Towards a self-force calculation of the scatter angle in hyperbolic encounters

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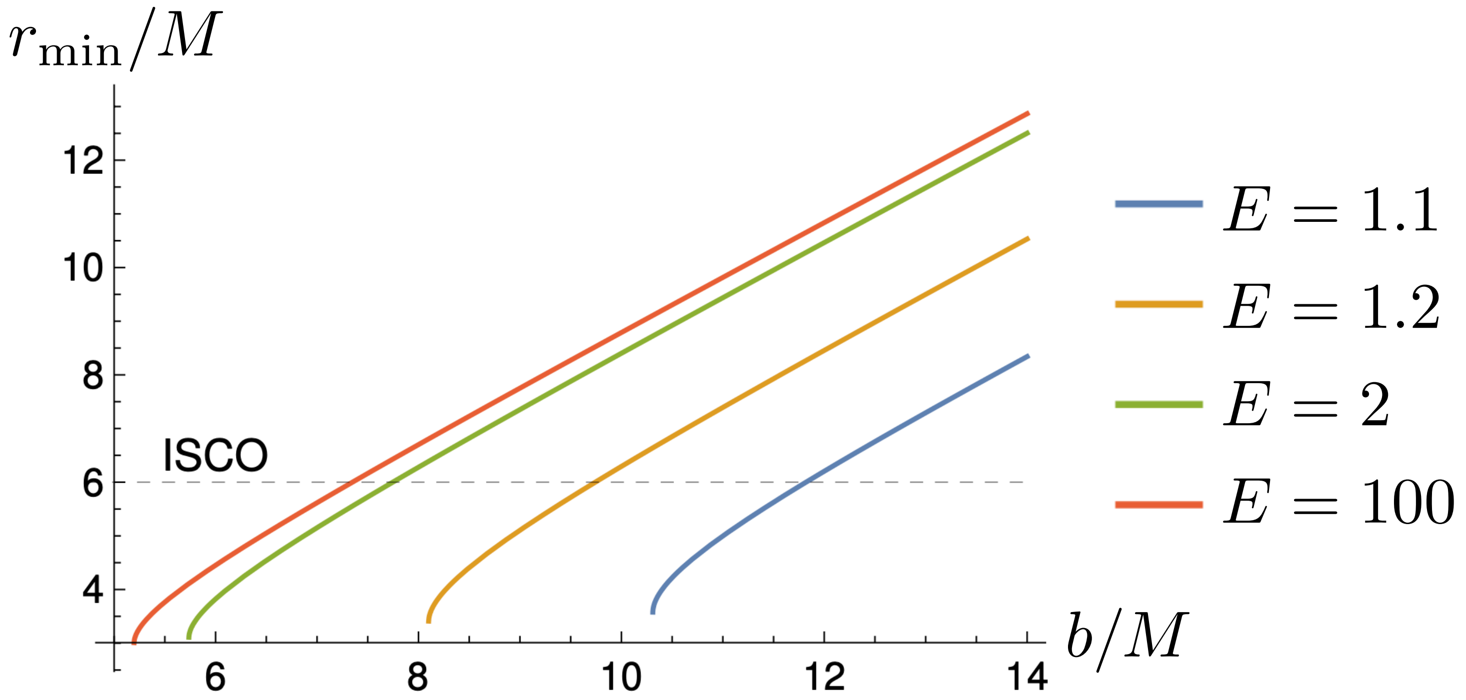
Motivation

- **Exact** post-Minkowskian calculations [Damour '19; Bini, Damour, Geralico '20]
  - 1[st] order self-force exactly determines 2 body Hamiltonian up to 4PM
  - 2[nd] order self-force exactly determines 2 body Hamiltonian up to 6PM
- Comparison with scatter amplitude calculations
- Strong-field benchmarking exact at $O(\eta)$ (no PN or PM expansions)



Scatter geodesics in Schwarzschild

Energy and angular momentum

$$E > 1 \qquad L > L_{\text{crit}}(E)$$

Eccentricity and semi-latus rectum

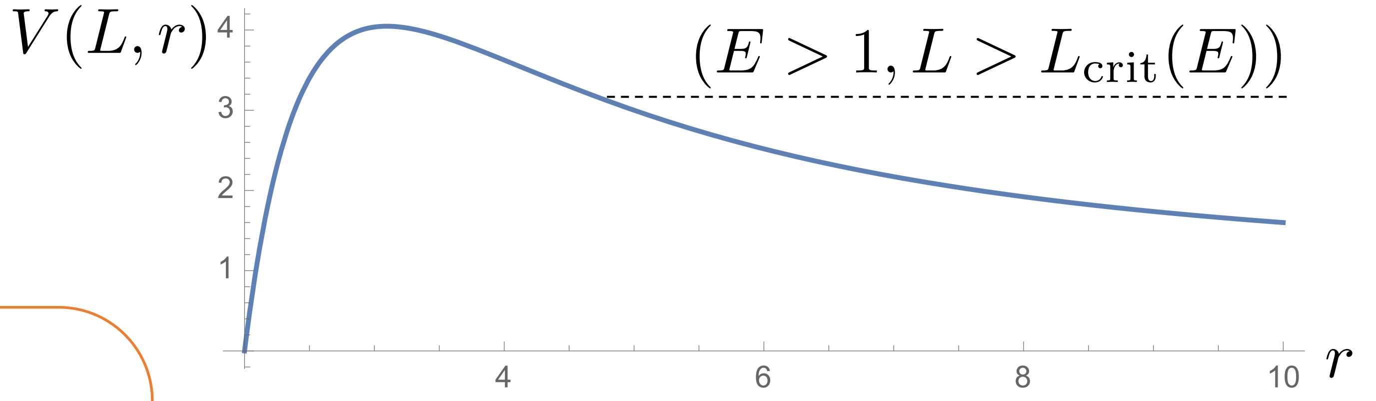
$$e \geq 1 \qquad p > p_{\text{crit}}(e)$$

