Calculating the scalar self-force during $r\theta$-resonances

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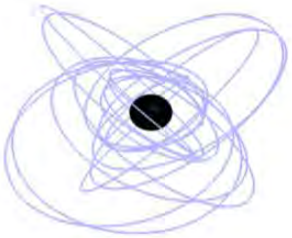
Motivation

- Model evolution of EMRIs
  - Parameterize motion with slowly evolving ‘constants’
    - Example: (p, e, x); (E, L_z, Q); $(\Upsilon_r, \Upsilon_\theta, \Upsilon_\varphi)$



- Orbital $r\theta$-resonances in EMRIs
  - Radial and polar motion are commensurate

$$\Upsilon_r / \Upsilon_\theta = \beta_r / \beta_\theta \quad (\beta_i \in \mathbb{Z})$$

    - Example: $(\beta_r, \beta_\theta) = (1,2) \Rightarrow$ 1:2 $r\theta$-resonance
  - Almost all EMRIs encounter at least one strong (1:3, 1:2, 2:3) $r\theta$-resonance [Ruangsri & Hughes (2014)]

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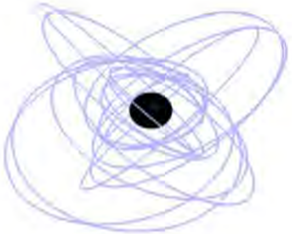

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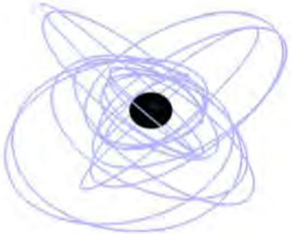

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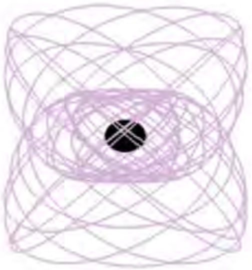

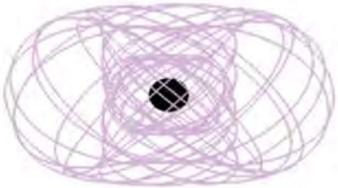

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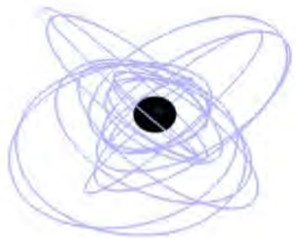

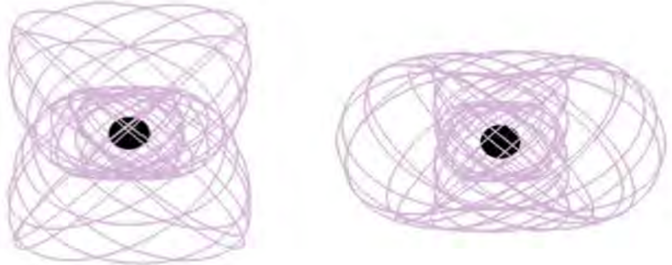

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- Importance of transient orbital $r\theta$-resonances
  - 'Kicks' in the orbital 'constants', e.g., (E, L_z, Q)
    - Scale as $\sim \epsilon^{1/2}$ $(\epsilon \equiv \mu/M)$
    - Sensitive to initial conditions/phases at resonance
  - Impact EMRI waveform phase ϕ @ 1/2-order

$$\phi = \phi^{(-1)}\epsilon^{-1} + \phi^{(-1/2)}\epsilon^{-1/2} + \phi^{(0)}\epsilon^{0} + \cdots$$

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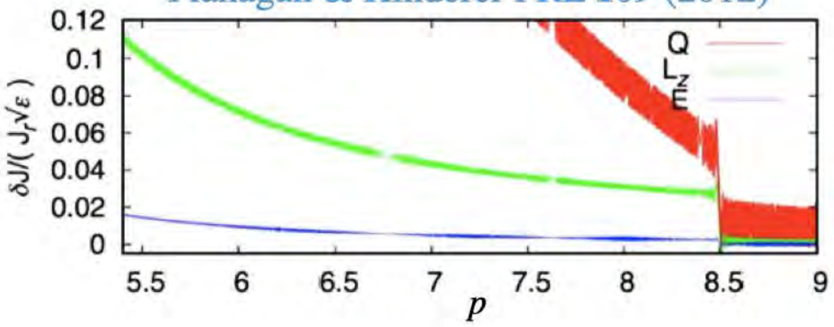

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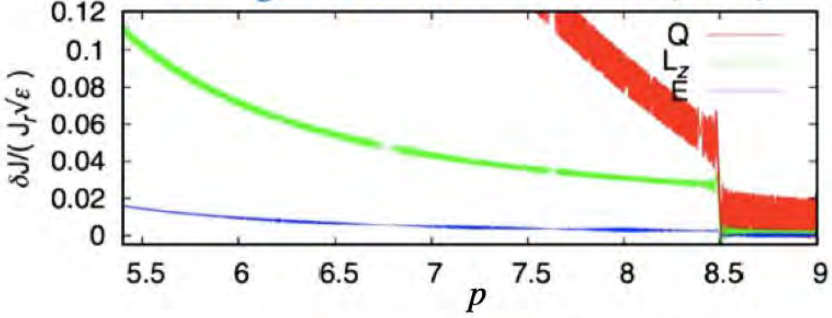

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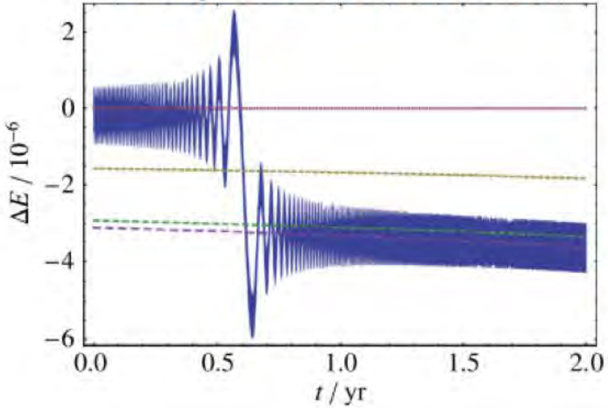

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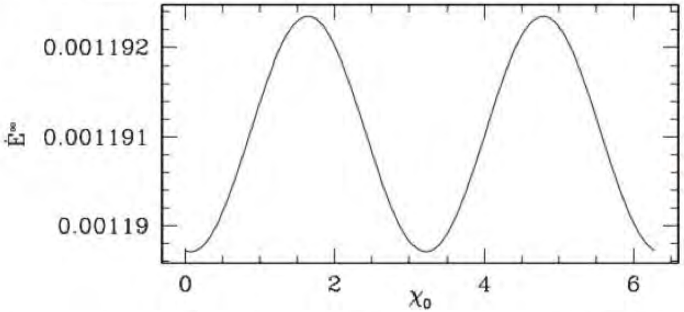

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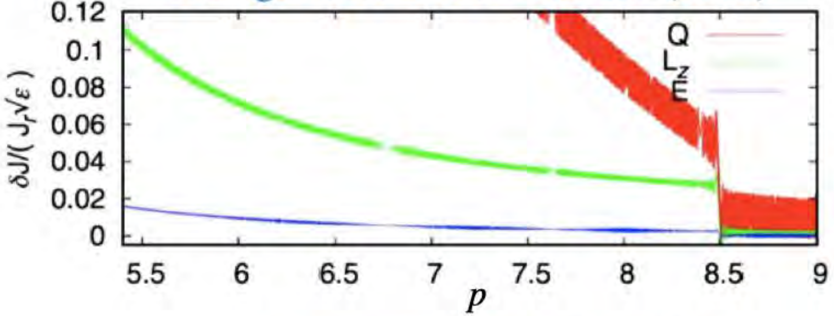

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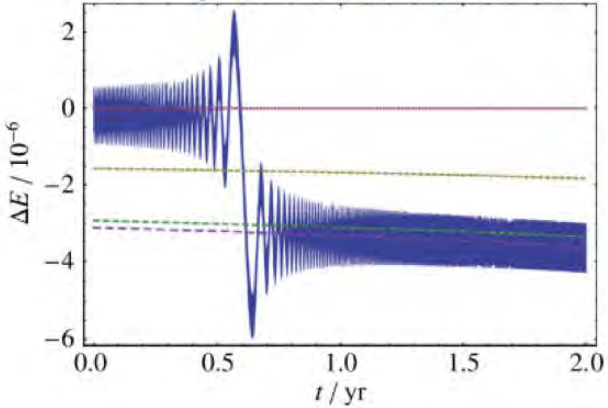

