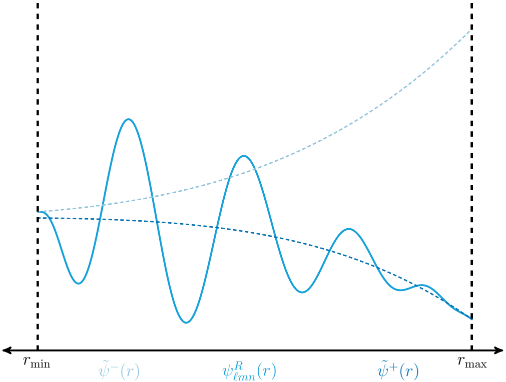

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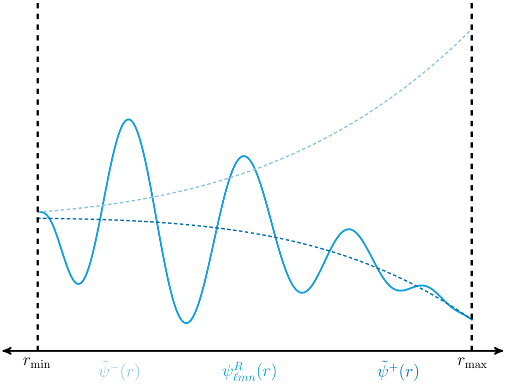

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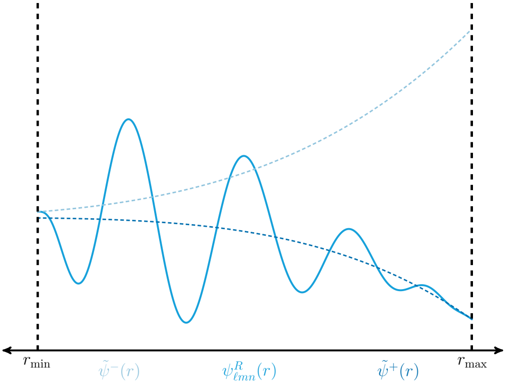

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Restoring Exponential Convergence with EES

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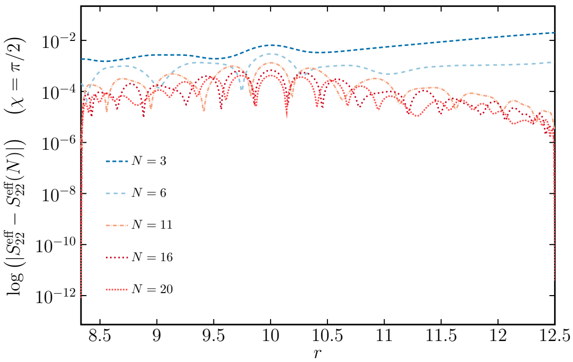

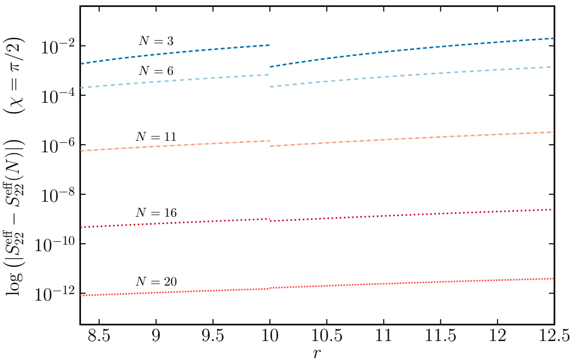

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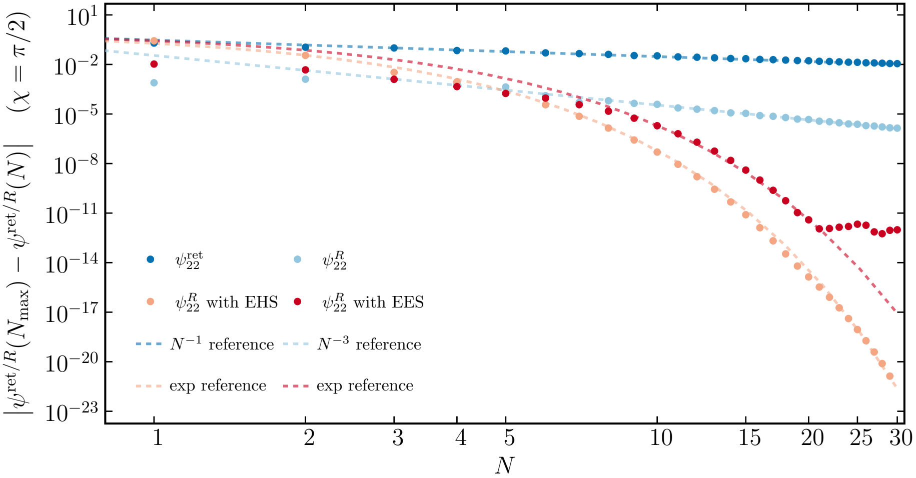