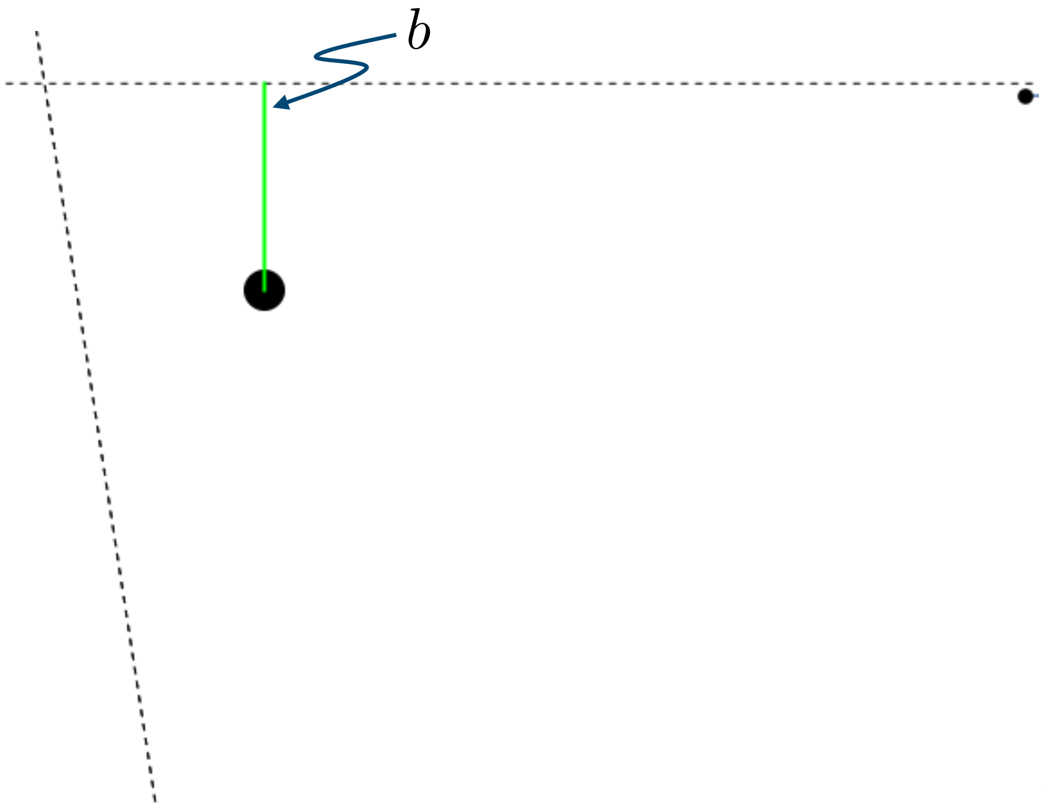
$$r = \frac{Mp}{1 + e\cos\chi}$$

$$-\chi_\infty \leq \chi \leq \chi_\infty \text{ where } \chi_\infty := \arccos(-1/e)$$

Velocity at infinity and impact parameter

$$v_\infty := \left.\frac{dr}{dt}\right|_{r\to\infty} \qquad b := \lim_{r\to\infty} r \sin|\varphi(r) - \varphi(\infty)|$$





Scatter angle: geodesic case

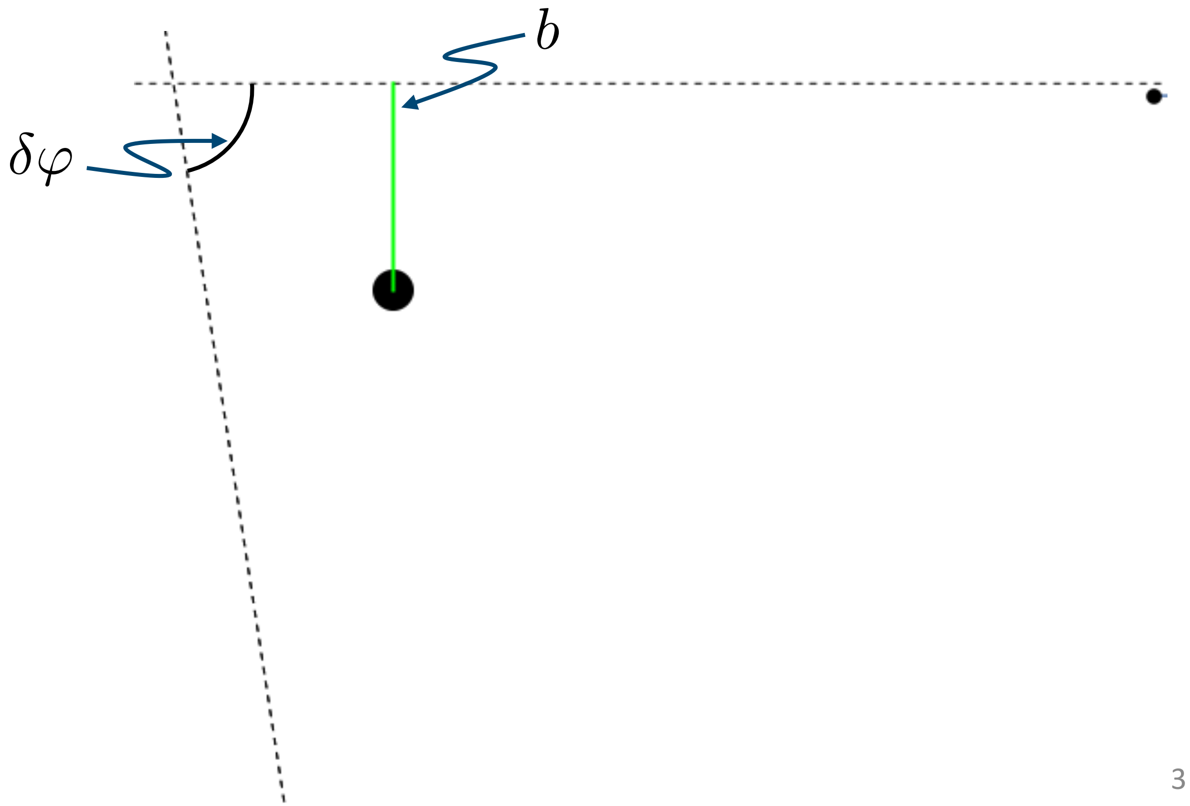
Definition

$$\delta\varphi := \int_{-\chi_\infty}^{\chi_\infty} \frac{d\varphi}{d\chi} d\chi - \pi$$

$$\delta\varphi = 2k\sqrt{p/e}\,\hat{F}\left(\frac{\chi_\infty}{2}; -k^2\right) - \pi$$

$$k = 2\sqrt{\frac{e}{p-6-2e}}$$

$$\hat{F}(\varphi;k) = \int_0^\varphi (1 - k\sin^2 x)^{-1/2} dx$$



First order self-force correction to the scatter angle

Expansion of scatter angle

$$\delta\varphi = \delta\varphi^{(0)}(v_\infty, b) + \eta\delta\varphi^{(1)}(v_\infty, b)$$

