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DISCONTINUOUS COLLOCATION METHODS FOR SELF-FORCE APPLICATIONS

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TEUKOLSKY (OR REGGE-WHEELER) EQUATION

$$[(\nabla^{\mu} + A^{\mu})\nabla_{\mu} + V]\psi_0(r) = S$$

- Strongly hyperbolic PDE describing linear perturbations in Kerr (or Schwarzschild)
- Typically solved as ODE in frequency domain
- Used to compute GSF in radiation (or Regge-Wheeler) gauge
  (Shah, Friedman, Whiting; Barack, Pound, Merlin; Kohen, Kegeles; Barack, Ori et al)
- For highly eccentric or unbound orbits, time-domain self-consistent evolution is ideal
- Time-domain Teukolsky solvers have been increasing in sofistication
  (Burko, Khanna, Pullin, Hughes, Poisson, Lousto, Zenginoglu, Harms, Bernuzzi, Brügmann

METHOD OF LINES

- Reduce to first-order in time

$$\partial_t U_A = L^{AB} U_B + S_B, \quad U_A = \left(\psi_0, \dot{\psi}_0\right)$$

- Discretize **U**, **L** in space, evolved as coupled ODE system in time

$$\frac{d\mathbf{U}}{dt} = \mathbf{L}\mathbf{U} + \mathbf{S}$$

- **L** contains spatial differentiation matrices, which are typically valid only for smooth methods

TEUKOLSKY EQUATION WITH PARTICLE SOURCE

$$[(\nabla^{\mu} + A^{\mu})\nabla_{\mu} + V]\psi_0(r) = G\delta(r - \xi(t)) + F\delta'(r - \xi(t))$$

- Approximate δ as Gaussian pulse (Harms, Bernuzzi and Brügmann)
- Construct finite-difference representation of δ (Hughes et al)
- "Lagrangian" picture: Domain decomposition + time-dependent grid + jump conditions across particle (Canizares and Sopuerta; Field, Hesthaven and Lau, Diener et al).
- "Eulerian" picture: Discontinuous collocation method + fixed grid + jump conditions across particle (Markakis, Barack et al).

C. Markakis and L. Barack [arXiv:1406.4865]

DISCONTINUOUS COLLOCATION METHOD

- Jumps in solution ψ and its derivatives across particle known a priori from field equation

$$\psi^{(k)}(\xi^+) - \psi^{(k)}(\xi^-) = J_k, \quad k = 0,1,2,\ldots$$

- Collocation (finite-difference and pseudo-spectral) methods are based on Lagrange interpolation
- Construct discontinuous generalization to Lagrange interpolation that uses $\{J_i\}$ as input.

LAGRANGE INTERPOLATION

- N^{th} order polynomial

$$p(x) = \sum_{j=0}^{N} c_j x^j$$

- Collocation conditions

$$p(x) = f_i, \quad i = 0,1,\ldots,N$$

- Solution: Lagrange interpolating polynomial

$$p(x) = \sum_{j=0}^{N} f_j(x)\pi_j(x), \quad \pi_j = \prod_{k=0}^{N} \frac{x - x_k}{x_j - x_k}$$