Effective Source Method for Teukolsky

- Need Teukolsky formalism for second-order SF in Kerr. See Andrew's talk...
- Require quick sampling over libration region
- Use Hyperboloidal slicing based on the framework by Zenginoglu (2011)

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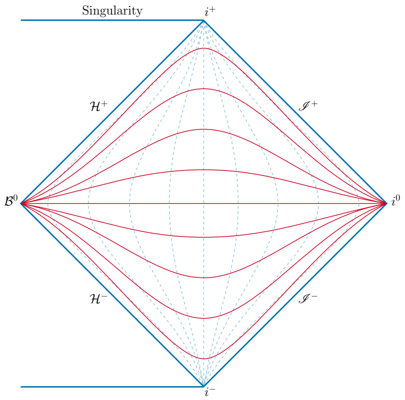

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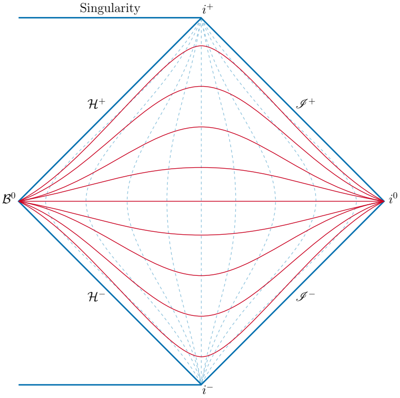

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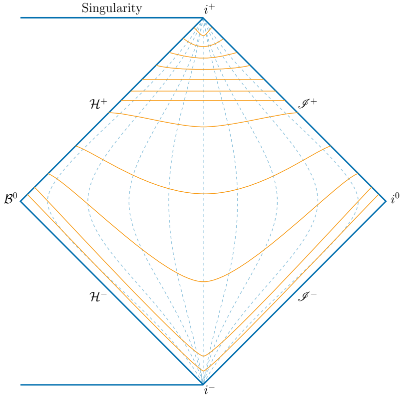

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Regularized Weyl Scalar

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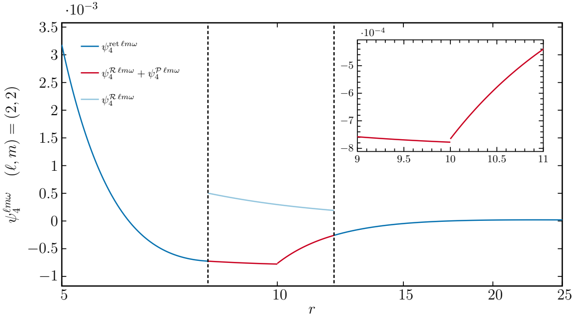

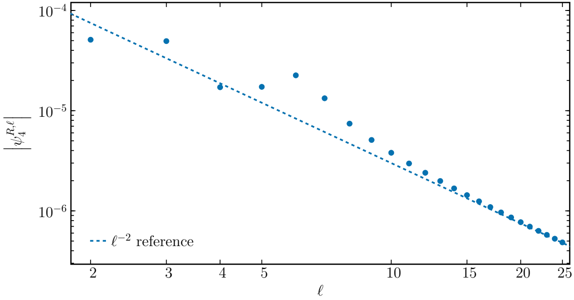

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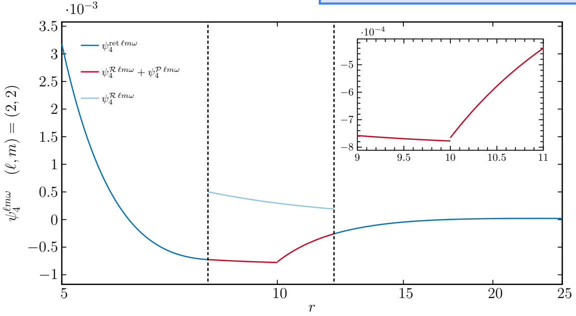
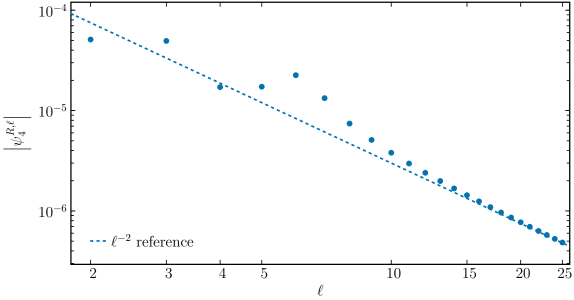
Bardeen-Press-Teukolsky differential operator, $s = -2$





Future Research

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Worldtube method for eccentric orbits with exponential convergence

- Resolve bottleneck in calculation of κ^+ and λ^-
- Extend to gravitational perturbations
- Lorenz Gauge eccentric formulation...
- Challenging

Effective source method for the Teukolsky equation (preliminary)

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