Solve 1st order self-force equations of motion at fixed v_∞ and b

$$\begin{aligned}\delta\varphi^{(1)} =& \mathcal{A}(e,p)\int_{-\chi_\infty}^{\chi_\infty} F_t \, d\chi + \mathcal{B}(e,p)\int_{-\chi_\infty}^{\chi_\infty} F_\varphi \, d\chi \\ &+ \int_{-\chi_\infty}^{\chi_\infty} \left[a(\chi; e,p)F_t + b(\chi; e,p)F_\varphi\right] d\chi\end{aligned}$$

F_α is the conservative part of the self-force

Self-force calculation method

- Time-domain metric reconstruction in a radiation gauge from Hertz potential, with mode sum regularization [Barack & Giudice '17]
- Main hurdle is the formulation of jump conditions for the Hertz potential
- Implementation using a 1+1 evolution in Eddington-Finkelstein coordinates

In the rest of this talk:

- Formulation of jump conditions for Hertz potential
- Implementation of jump conditions – instabilities and resolutions
- Evolution of Teukolsky equation: problem of divergent modes
- Resolution via transformation to a Regge-Wheeler-like equation

Brief review of metric reconstruction

Vacuum case

Teukolsky equation

$$\hat{\mathcal{O}}_\pm \psi_\pm = 0$$

'Inversion' relation

$$D_\pm^4 \phi_\pm = \psi_\pm$$

Hertz potential obeys adjoint Teukolsky

$$\hat{\mathcal{O}}_\pm^\dagger \phi_\pm = \hat{\mathcal{O}}_\mp \phi_\pm = 0$$

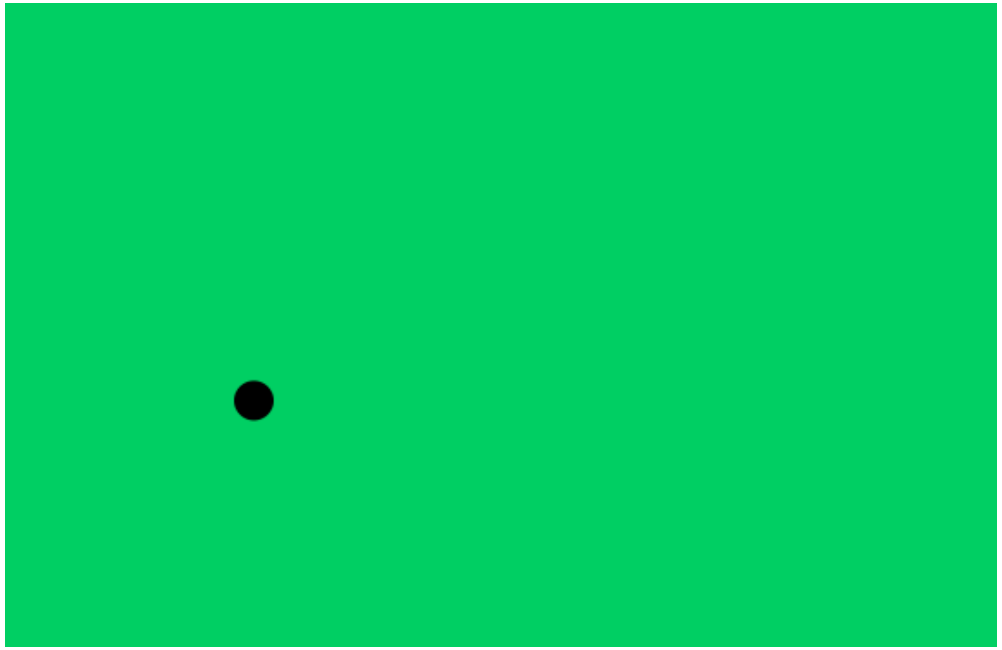
Metric reconstruction

$$h_{\alpha\beta} = \mathrm{Re}\left(\hat{S}_\pm^\dagger \phi_\pm\right)_{\alpha\beta}$$

$$\psi_+ := \psi_0$$
$$\psi_- := \rho^{-4}\psi_4$$
$$\phi := \text{Hertz potential}$$

Brief review of metric reconstruction

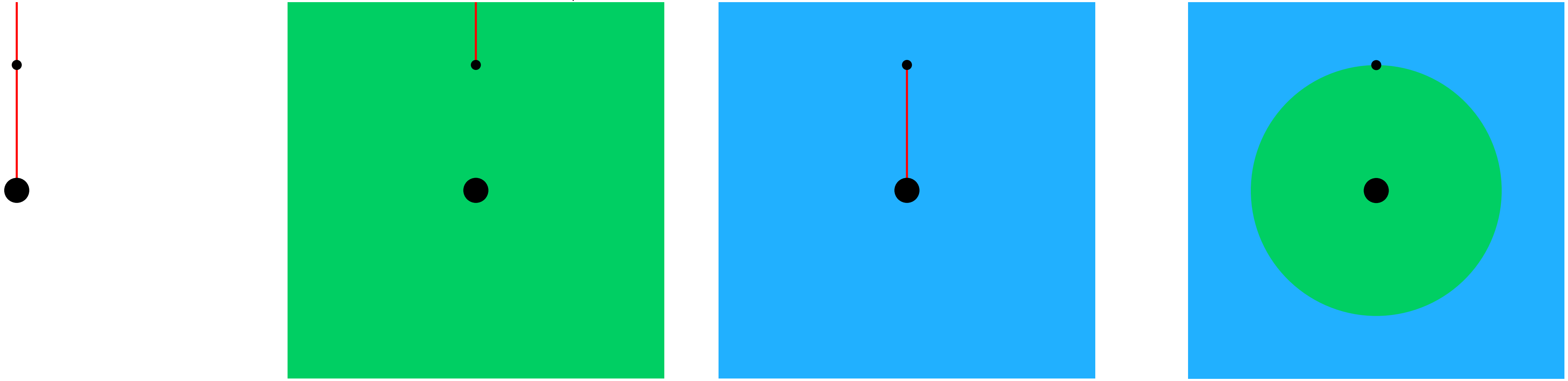
Particle case



Sourced Teukolsky equation

$$\hat{\mathcal{O}}_{\pm}\psi_{\pm} = T_{\pm} \propto \delta(r - r_{\mathrm{P}})$$