DISCONTINUOUS LAGRANGE INTERPOLATION

- N^th order piecewise polynomial

$$p(x) = \sum_{j=0}^{N} [\theta(x-\xi)c_j^+ + \theta(\xi - x)c_j^-]x^j$$

- Collocation conditions

$$p(x) = f_i, \quad i = 0,1,\ldots,N$$

- Jump conditions

$$p^{(k)}(\xi^+) - p^{(k)}(\xi^-) = J_k, \quad k = 0,1,2,\ldots$$

- Solution: interpolating piecewise polynomial

$$p(x) = \sum_{j=0}^{N} [f_j(x) + \Delta(x_j - \xi; x - \xi)]\pi_j(x)$$

DISCONTINUOUS INTERPOLATION, DIFFERENTIATION, INTEGRATION

- Interpolation

$$p(x) = \sum_{j=0}^{N} [f_j(x) + \Delta(x_j - \xi; x - \xi)]\pi_j(x)$$

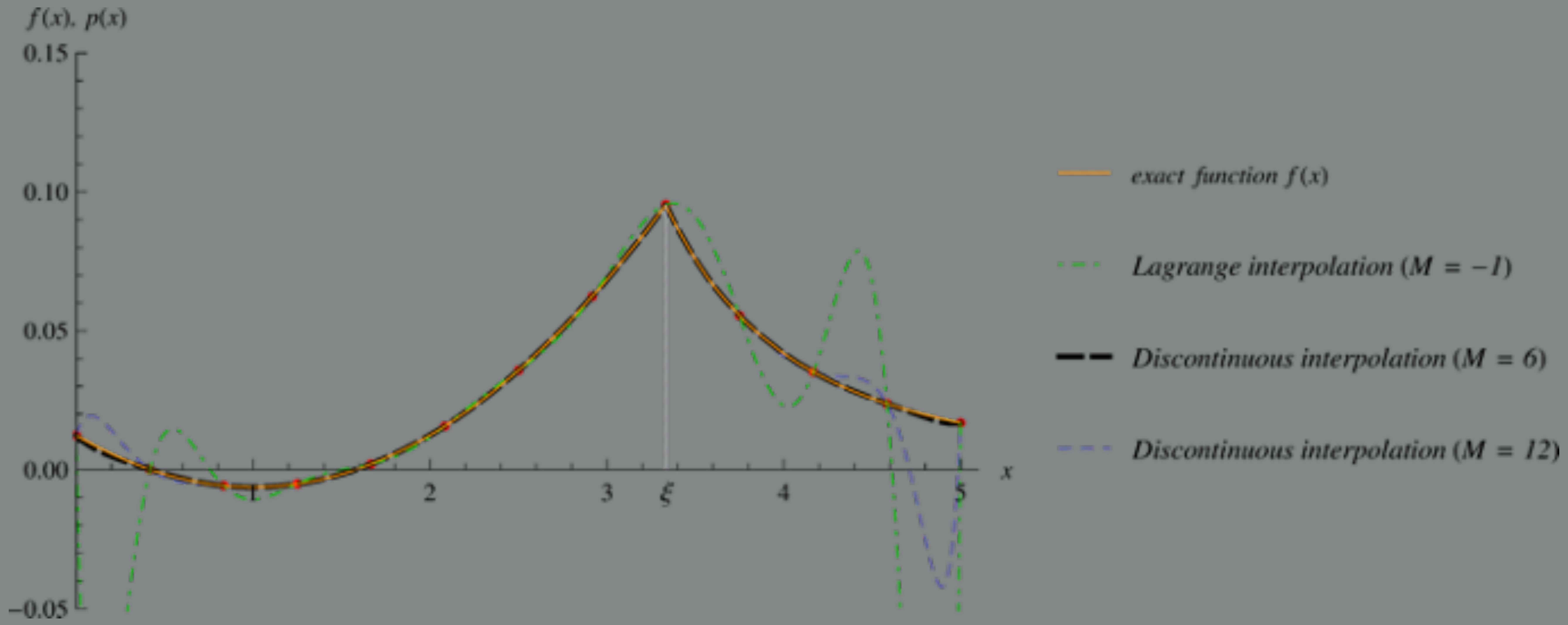
- Differentiation

$$p^{(n)}(x_i) = \sum_{j=0}^{N} D_{ij}^{(n)}[f_j(x) + \Delta(x_j - \xi; x_i - \xi)], \quad D_{ij}^{(n)} = \pi_j^{(n)}(x_i)$$

$$\Delta(x_j - \xi; x_i - \xi) = [\theta(x_i - \xi) - \theta(x_j - \xi)] \sum_k \frac{J_k}{k!}(x_j - \xi)^k$$