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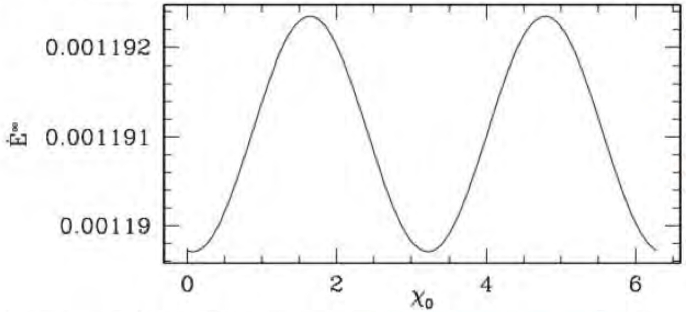

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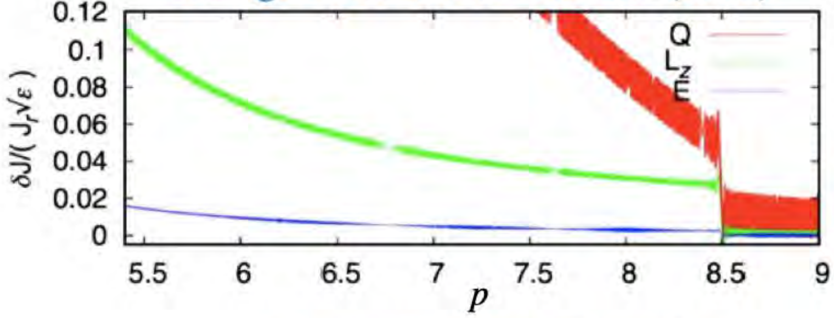

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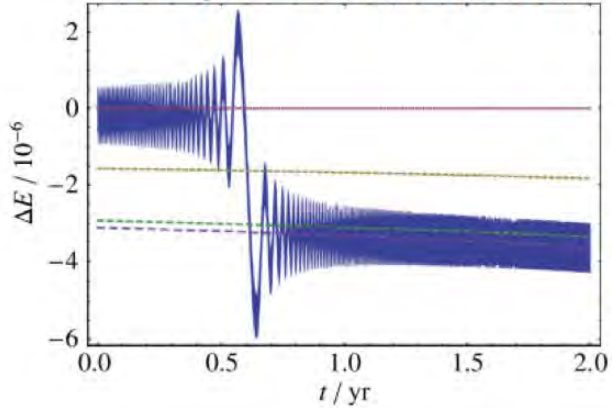

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    - Incorporating initial conditions in self-force calculations?
    - Conservative contributions? [Flanagan & Hinderer PRL 109 (2012)]
        - Hamiltonian formulation of $\langle \dot{Q} \rangle$ includes potential conservative terms [Isoyama et. al PTEP 013E01 (2019)]
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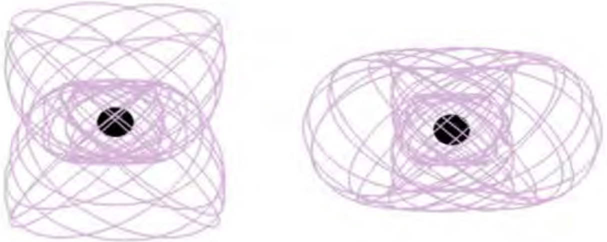

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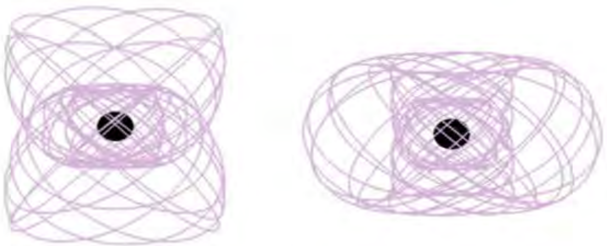

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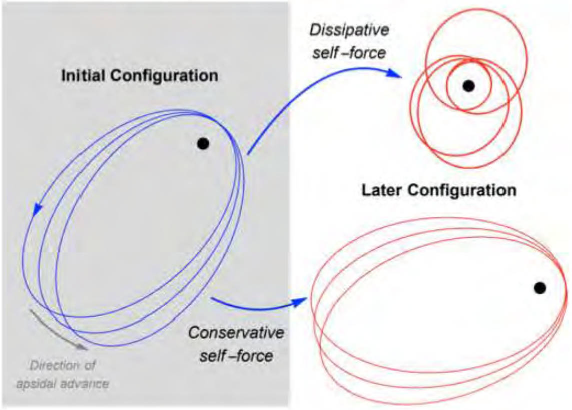

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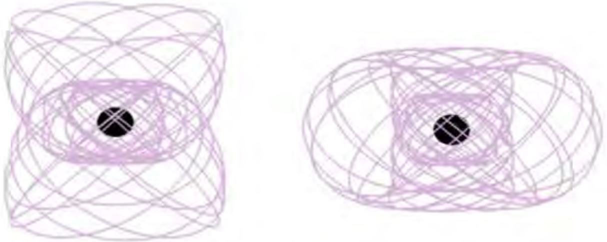

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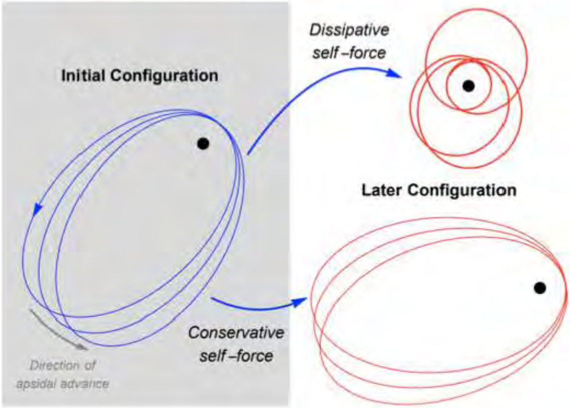

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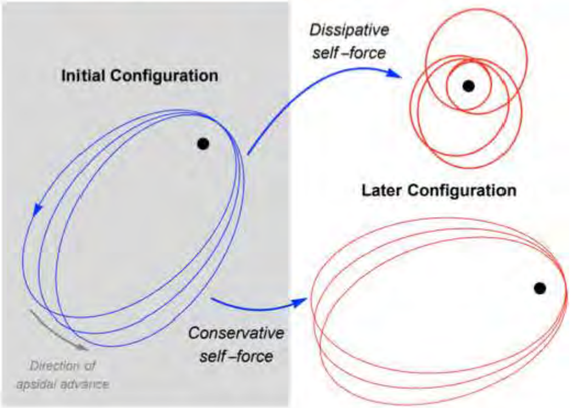

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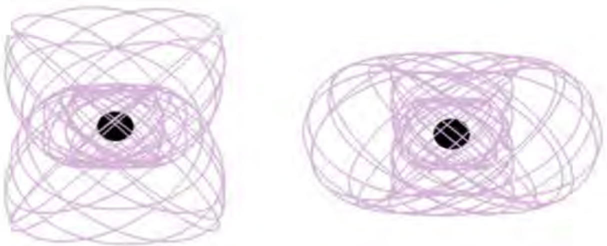

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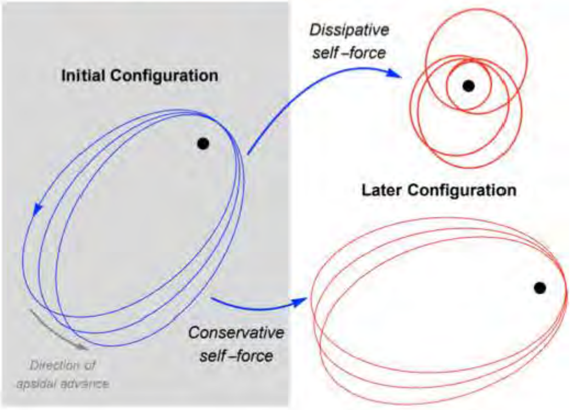

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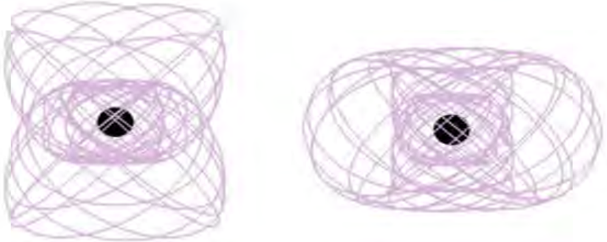