Point particle solutions for ϕ and $h_{\alpha\beta}$



Reduction of 1+1 inversion relation to 1$^{\text{st}}$ order

Inversion relation [schematic]

$$\partial_u \left[\phi_{,vvvv} = \psi_0\right]$$

1+1 mode decomposed Teukolsky equation

$$\phi_{,uv} + U(r)\phi_{,u} + V(r)\phi_{,v} + W(r)\phi = 0$$

3$^{\text{rd}}$ order ODE

$$\partial_u \left[\phi_{,vvv} + a_2\phi_{,vv} + a_1\phi_{,v} + a_0\phi = D_1\psi_0\right]$$

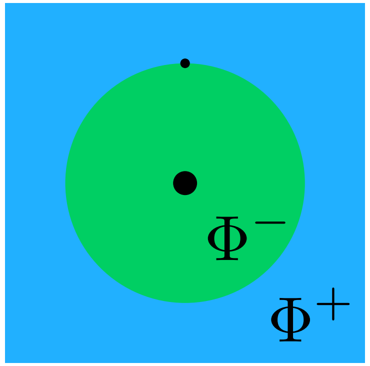
2$^{\text{nd}}$ order ODE

$$\partial_u \left[\phi_{,vv} + b_1\phi_{,v} + b_0\phi = D_2\psi_0\right]$$

1$^{\text{st}}$ order PDE

$$\boxed{c_v\phi_{,v} + c_u\phi_{,u} + c\phi = D_3\psi_0}$$

Jumps in the Hertz potential



Jump in a field

$$[\Phi] := \lim_{\epsilon\to 0}\left[\Phi^+(\tau, r_\mathrm{P}(\tau)+\epsilon) - \Phi^-(\tau, r_\mathrm{P}(\tau)-\epsilon)\right]$$

Proper time derivatives of jump

$$\begin{aligned}[\dot\phi] &= \dot v_\mathrm{P}[\phi_{,v}] + \dot u_\mathrm{P}[\phi_{,u}] \\ &= d_1[\phi_{,v}] + d_2[\phi] + [\psi_0] \text{ terms}\end{aligned}$$

$$\begin{aligned}[\ddot\phi] &= e_1[\phi_{,vv}] + e_2[\phi_{,uv}] + e_3[\phi_{,v}] + e_4[\phi] + e_5[\dot\phi] + [\psi_0] \text{ terms} \\ &= f_1[\phi_{,v}] + f_2[\phi] + f_3[\dot\phi] + [\psi_0] \text{ terms}\end{aligned}$$

Eliminate $[\phi_{,v}]$

$$\boxed{[\ddot\phi] + A(\tau)[\dot\phi] + B(\tau)[\phi] = \mathcal{F}(\tau, [\psi_0])}$$

