Gravitational Perturbations by a Spinning Secondary in the RW gauge

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CAPRA 23

‘Capra’ is the scientific name for the genus of goats.

Emphasis on:

1. Linearising the RWZ formalism in spin.
2. Deriving regularisation parameters for a spinning secondary.

Schwarzschild Perturbations in a Nutshell

The RWZ Formalism in the Frequency Domain

Looking for first order MP induced by secondary body's $T^{\alpha\beta}$.

$$\mathbf{g}_{\alpha\beta} = g_{\alpha\beta} + \epsilon h_{\alpha\beta} + O(\epsilon^2)$$

LEFE manipulated so instead solve the RWZ equation for 'master functions'.

$$\left[\frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega^2\right] R_{lm}(r) = Z_{lm}(r)$$

Spherical harmonic decomposition, put into frequency domain.

Reconstruct MP amplitudes from master functions.

$$h_{\alpha\beta} = \sum_{l,m} h_{\alpha\beta}^{lm} Y^{lm}$$

Full MP from sum over harmonic modes

Spinning Secondary Bodies

- MPD Equations. Using Tulczyjew SSC.
- $\frac{|S|}{\mu} \ll M \implies |\sigma| \equiv \frac{|S|}{\mu M} \ll 1$, neglecting $O(\sigma^2)$.
- RWZ, what changes? $T_{\alpha\beta} = \mu T^{(\mu)}_{\alpha\beta} + \mu\sigma T^{(\sigma)}_{\alpha\beta} + O(\epsilon^2)$ $\Rightarrow$ $Z_{lm} = \mu Z^{(\mu)}_{lm} + \mu\sigma Z^{(\sigma)}_{lm}$
- Specialised to circular orbit with (anti-)aligned spin. $S^\mu = S^\theta \delta^\mu_\theta$

Functional Form of the Field

$$L[\Psi(t,r)] = S(t,r)$$

- Non-spinning pp - weak solution.
- Spinning pp - ‘weaker’ solution.
- Can find jump in field, the amplitude of the singular delta function etc.

$$S_{lm}(t,r) = \tilde{G}_{lm}(t)\delta\left(r-r_0\right) + \tilde{F}_{lm}(t)\delta'\left(r-r_0\right)$$

$$+\tilde{H}_{lm}(t)\delta''\left(r-r_0\right)$$

$$\Psi_{lm}(t,r) = \Psi^{+}_{lm}(t,r)\theta\left[r-r_0\right] + \Psi^{-}_{lm}(t,r)\theta\left[r_0-r\right]$$

$$+X_{lm}(t)\delta\left[r-r_0\right]$$

A Computational Problem

Linearising in spin and neglecting $O(\sigma^2)$.

$$\left[\frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega^2\right] R_{lm}(r) = Z_{lm}(r) \qquad \omega = \omega_0 + \sigma\omega_\sigma$$

$$O(\sigma^2)$$

$$\left[\frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega_0^2 + 2\sigma\omega_0\omega_\sigma + \cancel{\sigma^2\omega_\sigma^2}\right](R_0 + \sigma R_\sigma) = Z_0 + \sigma Z_\sigma$$

$$\begin{cases} \mathcal{L}_{\omega_0} R_0 = Z_0 \\ \mathcal{L}_{\omega_0} R_\sigma = Z_\sigma - 2\omega_0\omega_\sigma R_0 \\ \mathcal{L}_{\omega_0} \equiv \left[\frac{\partial^2}{\partial r_*^2} - V_l(r) + \omega_0^2\right] \end{cases}$$

- Numerical linearisation not that expensive in circular case, looking to eccentric/generic.