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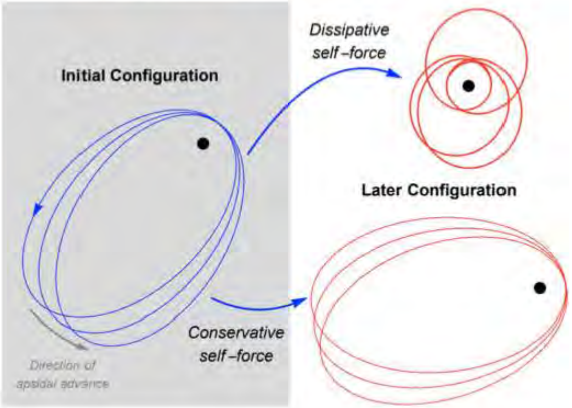

Impact of initial phases

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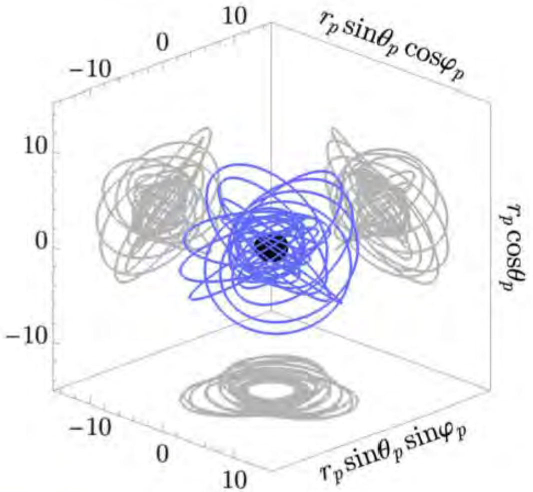

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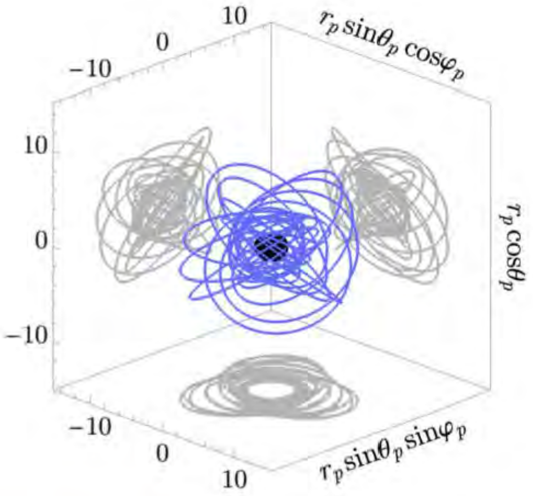

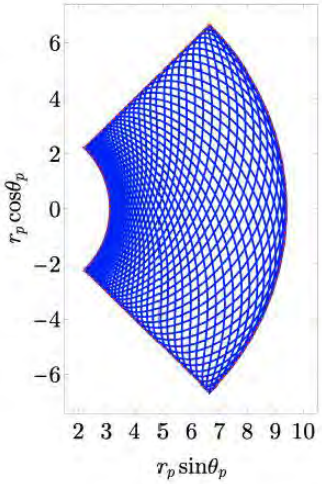

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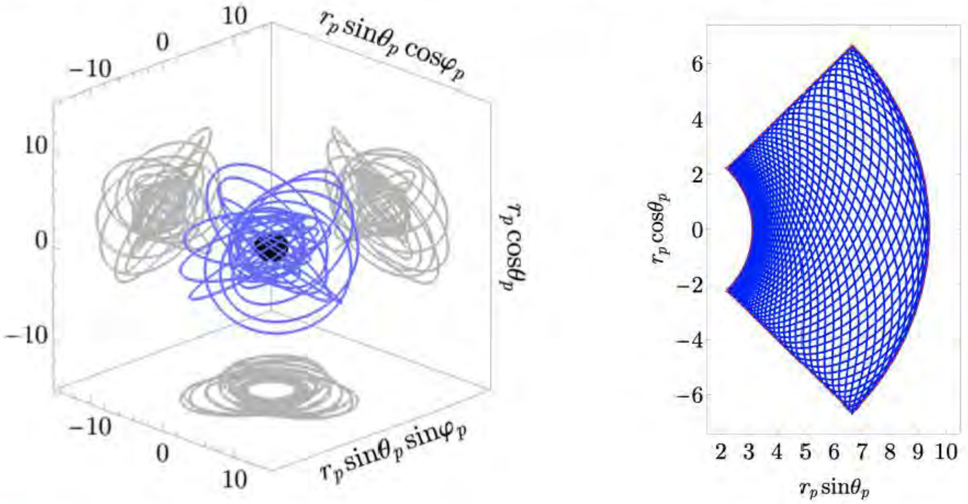

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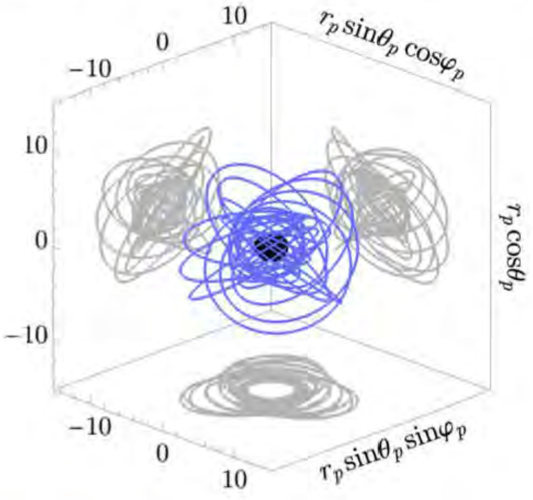

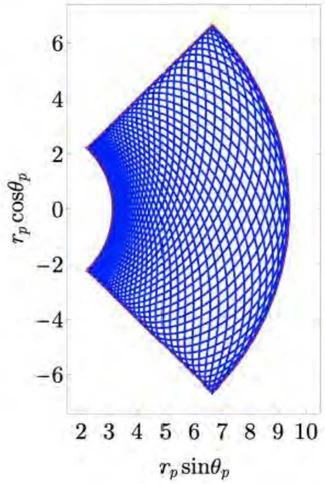

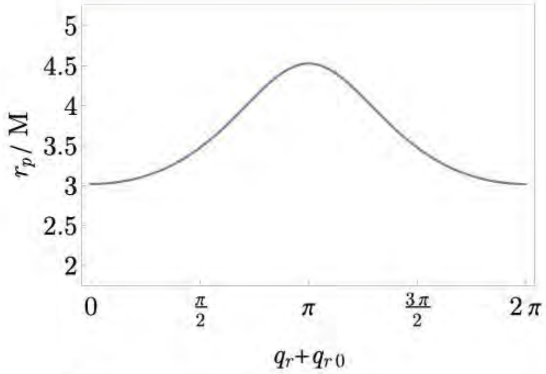

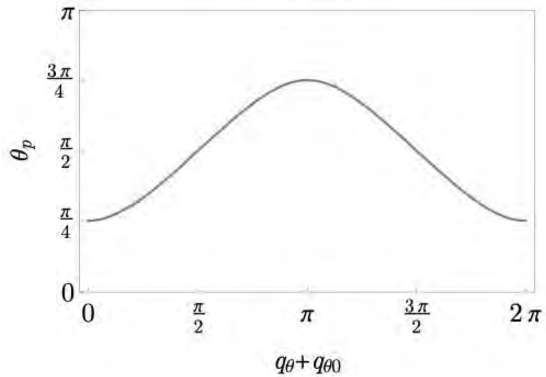

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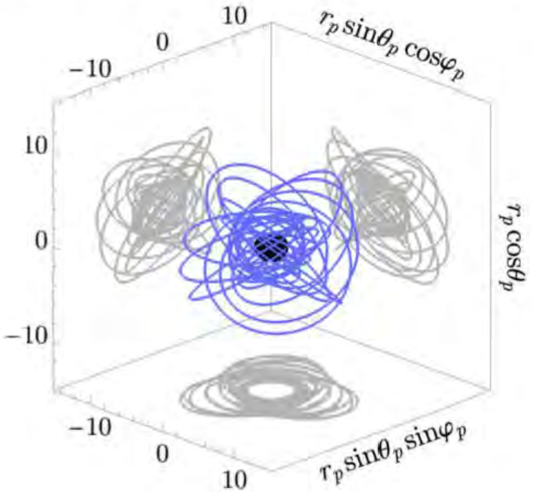