Jumps in the Hertz potential

Problem

Asymptotic homogeneous solution

$$[\phi] \sim C_1 e^{\kappa\tau} + C_2 e^{-\kappa\tau}$$

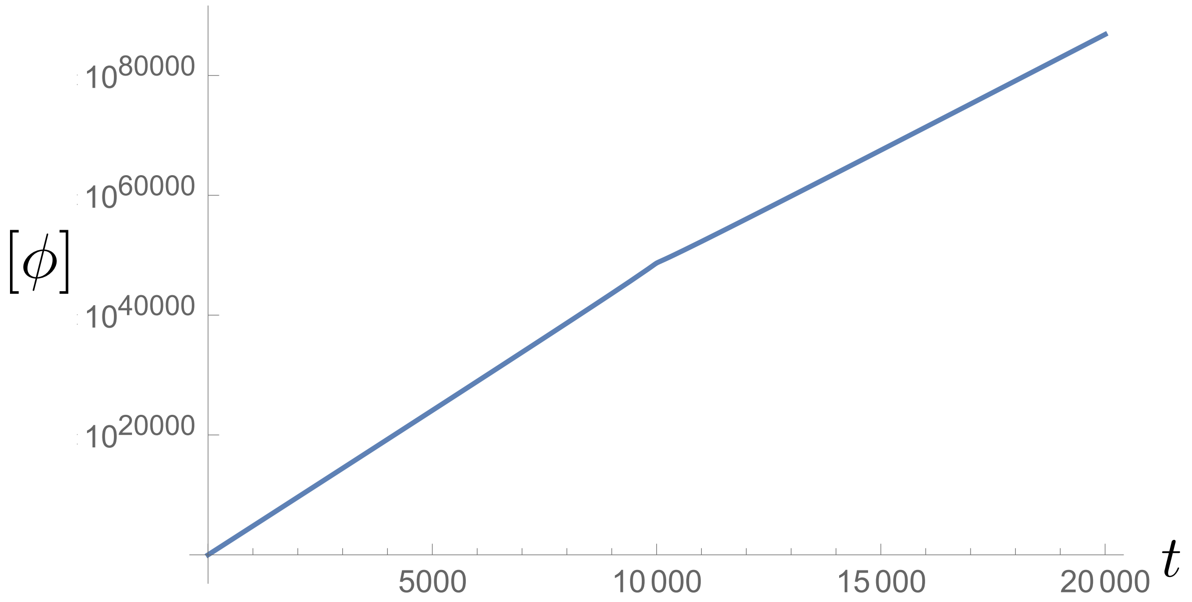
Resolution

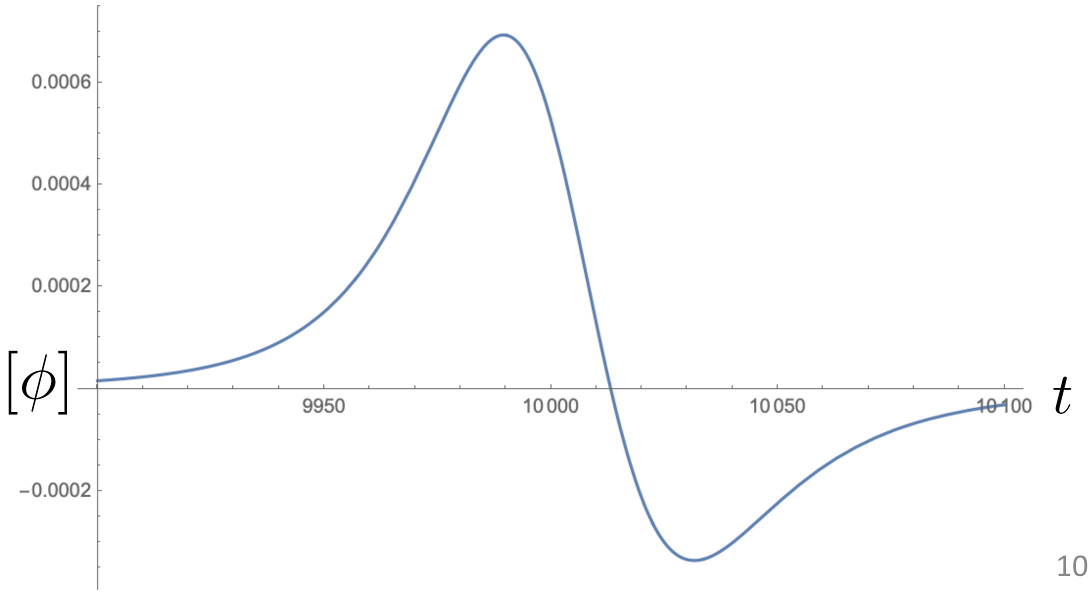
Convert to hierarchy of 1st order ODEs

$$\begin{aligned} [\dot{\phi}] - \alpha[\phi] &= \beta \\ \dot{\beta} + (\alpha + A)\beta &= \mathcal{F} \\ \dot{\alpha} + (\alpha + A)\alpha &= -B \end{aligned}$$

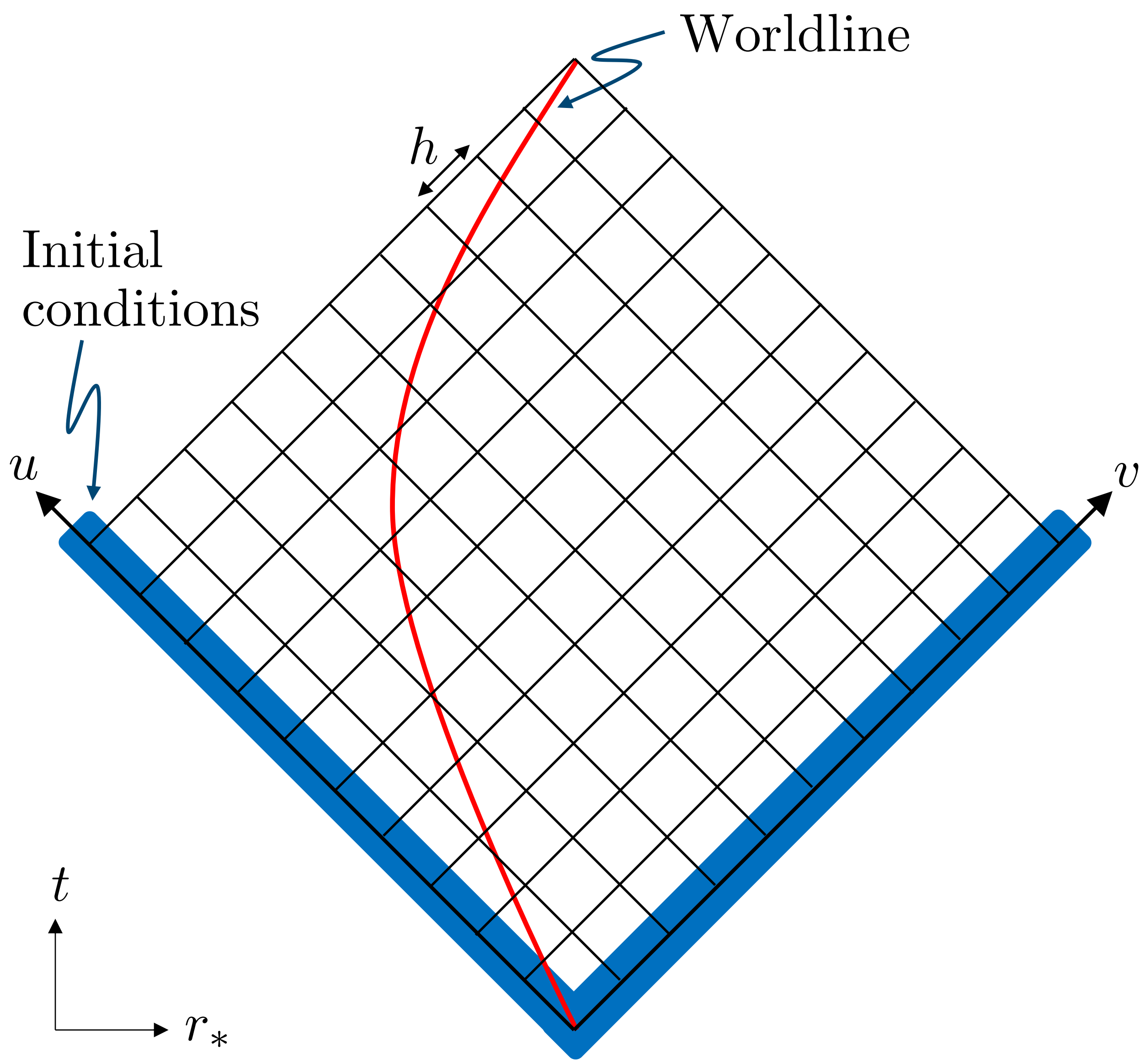
Asymptotic solutions of homogeneous ODEs

$$\alpha \sim C_\alpha e^{-\kappa_\alpha \tau} \quad \beta \sim C_\beta e^{-\kappa_\beta \tau} \quad [\phi] \sim C_{[\phi]} e^{\kappa_{[\phi]} \tau}$$



