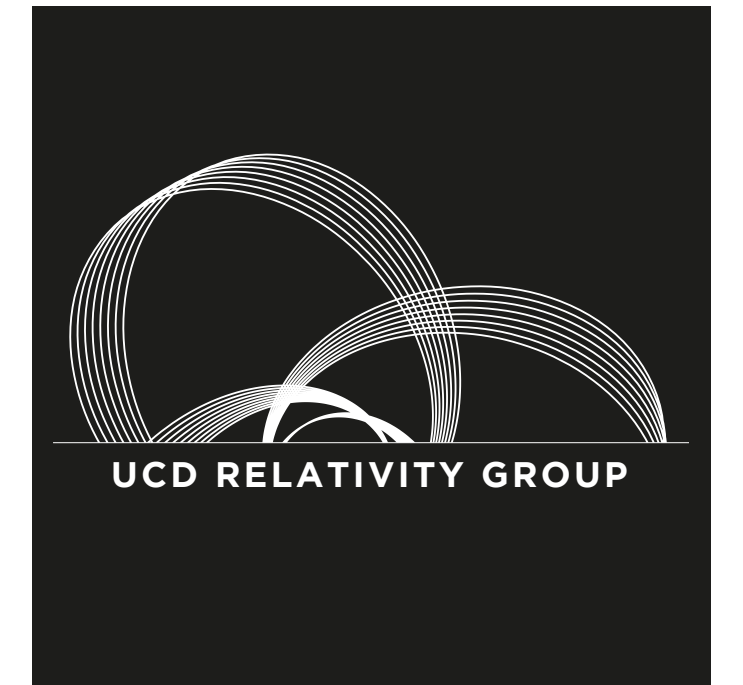




# Gravitational Perturbations by a Spinning Secondary in the RW gauge



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**CAPRA 23**

**'Capra' is the scientific name for the genus of goats.**

# Emphasis on:

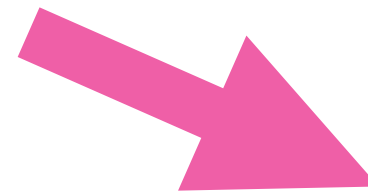
1. Linearising the RWZ formalism in spin.
2. Deriving regularisation parameters for a spinning secondary.

# Schwarzschild Perturbations in a Nutshell

## The RWZ Formalism in the Frequency Domain

Looking for first order MP induced by secondary body's  $T^{\alpha\beta}$ .

$$\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$$



LEFE manipulated so instead solve the RWZ equation for 'master functions'.

$$\left[ \frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega^2 \right] R_{lm}(r) = Z_{lm}(r)$$

Spherical harmonic decomposition, put into frequency domain.

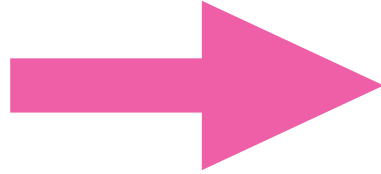


Reconstruct MP amplitudes from master functions.

$$h_{\alpha\beta} = \sum_{l,m} h_{\alpha\beta}^{lm} Y^{lm}$$

Full MP from sum over harmonic modes

# Spinning Secondary Bodies

- MPD Equations. Using Tulczyjew SSC.
- $\frac{|S|}{\mu} \ll M \implies |\sigma| \equiv \frac{|S|}{\mu M} \ll 1$  , neglecting  $O(\sigma^2)$ .
- RWZ, what changes?  $T_{\alpha\beta} = \mu T_{\alpha\beta}^{(\mu)} + \mu\sigma T_{\alpha\beta}^{(\sigma)} + O(\epsilon^2)$    $Z_{lm} = \mu Z_{lm}^{(\mu)} + \mu\sigma Z_{lm}^{(\sigma)}$
- Specialised to circular orbit with (anti-)aligned spin.  $S^\mu = S^\theta \delta_\theta^\mu$

# Functional Form of the Field

$$L[\Psi(t, r)] = S(t, r)$$

- Non-spinning pp - weak solution.
- Spinning pp - 'weaker' solution.
- Can find jump in field, the amplitude of the singular delta function etc.