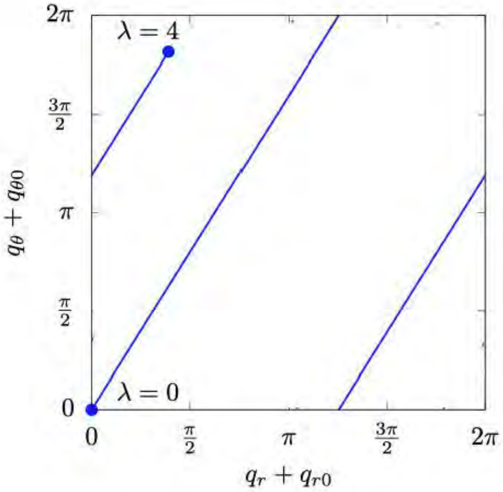

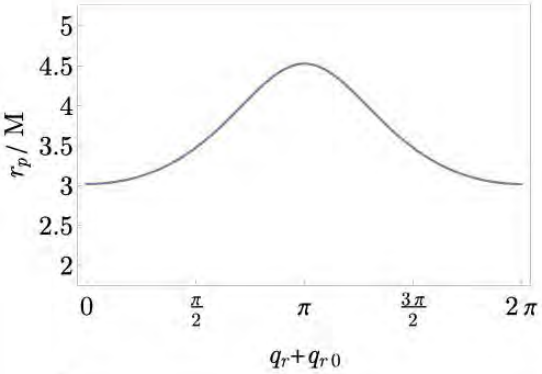

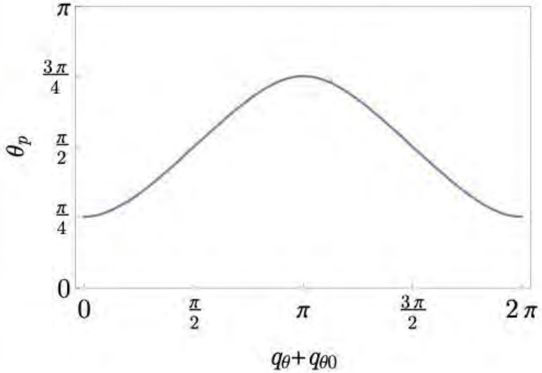

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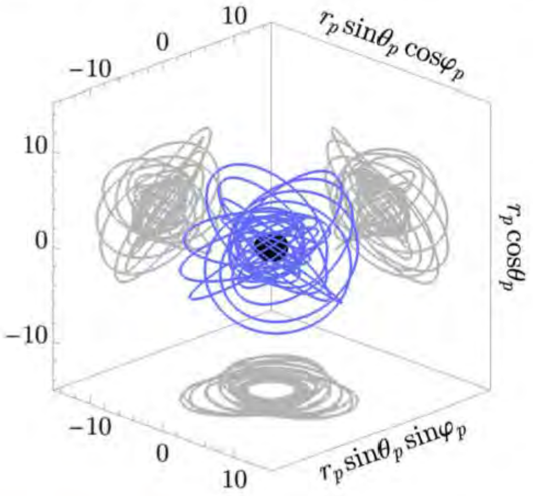

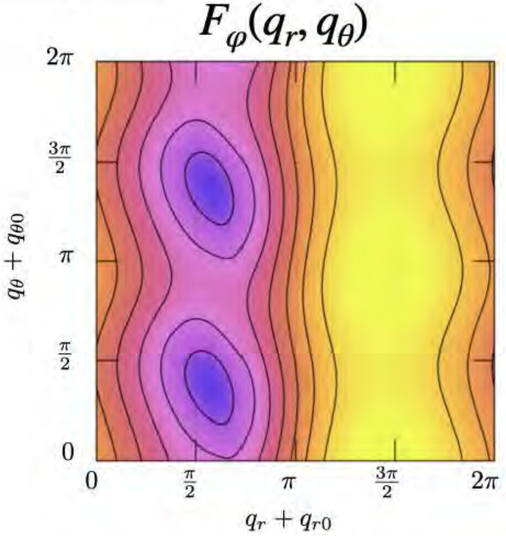

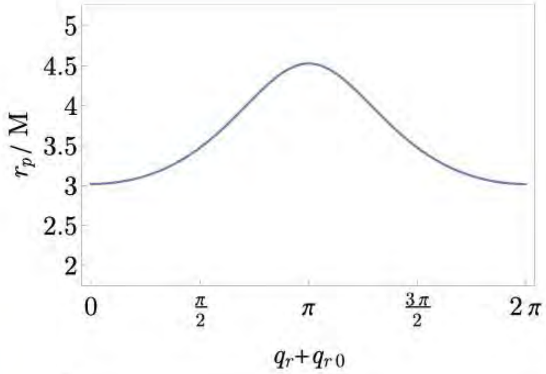

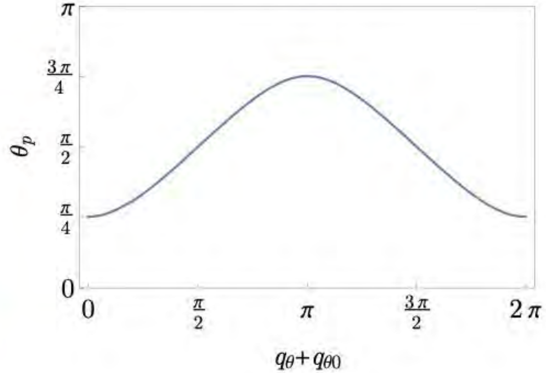

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- Consider resonant inclined, eccentric geodesic
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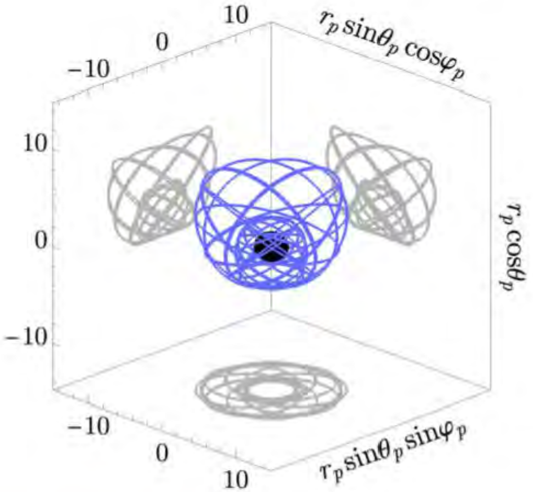

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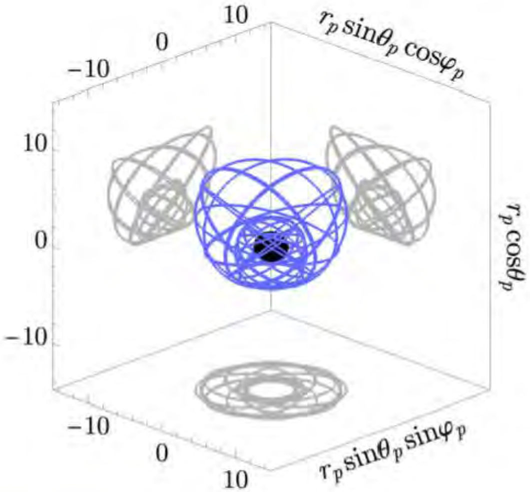

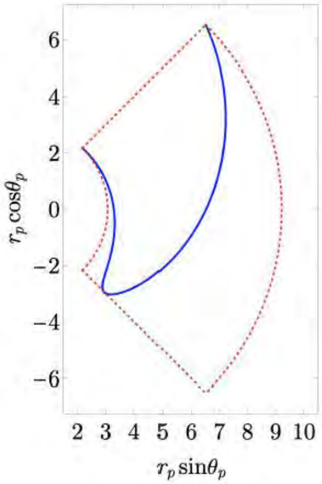

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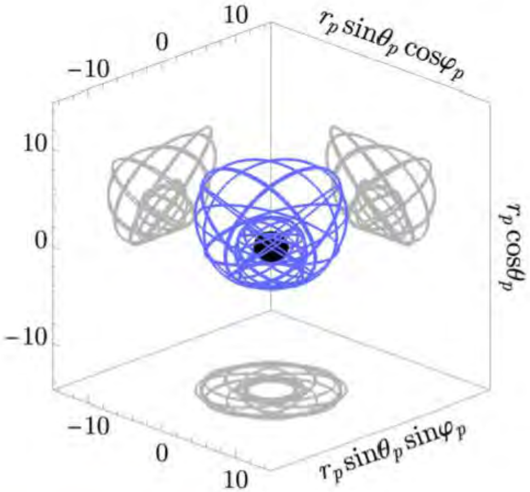