Partial Annihilator Method

$$\mathcal{L}_{\omega_0} R_0 = Z_0$$

$$\mathcal{L}_{\omega_0} R_\sigma = Z_\sigma - 2\omega_0\omega_\sigma R_0 \longrightarrow \begin{cases} \mathcal{L}_{\omega_0} R_{\sigma(c)} = Z_\sigma \\ \mathcal{L}_{\omega_0} R_{\sigma(ext)} = -2\omega_0\omega_\sigma R_0 \longrightarrow \mathcal{L}^2_{\omega_0} R_{\sigma(ext)} = -2\omega_0\omega_\sigma Z_0 \end{cases}$$

- Want a compact source.
- 2x 2nd order ODEs, 1x 4th order…
  Only four unique homogeneous solutions.
- Then use variation of parameters for the inhomogeneous solution (or EHS in extending to eccentric orbits)
  [Hopper + Evans 2010, 2013].

$$\hat{R}_2^-(r_* \to -\infty) = e^{-i\omega_0 r_*}$$

$$\hat{R}_4^-(r_* \to -\infty) = r_* e^{-i\omega_0 r_*}$$

$$\hat{R}_2^+(r_* \to \infty) = e^{i\omega_0 r_*}$$

$$\hat{R}_4^+(r_* \to \infty) = r_* e^{i\omega_0 r_*}$$

$$R(r) = R_0(r) + \sigma[R_{\sigma(c)}(r) + R_{\sigma(ext)}(r)]$$

Detweiler-Whiting Singular field

- Best understood in the Lorenz gauge.
- Derived from the singular Green function in the Lorenz gauge.
- Spin contribution not yet specialised to an SSC, orbital configuration or ST.

$$h_{\mu\nu} = h^R_{\mu\nu} + h^S_{\mu\nu}$$

$$\bar{h}^{S(\mu)}_{ab}(x) = 2\left[\frac{U\left(x,x'\right)_{aba'b'}\,u^{a'}u^{b'}}{\left|\sigma_{c'}\left(x,x'\right)u^{c'}\right|}\right]\Bigg|_{x'=x_{(adv)},x_{(ret)}} + 2\int_{\tau_{(ret)}}^{\tau_{(adv)}} V\left(x,z\left(\tau'\right)\right)_{aba'b'}\,u^{a'}u^{b'}d\tau'$$

$$\begin{aligned}\bar{h}^{S(\sigma)}_{ab}(x) = {}& 2\left[\frac{u^{p'}\nabla_{p'}U\left(x,x'\right)_{aba'b'}\,\sigma_{\rho'}u^{(a'}\,\tilde{S}^{b')\rho'} + U\left(x,x'\right)_{aba'b'}\,u^{q'}\sigma_{\rho'q'}u^{(a'}\,\tilde{S}^{b')\rho'}\Big]}{\left|\sigma_{c'}u^{c'}\right|\sigma_{d'}u^{d'}}\right]\Bigg|_{x'=x_{(adv)},x_{(ret)}} \\ & - 2\left[\frac{\left[\nabla_{\rho'}U\left(x,x'\right)_{aba'b'} + V\left(x,x'\right)_{aba'b'}\,\sigma_{\rho'}\right]u^{(a'}\,\tilde{S}^{b')\rho'}\Big]}{\left|\sigma_{c'}u^{c'}\right|}\right]_{x'=x_{(adv)},x_{(ret)}} \\ & - 2\left[\frac{U\left(x,x'\right)_{aba'b'}\,\sigma_{\rho'}u^{(a'}\,\tilde{S}^{b')\rho'}\sigma_{p'q'}u^{p'}u^{q'}}{\left|\sigma_{c'}u^{c'}\right|\left(\sigma_{d'}u^{d'}\right)^2}\right]\Bigg|_{x'=x_{(adv)},x_{(ret)}} \\ & - 2\int_{\tau_{(ret)}}^{\tau_{(adv)}} \nabla_{\rho'}V\left(x,z\left(\tau'\right)\right)_{aba'b'}\,u^{(a'}\,\tilde{S}^{b')\rho'}d\tau'\end{aligned}$$

Regularisation Parameters

- Regularisation parameters from suitable expansion and harmonic decomposition of the singular field.

$$F^{(R)a} = \sum_{l=0}^{\infty} \left[F_l^a - F_l^{(S)a} \right] = \sum_{l=0}^{\infty} \left[F_l^a - F_{[-1]}^a (2l+1) - F_{[0]}^a - ... \right]$$

- RPs included - mode-sum converges.