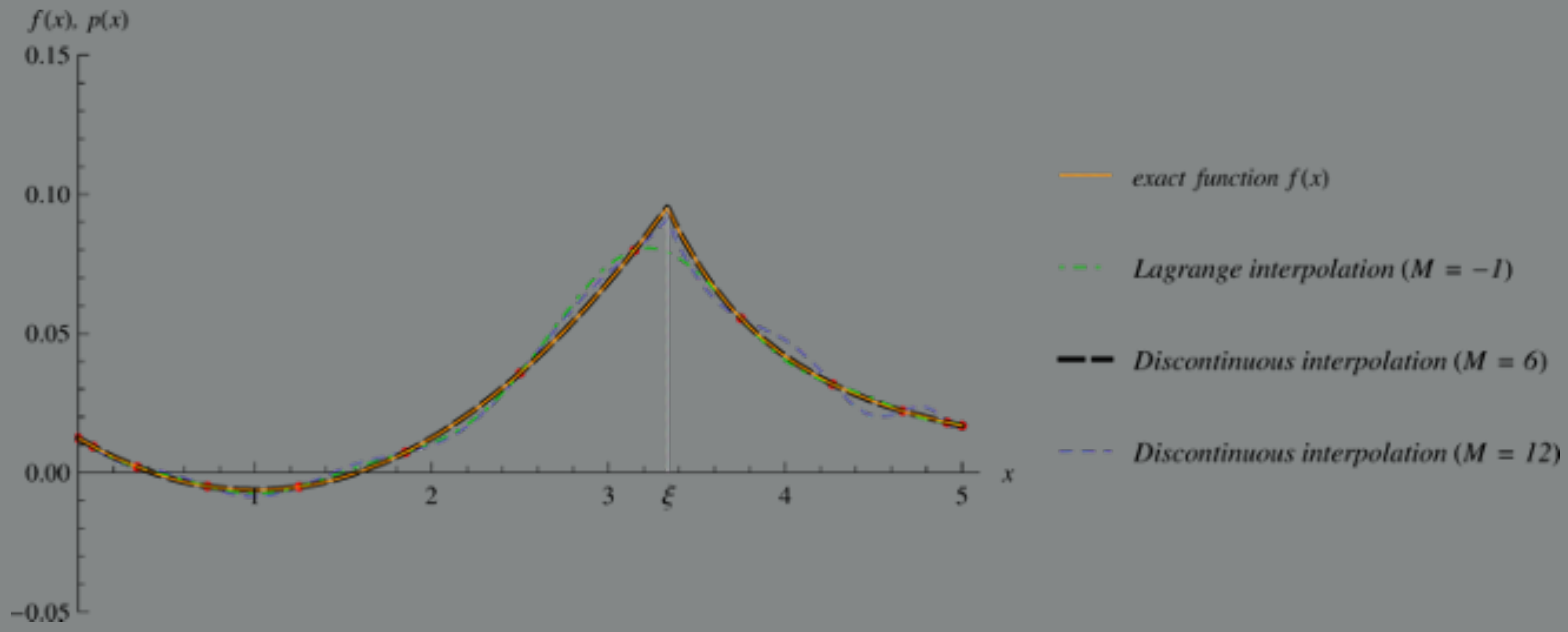
- Integration...

SELF-FORCE ON STATIC SCALAR PARTICLE (SCHWARZSCHILD)