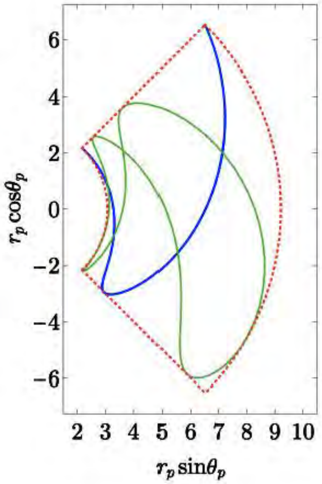

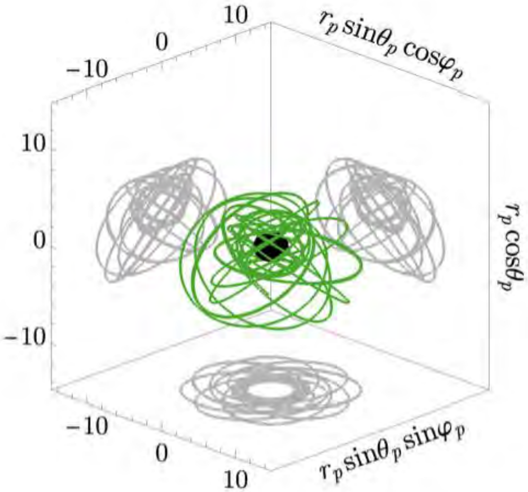

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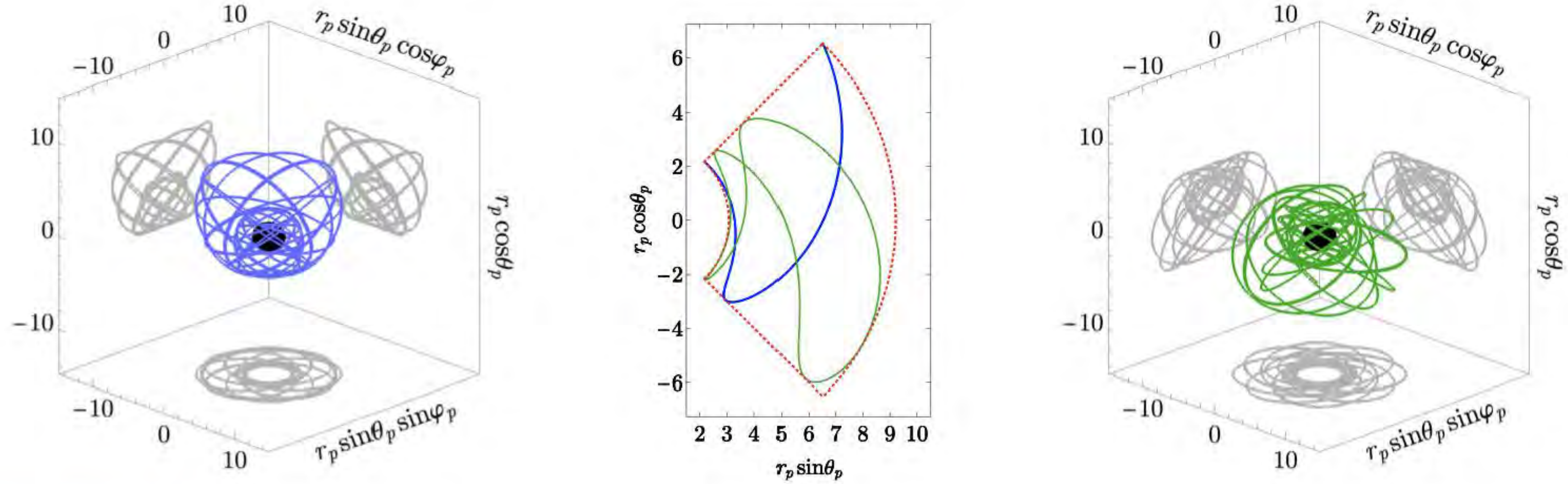

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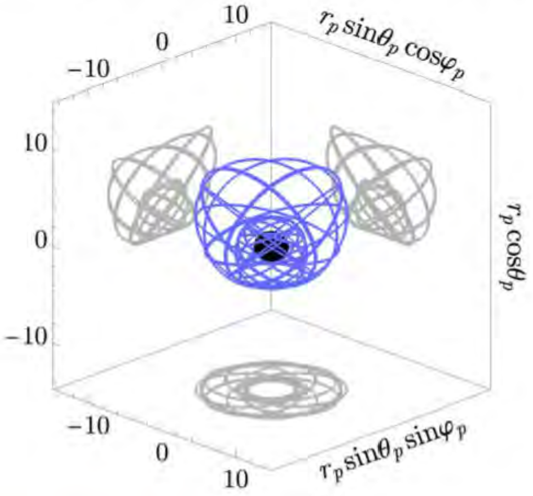

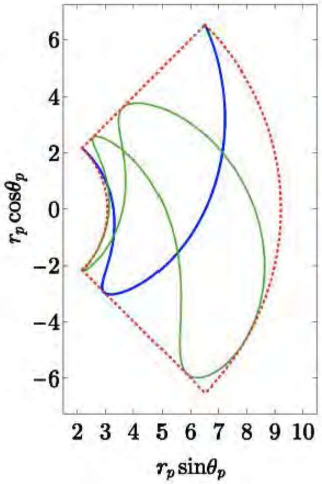

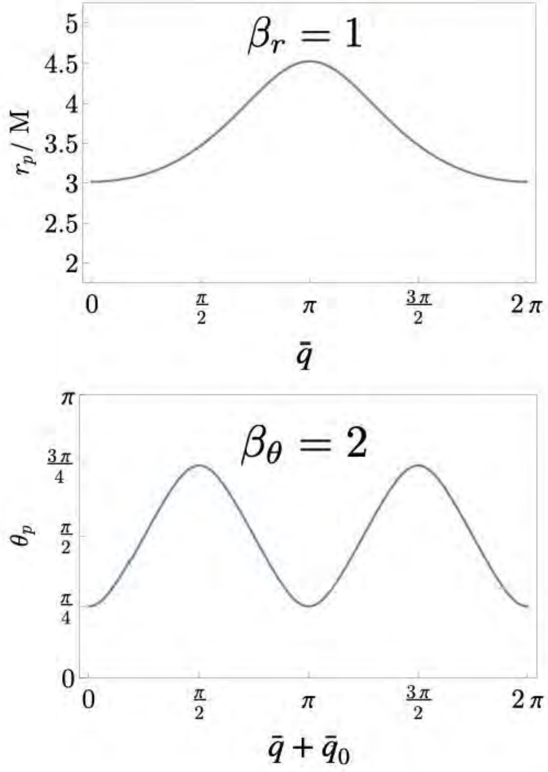

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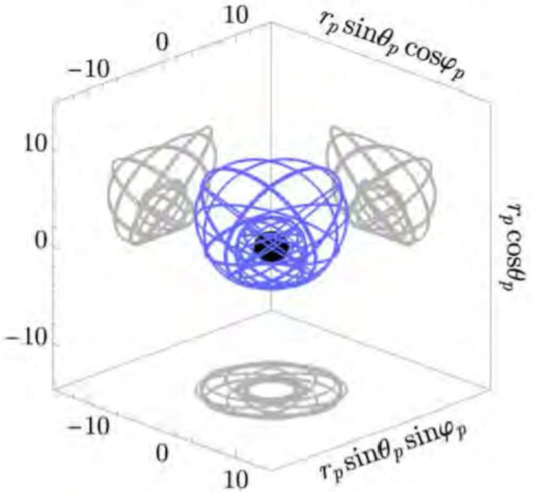