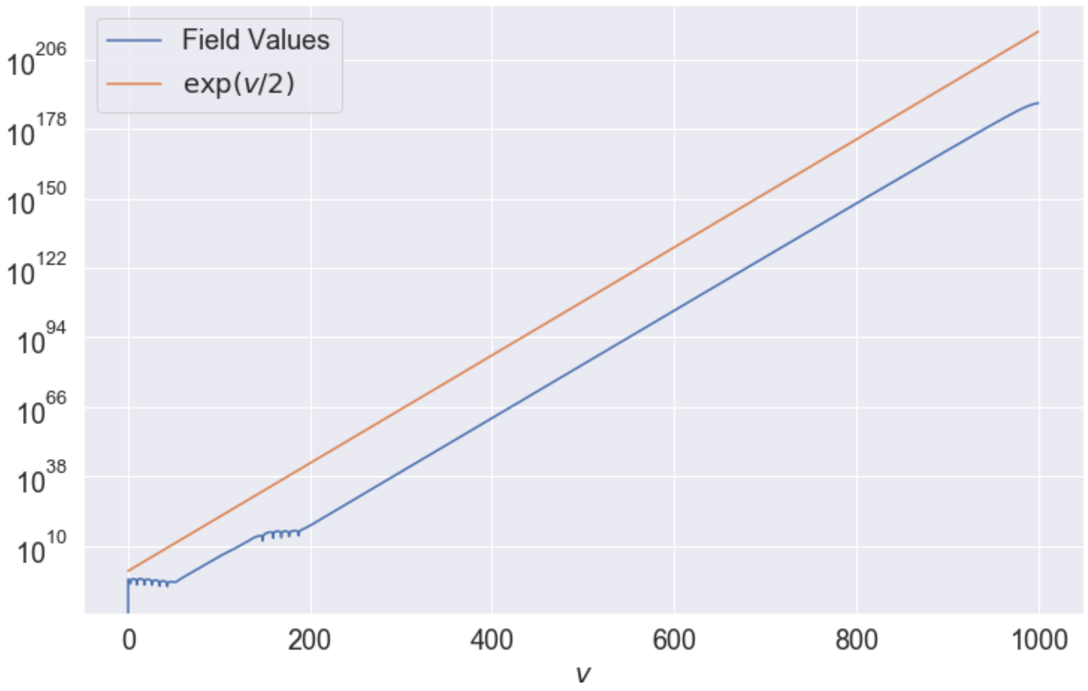


Teukolsky evolution code

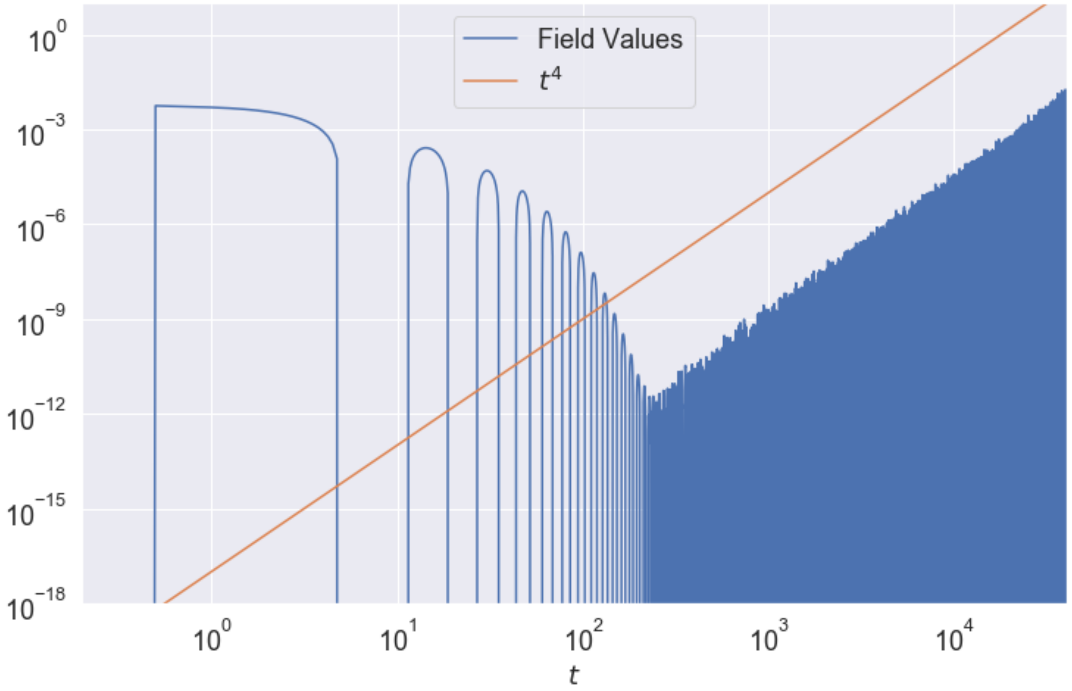


Problem of divergent modes

$s = +2$



$s = -2$



There exist non-physical asymptotic solutions:

$$\psi_{\mathcal{H}^+} \sim C_1(u)\exp(v\, s/4M) + C_2(v) \qquad \psi_{\mathcal{I}^+} \sim C_3 t^{-2s} e^{-i\omega v}$$

Non-physical terms usually removed by physical boundary conditions

Both terms originate from the $\phi_{,t}$ term of the Teukolsky equation

How to avoid the divergent modes?

Other Teukolsky codes:

- Impose boundary conditions (Khanna ‘04)
- Hyperboloidal slicing (Harms et al ‘15)