$$S_{lm}(t, r) = \tilde{G}_{lm}(t)\delta(r - r_0) + \tilde{F}_{lm}(t)\delta'(r - r_0) + \tilde{H}_{lm}(t)\delta''(r - r_0)$$

$$\Psi_{lm}(t, r) = \Psi_{lm}^+(t, r)\theta[r - r_0] + \Psi_{lm}^-(t, r)\theta[r_0 - r] + X_{lm}(t)\delta[r - r_0]$$

# A Computational Problem

Linearising in spin and neglecting  $O(\sigma^2)$ .

$$\left[ \frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega^2 \right] R_{lm}(r) = Z_{lm}(r) \quad \omega = \omega_0 + \sigma\omega_\sigma$$

$$\left[ \frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega_0^2 + 2\sigma\omega_0\omega_\sigma + \cancel{\sigma^2\omega_\sigma^2} \right] (R_0 + \sigma R_\sigma) = Z_0 + \sigma Z_\sigma$$

$O(\sigma^2)$

$$\left\{ \begin{array}{l} \mathcal{L}_{\omega_0} R_0 = Z_0 \\ \mathcal{L}_{\omega_0} R_\sigma = Z_\sigma - 2\omega_0\omega_\sigma R_0 \\ \mathcal{L}_{\omega_0} \equiv \left[ \frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega_0^2 \right] \end{array} \right.$$

- Numerical linearisation not that expensive in circular case, looking to eccentric/generic.

# Partial Annihilator Method

$$\mathcal{L}_{\omega_0} R_0 = Z_0 \quad \checkmark$$

$$\mathcal{L}_{\omega_0} R_\sigma = Z_\sigma - 2\omega_0\omega_\sigma R_0 \quad \rightarrow \quad \mathcal{L}_{\omega_0} R_{\sigma(c)} = Z_\sigma \quad \checkmark$$

$$\mathcal{L}_{\omega_0} R_{\sigma(ext)} = -2\omega_0\omega_\sigma R_0 \quad \rightarrow \quad \mathcal{L}_{\omega_0}^2 R_{\sigma(ext)} = -2\omega_0\omega_\sigma Z_0 \quad \checkmark$$

- Want a compact source.
- 2x 2<sup>nd</sup> order ODEs, 1x 4<sup>th</sup> order...  
Only four unique homogeneous solutions.
- Then use variation of parameters for the inhomogeneous solution (or EHS in extending to eccentric orbits)

[Hopper + Evans 2010, 2013].

$$\hat{R}_2^-(r_* \rightarrow -\infty) = e^{-i\omega_0 r_*}$$

$$\hat{R}_4^-(r_* \rightarrow -\infty) = r_* e^{-i\omega_0 r_*}$$

$$\hat{R}_2^+(r_* \rightarrow \infty) = e^{i\omega_0 r_*}$$

$$\hat{R}_4^+(r_* \rightarrow \infty) = r_* e^{i\omega_0 r_*}$$

$$R(r) = R_0(r) + \sigma [R_{\sigma(c)}(r) + R_{\sigma(ext)}(r)]$$

# Detweiler-Whiting Singular field

- Best understood in the Lorenz gauge.
- Derived from the singular Green function in the Lorenz gauge.
- Spin contribution not yet specialised to an SSC, orbital configuration or ST.

$$h_{\mu\nu} = h_{\mu\nu}^R + h_{\mu\nu}^S$$

$$\bar{h}_{ab}^{S(\mu)}(x) = 2 \left[ \frac{U(x, x')_{aba'b'} u^{a'} u^{b'}}{|\sigma_{c'}(x, x') u^{c'}|} \right] \Big|_{x'=x_{(adv)}, x_{(ret)}} + 2 \int_{\tau_{(ret)}}^{\tau_{(adv)}} V(x, z(\tau'))_{aba'b'} u^{a'} u^{b'} d\tau'$$

$$\bar{h}_{ab}^{S(\sigma)}(x) = 2 \left[ \frac{u^{p'} \nabla_{p'} U(x, x')_{aba'b'} \sigma_{\rho'} u^{(a'} \tilde{S}^{b')\rho'} + U(x, x')_{aba'b'} u^{q'} \sigma_{\rho'q'} u^{(a'} \tilde{S}^{b')\rho'}}{|\sigma_{c'} u^{c'}| \sigma_{d'} u^{d'}} \right] \Big|_{x'=x_{(adv)}, x_{(ret)}}$$