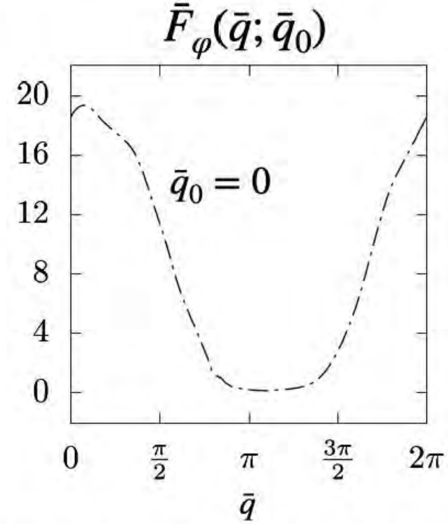

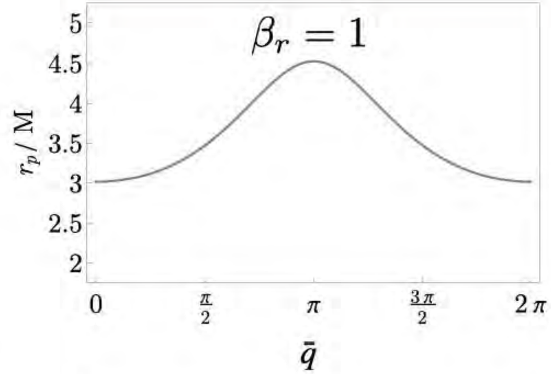

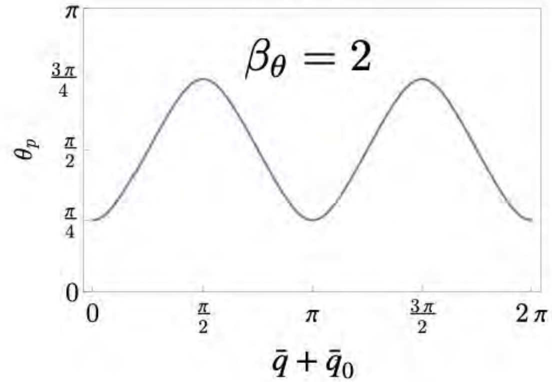

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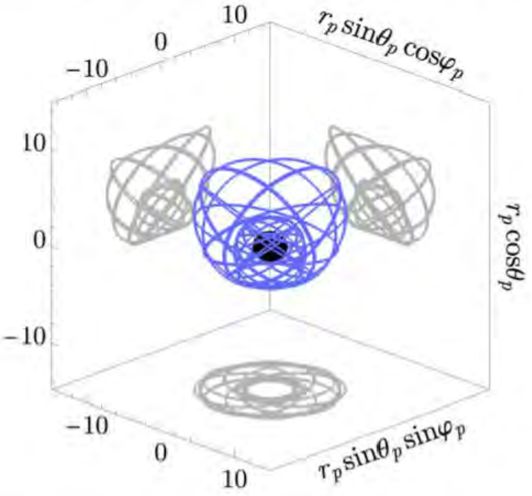

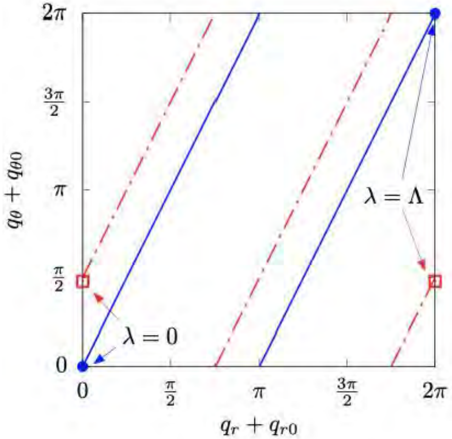

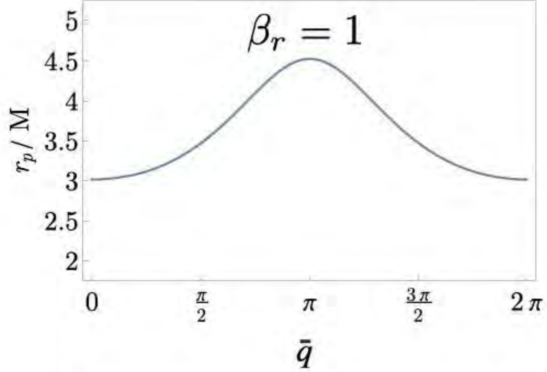

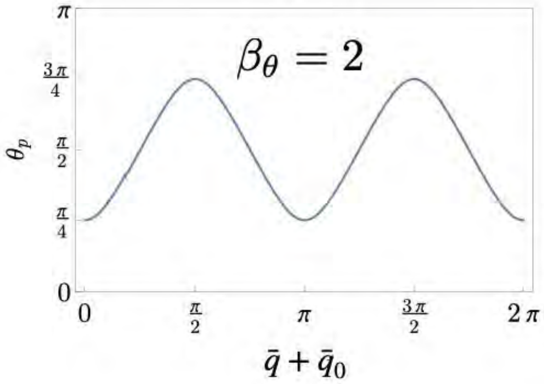

Scalar self-force for resonant orbits

- Constructing SSF
  - Calculate SSF just as if resonance is a non-resonant geodesic with $q_{r0} = q_{\theta 0} = 0$
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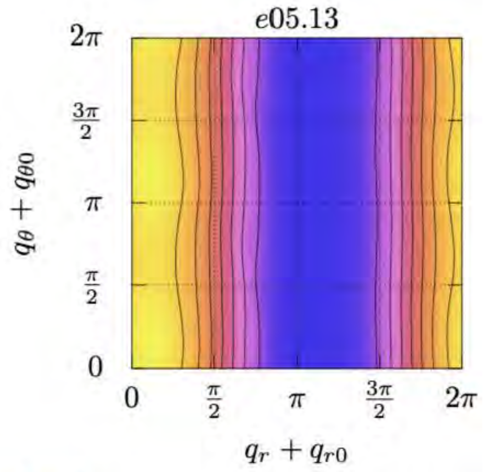

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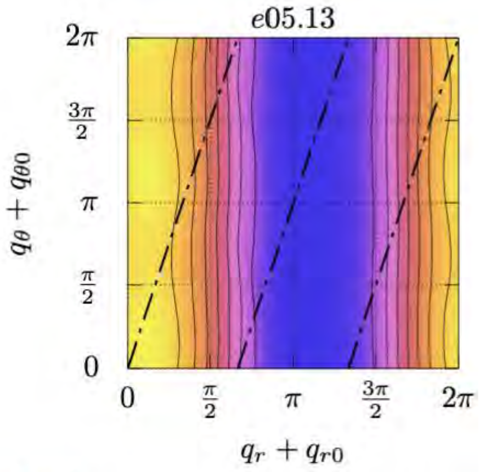

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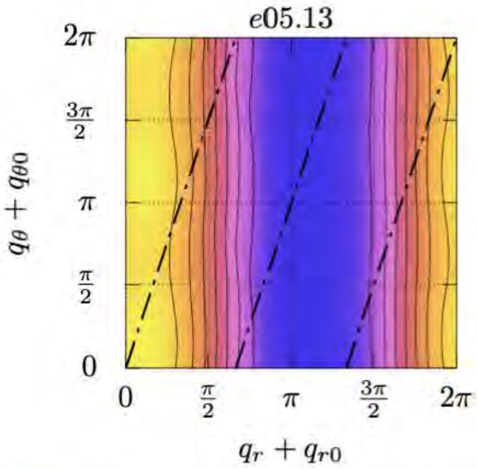

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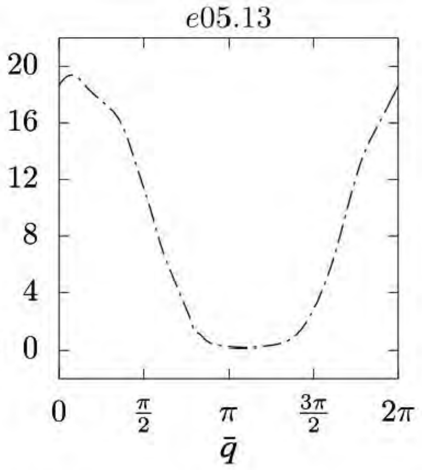

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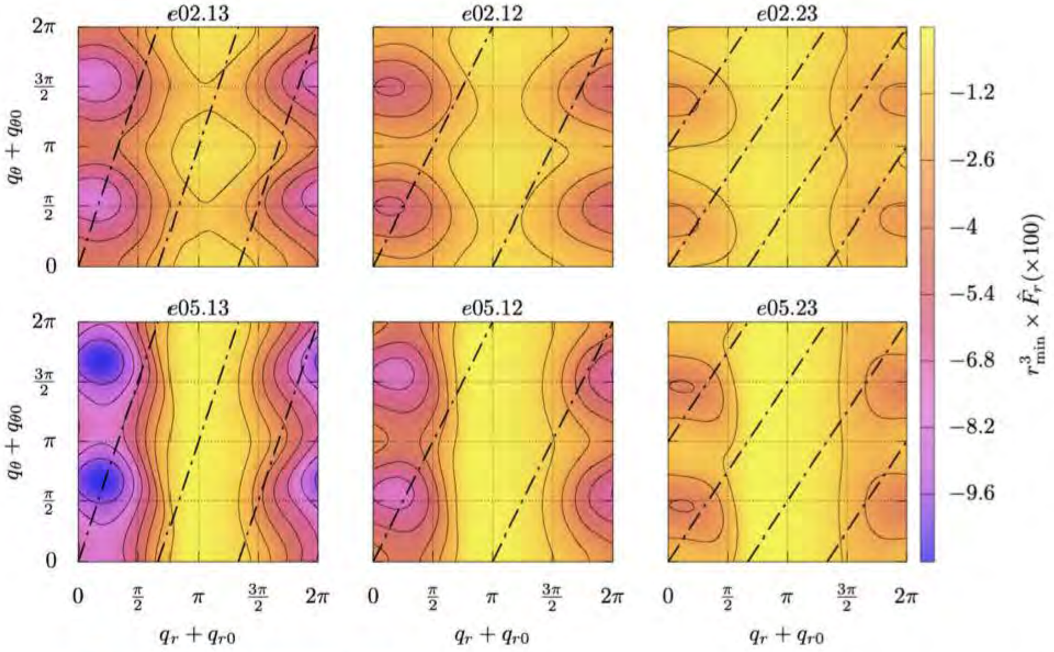