- Spinning case is work in progress.

Detweiler's Redshift Invariant

(For a spin-aligned secondary in a circular orbit.)

Conservative (requires regularisation), gauge invariant GSF quantity.

$$h_{uu} = \frac{1}{2} h_{\mu\nu} u^\mu u^\nu$$

$$z_1(y) \equiv z_1^{(0)}(y) + q\left(z_1^{(1)\sigma^0}(y) + \sigma z_1^{(1)\sigma^1}(y)\right)$$

$$z_1^{(1)\sigma^1}(y) = -\frac{1}{2\sqrt{1-3y}}\left[h_{kk\sigma}(y) + My^{1/2}\partial_r h_{kk(0)}(y)\right]$$

$$z_1^{(1)\sigma^0}(y) = -\frac{1}{2\sqrt{1-3y}} h_{kk(0)}(y)$$

(Bini et al 2018)

Preliminary Regularisation Results with a Spinning Secondary

- Need first two RPs for finite h_{uu}.

$$h^{(S)}_{uu[-1]} = 0$$

- The 'expected' leading RP for h_{uu} is zero. (As in *[Bini et al 2018]* in the RW, Rad gauges).

$$h^{(S)}_{uu[0]} = \mu C_0(r_0)\mathcal{K} + \mu\sigma[C_1(r_0)\mathcal{K} + C_2(r_0)\mathcal{E}]$$

- Comment: Want a SSC in which $S^{ab}u_b=0$ to linear order in spin. Simplify the leading spin contributions to the singular field. (T-SSC for example).

$$\cancel{\frac{2\, s\, \mu\, g_{a'}{}^{b}\, g_{b'}{}^{c}\, \bar{r}\, \tilde{S}_{c}{}^{\delta}\, u_b\, u_\delta}{\underline{s}^3} + \frac{2\, s\, \mu\, g_{a'}{}^{b}\, g_{b'}{}^{c}\, \bar{r}\, \tilde{S}_{b}{}^{\delta}\, u_c\, u_\delta}{\underline{s}^3} - \frac{2\, s\, \mu\, g_{c'}{}^{b}\, g_{d'}{}^{c}\, g_{a'b'}\, g^{c'd'}\, \bar{r}\, \tilde{S}_{b}{}^{\delta}\, u_c\, u_\delta}{\underline{s}^3}} +$$

$$\frac{2\, s\, \mu\, g_{a'}{}^{b}\, g_{b'}{}^{c}\, \tilde{S}_{c}{}^{\delta}\, u_b\, \sigma_\delta}{\underline{s}^3} + \frac{2\, s\, \mu\, g_{a'}{}^{b}\, g_{b'}{}^{c}\, \tilde{S}_{b}{}^{\delta}\, u_c\, \sigma_\delta}{\underline{s}^3} - \frac{2\, s\, \mu\, g_{c'}{}^{b}\, g_{d'}{}^{c}\, g_{a'b'}\, g^{c'd'}\, \tilde{S}_{b}{}^{\delta}\, u_c\, \sigma_\delta}{\underline{s}^3}$$

Other Notes/ Future Work

- Flux balance law *(Akcay et al 2019)* held to a relative error of ~10^{-9} in the spin contribution in this work.
- Full strong field GSF calculation with a spinning secondary.
- Will improve performance of solving for the RW squared homogeneous solutions.
- Will increasingly generalise the EMRI model (next to Kerr, then move away from aligned spin, to generic orbits etc.).

Any Questions?