In Schwarzschild, eliminate $\phi_{,t}$ term

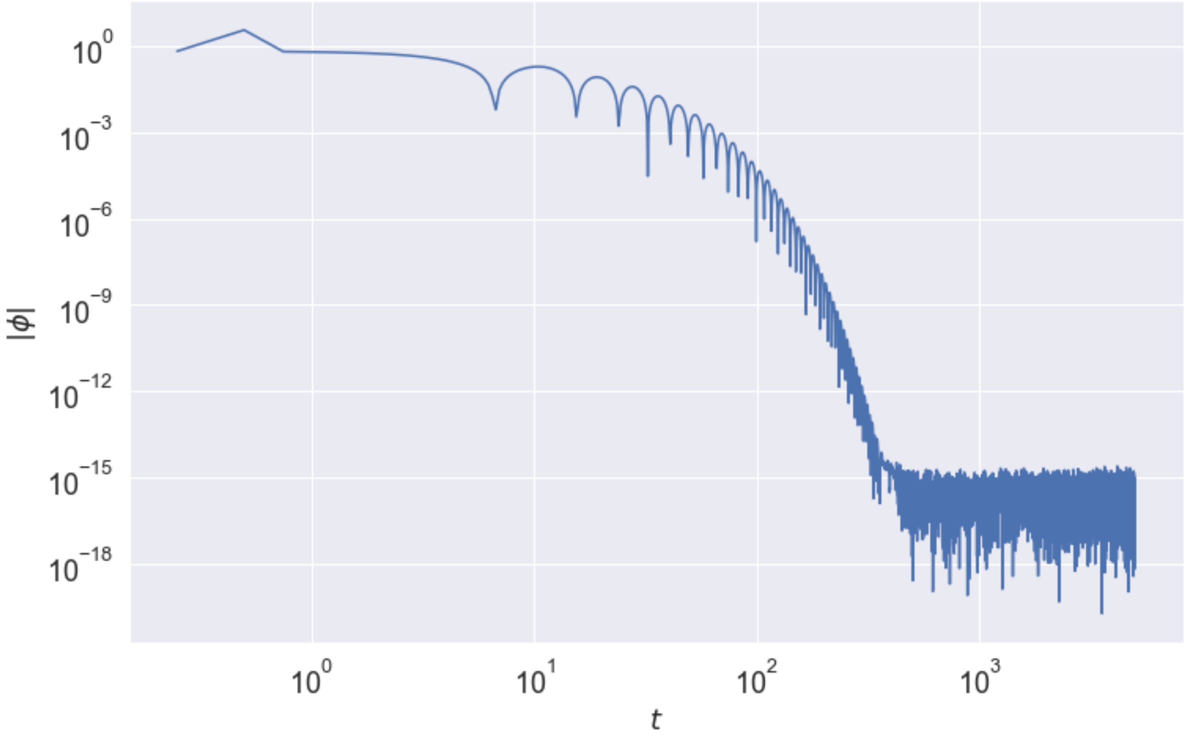
Transformation to Regge-Wheeler form

Time-domain Chandrasekhar transformation

$$\phi = f^{-2}\left(X_{,uu} - \frac{r-3M}{r^2}X_{,u}\right)$$

$$X_{,uv} + \frac{f}{4}\left(\frac{l(l+1)}{r^2} - \frac{6M}{r^3}\right)X = 0$$

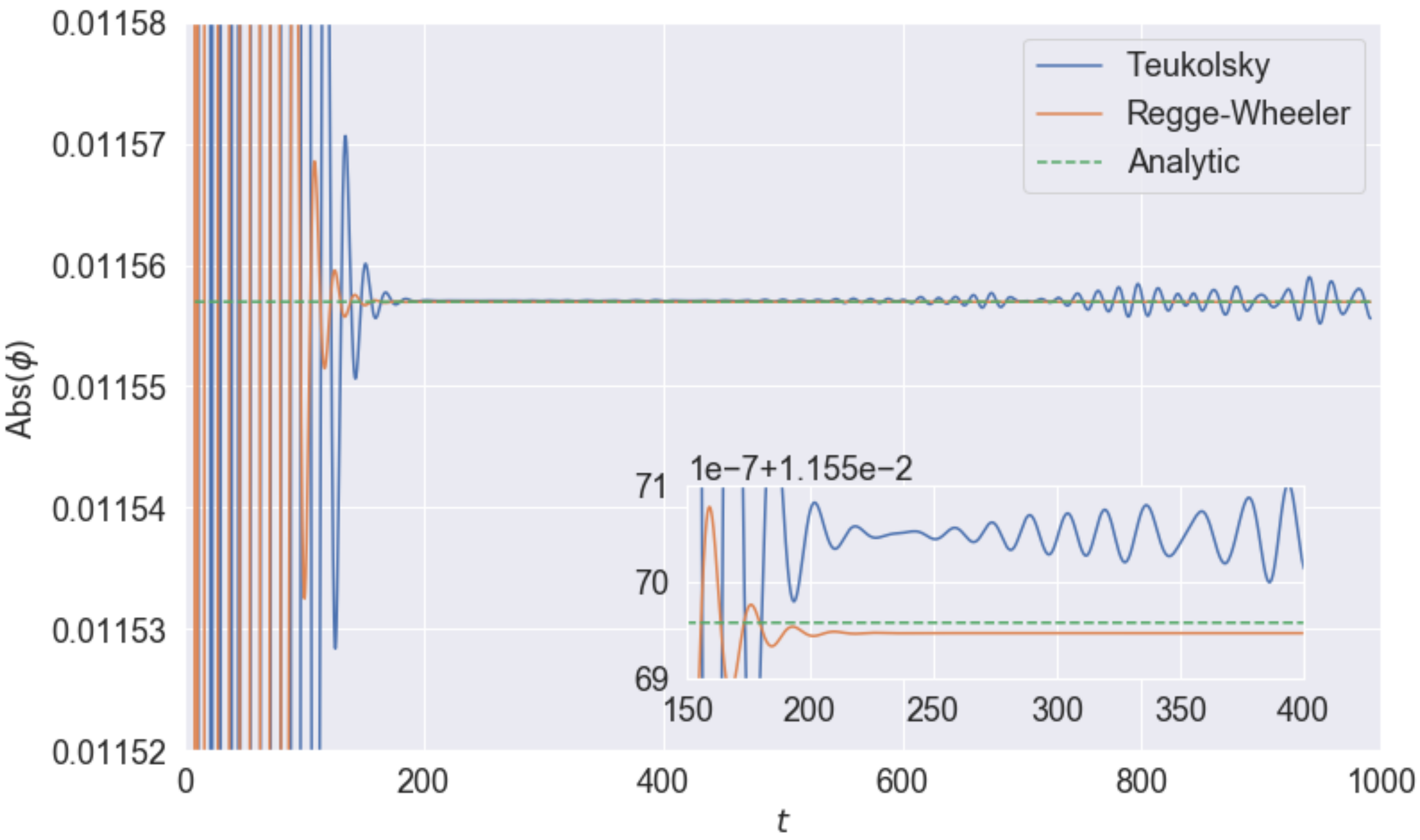
Use similar method to inversion relation to obtain ODE for jumps in the Regge-Wheeler Field