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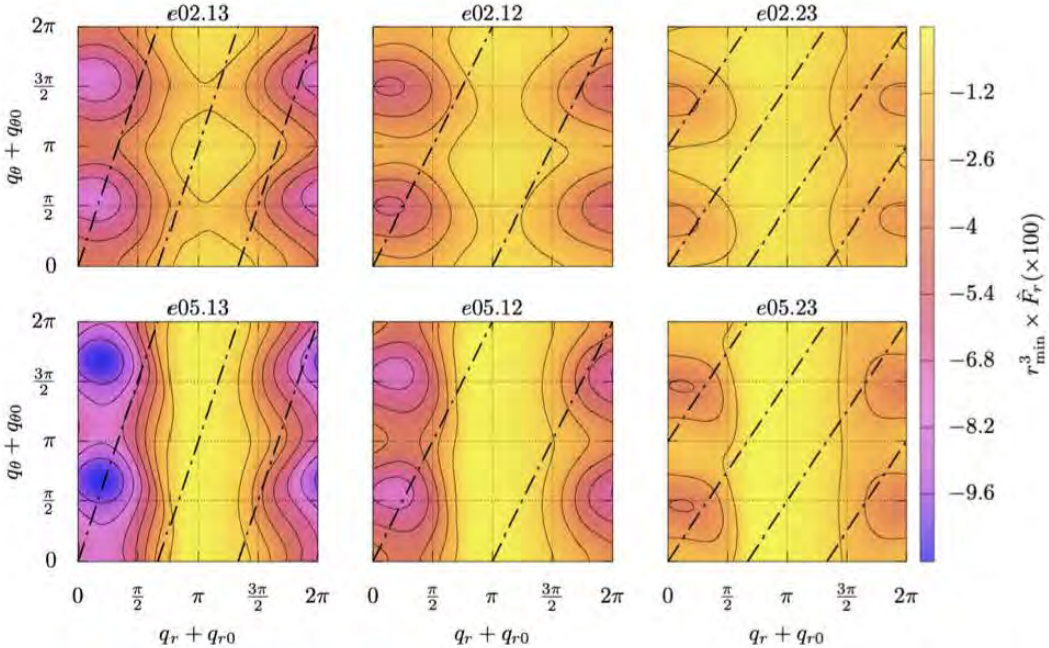
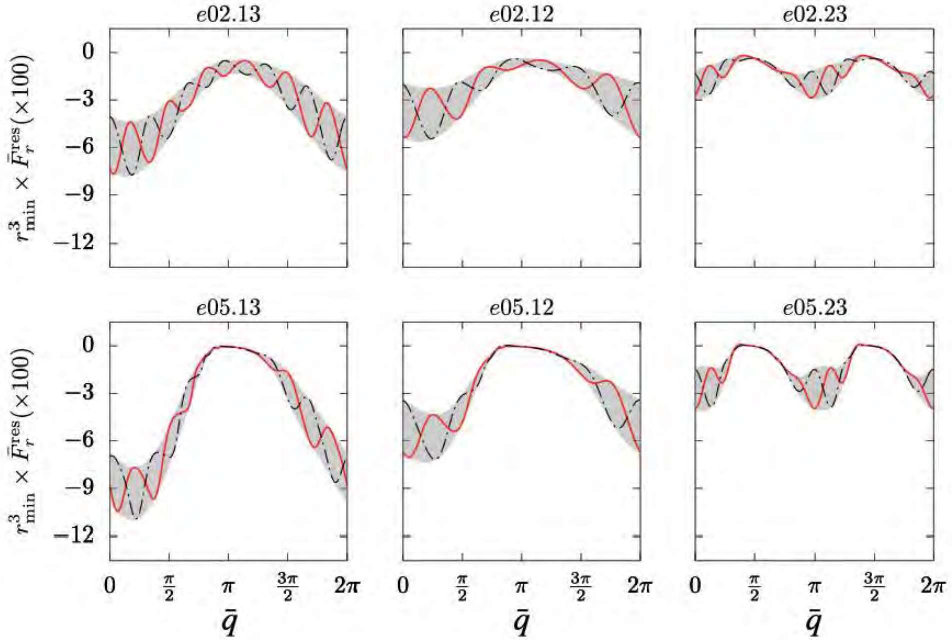
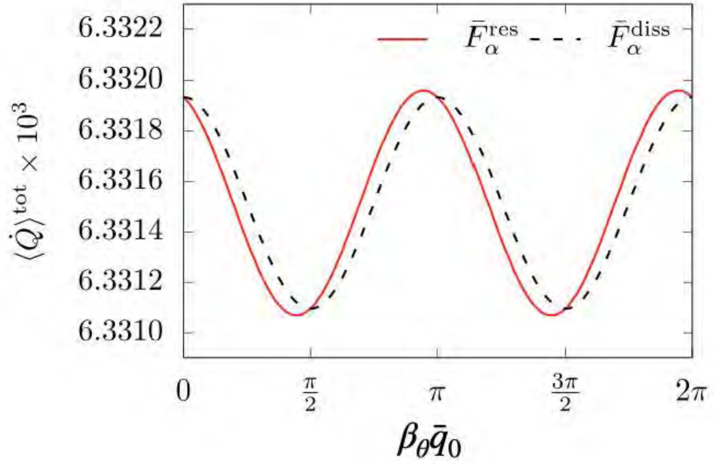




Testing the integrability conjecture

- Conservative contributions to $\langle \dot{Q} \rangle^{\mathrm{tot}}$
  - Potential conservative contribution suppressed by 4 orders of magnitude
  - Numerical error estimated to be 5×10^{-7}; consistent w/ zero



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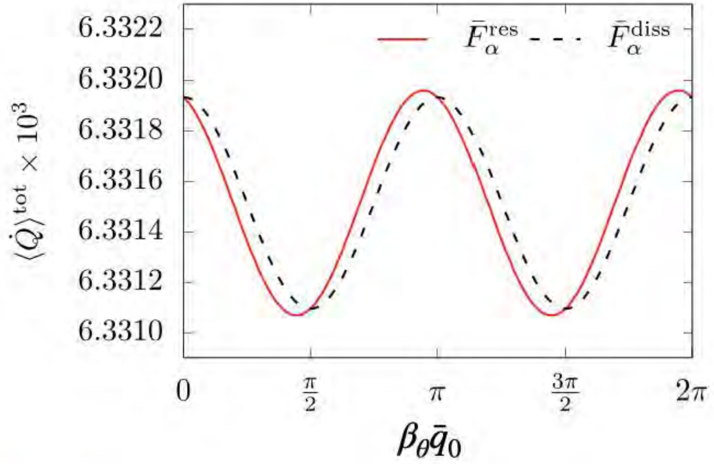

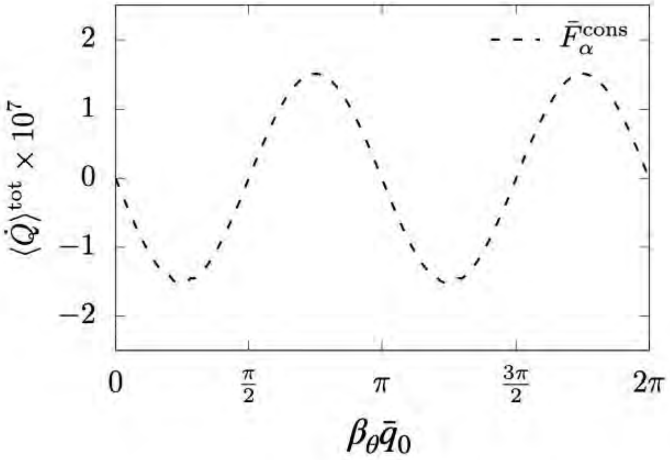

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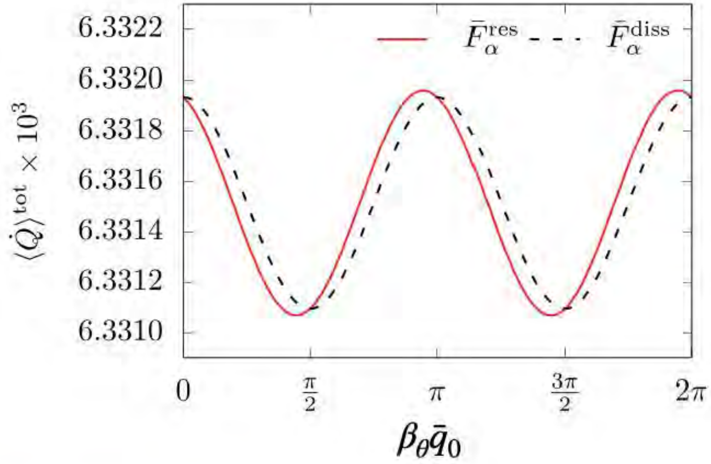