$$- 2 \left[ \frac{[\nabla_{\rho'} U(x, x')_{aba'b'} + V(x, x')_{aba'b'} \sigma_{\rho'}] u^{(a'} \tilde{S}^{b')\rho'}}{|\sigma_{c'} u^{c'}|} \right] \Big|_{x'=x_{(adv)}, x_{(ret)}}$$

$$- 2 \left[ \frac{U(x, x')_{aba'b'} \sigma_{\rho'} u^{(a'} \tilde{S}^{b')\rho'} \sigma_{p'q'} u^{p'} u^{q'}}{|\sigma_{c'} u^{c'}| (\sigma_{d'} u^{d'})^2} \right] \Big|_{x'=x_{(adv)}, x_{(ret)}}$$

$$- 2 \int_{\tau_{(ret)}}^{\tau_{(adv)}} \nabla_{\rho'} V(x, z(\tau'))_{aba'b'} u^{(a'} \tilde{S}^{b')\rho'} d\tau'$$



# Regularisation Parameters

- Regularisation parameters from suitable expansion and harmonic decomposition of the singular field.

$$F^{(R)a} = \sum_{l=0}^{\infty} \left[ F_l^a - F_l^{(S)a} \right] = \sum_{l=0}^{\infty} \left[ F_l^a - F_{[-1]}^a (2l + 1) - F_{[0]}^a - \dots \right]$$

- RPs included - mode-sum converges.
- Spinning case is work in progress.

# Detweiler's Redshift Invariant

(For a spin-aligned secondary in a circular orbit.)

Conservative (requires regularisation),  
gauge invariant GSF quantity.

$$h_{uu} = \frac{1}{2} h_{\mu\nu} u^\mu u^\nu$$

$$z_1(y) \equiv z_1^{(0)}(y) + q \left( z_1^{(1)\sigma^0}(y) + \sigma z_1^{(1)\sigma^1}(y) \right)$$

$$z_1^{(1)\sigma^1}(y) = -\frac{1}{2\sqrt{1-3y}} \left[ h_{kk\sigma}(y) + My^{1/2} \partial_r h_{kk(0)}(y) \right]$$

$$z_1^{(1)\sigma^0}(y) = -\frac{1}{2\sqrt{1-3y}} h_{kk(0)}(y)$$

*(Bini et al 2018)*

# Preliminary Regularisation Results with a Spinning Secondary

- Need first two RPs for finite  $h_{uu}$ .
- The ‘expected’ leading RP for  $h_{uu}$  is zero. (As in [Bini et al 2018] in the RW, Rad gauges).

$$h_{uu[-1]}^{(S)} = 0$$

$$h_{uu[0]}^{(S)} = \mu C_0(r_0) \mathcal{K} + \mu \sigma [C_1(r_0) \mathcal{K} + C_2(r_0) \mathcal{E}]$$

- Comment: Want a SSC in which  $S^{ab}u_b=0$  to linear order in spin. Simplify the leading spin contributions to the singular field. (T-SSC for example).

$$\frac{2s\mu g_{a'}^b g_{b'}^c \tilde{r} \tilde{S}_c^\delta u_b u_\delta}{\varsigma^3} + \frac{2s\mu g_{a'}^b g_{b'}^c \tilde{r} \tilde{S}_c^\delta u_b u_\delta}{\varsigma^3} - \frac{2s\mu g_{c'}^b g_{d'}^c g_{a'b'} g^{c'd'} \tilde{r} \tilde{S}_b^\delta u_c u_\delta}{\varsigma^3} +$$

$$\frac{2s\mu g_{a'}^b g_{b'}^c \tilde{S}_c^\delta u_b \sigma_\delta}{\varsigma^3} + \frac{2s\mu g_{a'}^b g_{b'}^c \tilde{S}_b^\delta u_c \sigma_\delta}{\varsigma^3} - \frac{2s\mu g_{c'}^b g_{d'}^c g_{a'b'} g^{c'd'} \tilde{S}_b^\delta u_c \sigma_\delta}{\varsigma^3}$$

# Other Notes/ Future Work

- Flux balance law (*Akcay et al 2019*) held to a relative error of  $\sim 10^{-9}$  in the spin contribution in this work.
- Full strong field GSF calculation with a spinning secondary.
- Will improve performance of solving for the RW squared homogeneous solutions.
- Will increasingly generalise the EMRI model (next to Kerr, then move away from aligned spin, to generic orbits etc.).

# Any Questions?