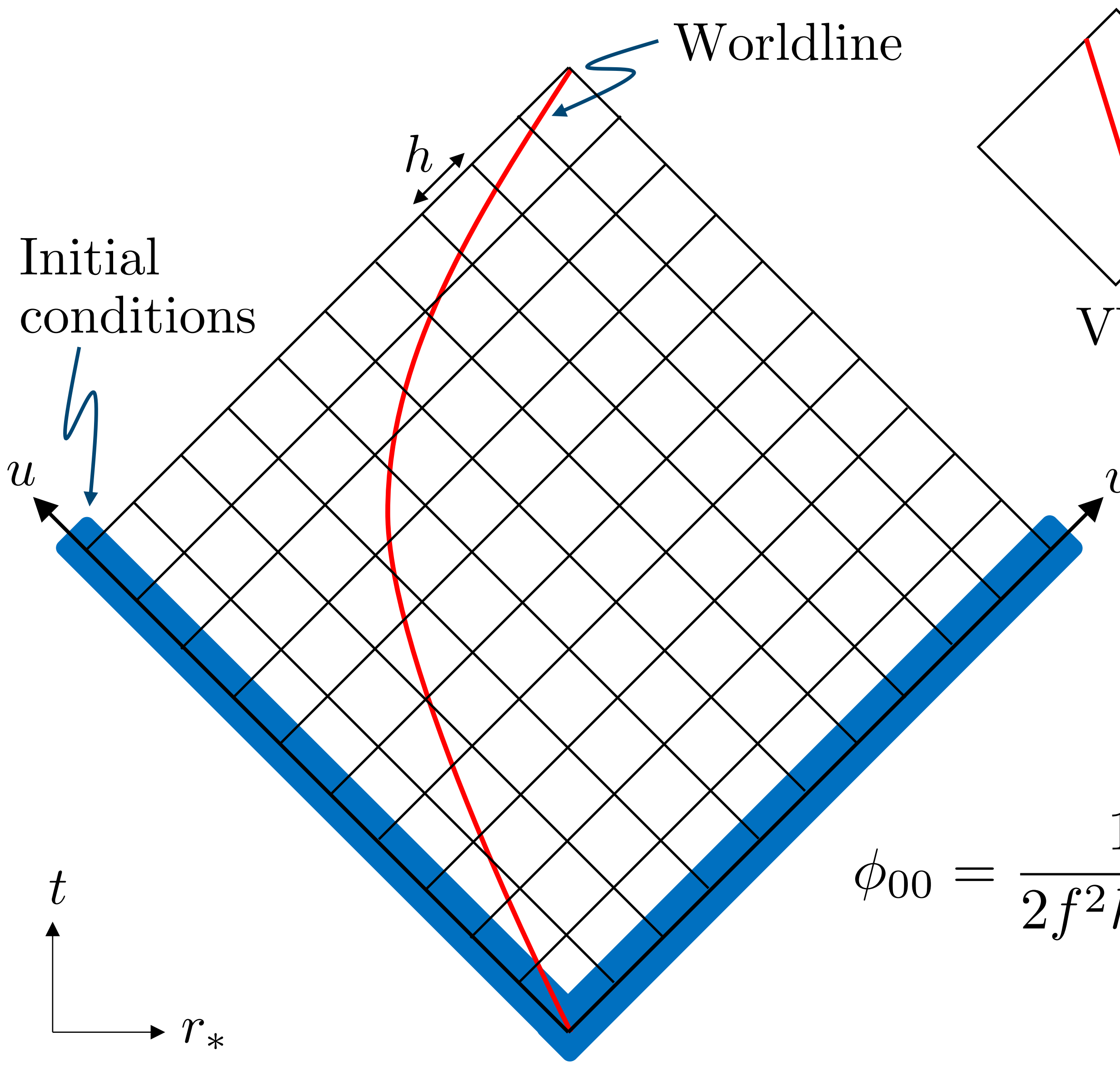
$$[\dot{X}] + \mathcal{B}(\tau)[X] = \mathcal{C}\left(\tau, [\phi]\right)$$

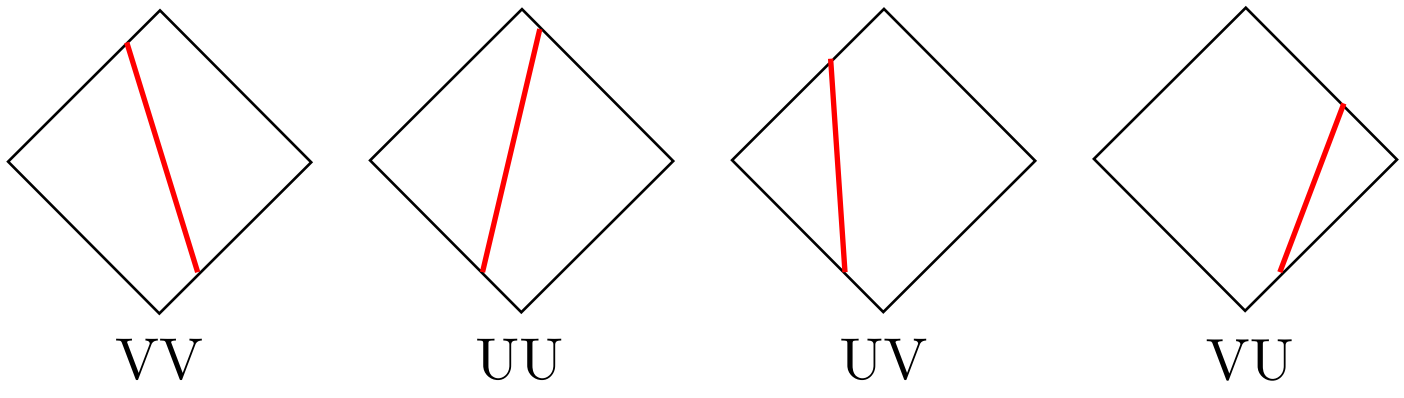
Asymptotic homogeneous solution $[X] \sim C_{[X]}e^{-\kappa_{[X]}\tau}$

Regge-Wheeler and Teukolsky circular point particle solutions



Scatter Regge-Wheeler evolution code





$$X_{00} = \sum_{i,j=0}^{1} H_{ij}(h;r)X_{ij} + J_{YY}$$

$$YY \in \{VV, UU, UV, VU\}$$

$$\phi_{00} = \frac{1}{2f^2h^2r^2}\Big[-4r^2X_{00} + \left(h(r-3M)+2r^2\right)X_{10} + \left(-h(r-3M)+2r^2\right)X_{-10}\Big] + \mathcal{J}_{YY}$$

Conclusion

- Self-force scattering is needed to give information for exact PM calculations, comparison with scattering amplitudes and benchmarking in the strong field regime
- The 4^{th} order inversion relation in 1+1 can be written as a 2^{nd} order ODE or 1^{st} order PDE
- The Teukolsky equation is numerically unstable without boundary conditions
- Transforming to Regge-Wheeler removes the numerical instability

- **Work plan:**
  - Full evolution of Hertz potential
  - Evaluation of self-force
  - Correction to scatter angle