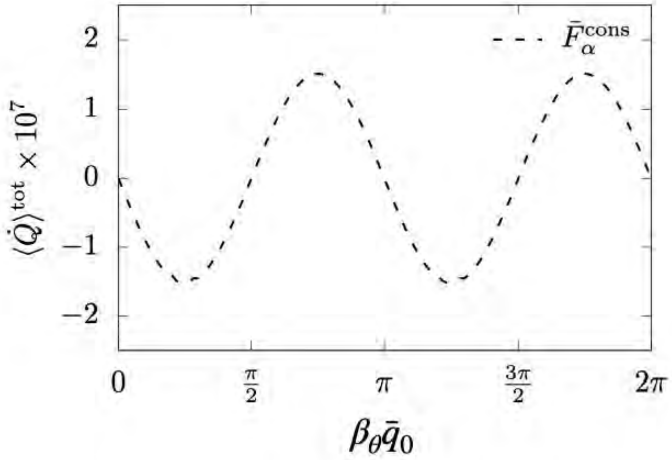

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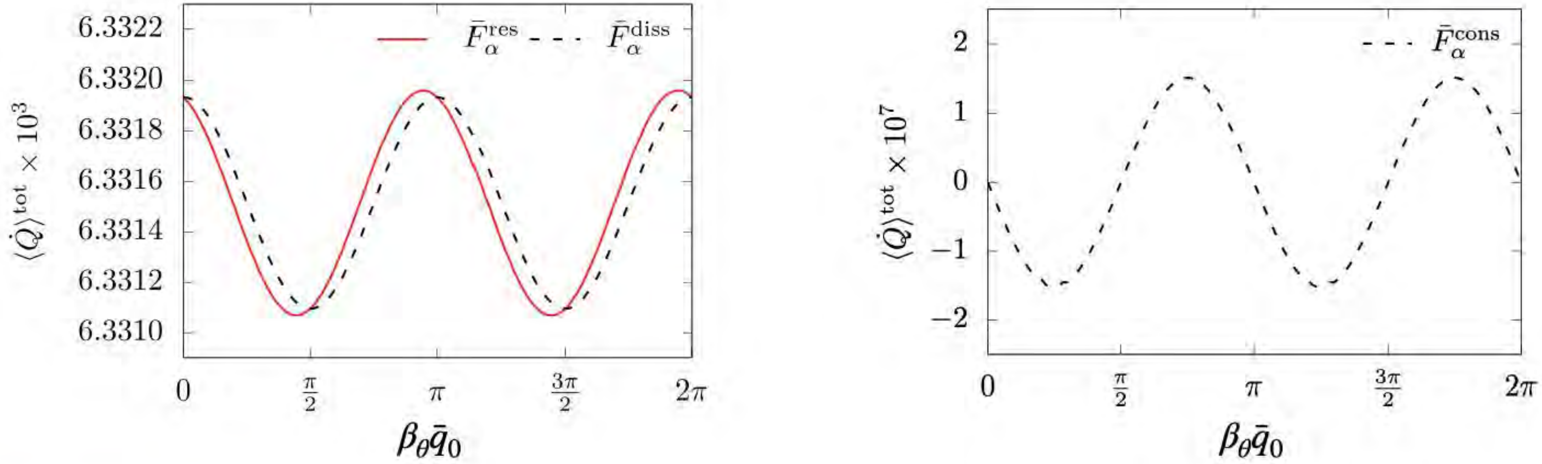

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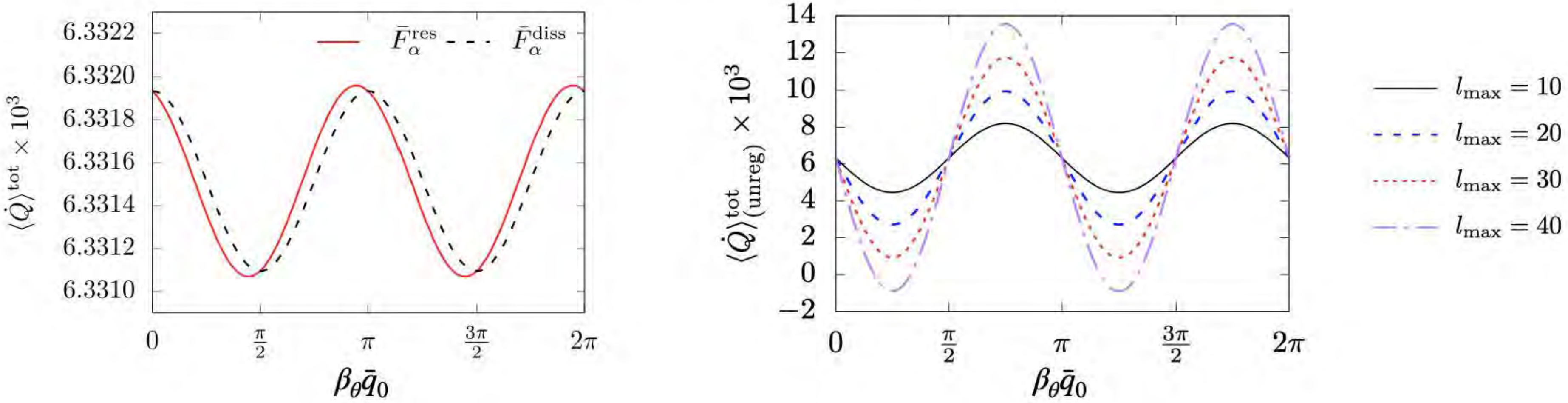

Summary

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Questions?

Scalar self-force (SSF) problem

- Gravitational case $\left(g_{\mu\nu} = g_{\mu\nu}^{\text{Kerr}} + \epsilon h_{\mu\nu}\right)$
  - Equations of motion
    $$\mu u^{\beta}\nabla_{\beta}u^{\alpha} = \epsilon F^{\alpha}_{\text{GSF}}$$
  - Gravitational self-force (GSF) equations
    $$F^{\alpha}_{\text{GSF}} = \mu\,\nabla^{\alpha\beta\gamma}h^{\text{R}}_{\beta\gamma}$$
  - Field equations & metric reconstruction
    $$\hat{\mathcal{O}}_{\pm 2}\Psi_{\pm 2} = T_{\pm 2} \quad\Longrightarrow\quad h^{\text{ret}}_{\alpha\beta} = \hat{\mathcal{S}}^{\dagger}_{\pm 2}\Psi^{\text{Hertz}}_{\pm 2} + h^{\text{comp}}_{\alpha\beta}$$
  - Regularization:
    $$h^{\text{R}}_{\alpha\beta} = h^{\text{ret}}_{\alpha\beta} - h^{\text{S}}_{\alpha\beta}$$

- Scalar case $\left(h_{\mu\nu} \to 0,\ q\right)$
  - Equations of motion
    $$u^{\beta}\nabla_{\beta}(\mu u^{\alpha}) = (q/M)F^{\alpha}$$
  - Scalar self-force (SSF) equations
    $$F^{\alpha} = q\,\nabla^{\alpha}\Phi^{\text{R}}$$
  - Field equations
    $$\hat{\mathcal{O}}_{0}\Psi_{0} = T_{0} \quad\Longrightarrow \Box_{g}\Phi = -4\pi T_{0}$$
  - Regularization:
    $$\Phi^{\text{R}} = \Phi^{\text{ret}} - \Phi^{\text{S}}$$

Impact of initial phases

- Self-force for generic orbits
  - Tri-periodic discrete spectrum
    $$\Upsilon_{mkn} = m\Upsilon_\varphi + k\Upsilon_\theta + n\Upsilon_r$$
  - Separate radial and polar dependence
    $$F^\alpha(q_r, q_\theta) = \sum_{\hat{l}mkn} Z_{\hat{l}mkn} f^\alpha_{\hat{l}lmkn}(q_r) g^\alpha_{\hat{l}lmkn}(q_\theta)$$
    - $Z_{\hat{l}mkn}$ related to source integration
  - Can express self-force as function of λ
    $$F^\alpha(\lambda) = F^\alpha(\Upsilon_r\lambda, \Upsilon_\theta\lambda)$$

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  - Can express resonant self-force in terms of non-resonant case
    $$\bar{F}^\alpha(\bar{q}; \bar{q}_0) = F^\alpha(\beta_r\bar{q}, \beta_\theta\bar{q} + \beta_\theta\bar{q}_0)$$