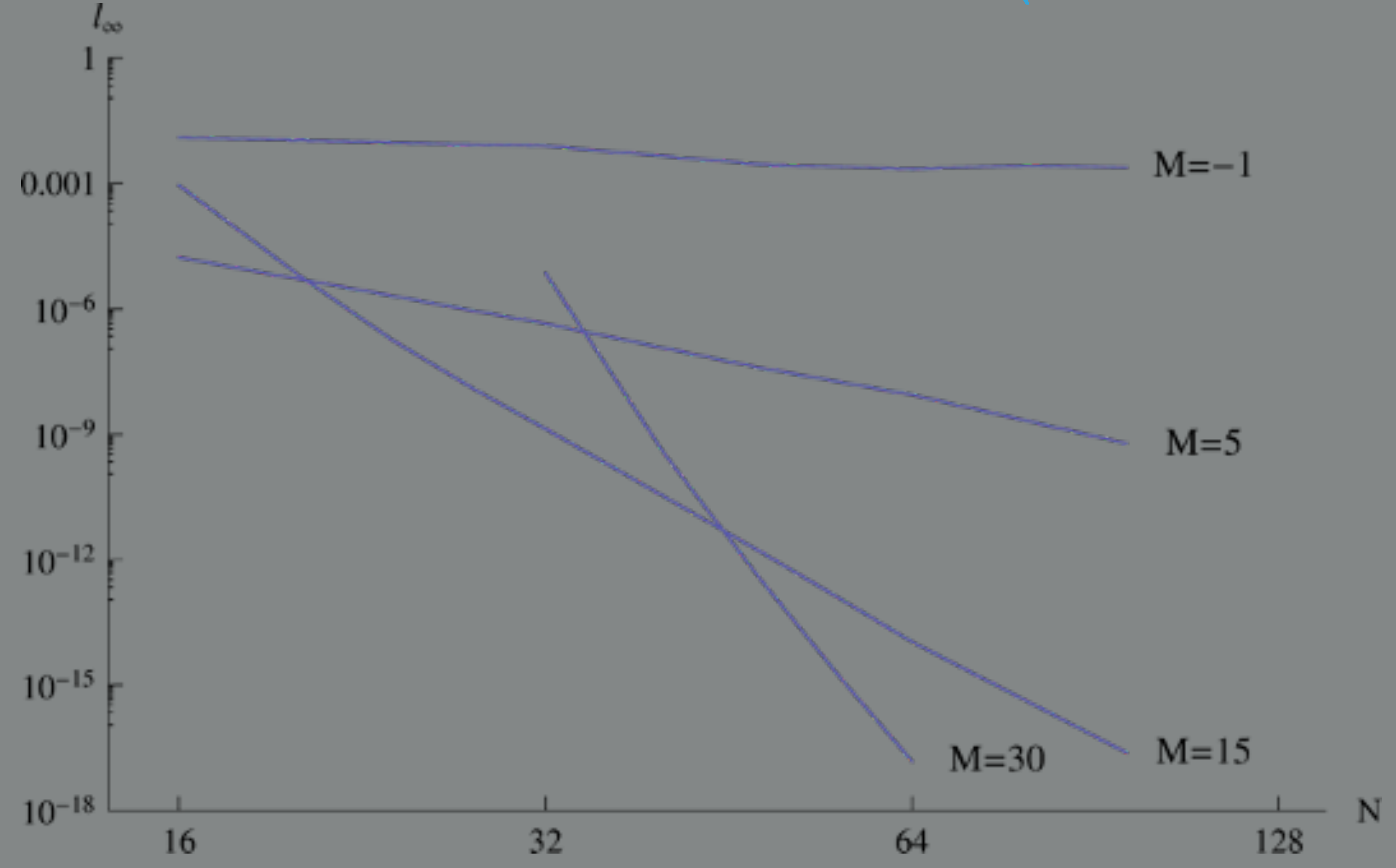
Equidistant nodes

SELF-FORCE ON STATIC SCALAR PARTICLE (SCHWARZSCHILD)



Chebyschev-Gauss-Lobatto nodes

SELF-FORCE ON STATIC SCALAR PARTICLE (SCHWARZSCHILD)



Chebyschev-Gauss-Lobatto nodes

TIME INTEGRATION

- Explicit methods (e.g. Runge-Kutta) are not time-symmetric.
  Energy not conserved. Conditionally stable (CFL limited).
- Implicit methods (e.g. trapezium rule, Hermite rule, Lotkin rule and their higher compositions, Suzuki-Yoshida).
  Energy conserved. Unconditionally stable (no CFL limit).
- For linear PDEs, implicit methods do not incur extra cost (matrix inversion).
- For GSF computations in the time domain, avoid RK. Opt for Crank-Nicolson and higher order generalizations. Invert and store matrices to make scheme 'explicit'.

C. Markakis et al. [arXiv:1901.09967]

CONCLUSIONS

- High accuracy for pseudospectral nodes
- 'Eulerian' picture. Grid fixed/stored, particle can move freely inside domain
- Avoids domain decomposition and coordinate mapping
- Can use a single hyperboloidal slice for the whole spacetime
- Efficient and easy to implement
- Must use relatively large number of jumps
- Must update $\Delta(x_j-\xi;x_i-\xi)$ as particle moves

- Solving Teukolsky equation for Hertz potential Ψ can be used to directly reconstruct GSF (Barack and Giudice)
- Self-consistent evolution under influence of GSF possible

C. Markakis and L. Barack [arXiv:1406.4865]
L. Barack and P. Giudice, rXiv:1702.04204
C. Markakis et al. [arXiv:1901.09967]