Progress toward post-adiabatic waveforms

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with Flanagan, Hinderer, Miller, Moxon, Warburton, and Wardell

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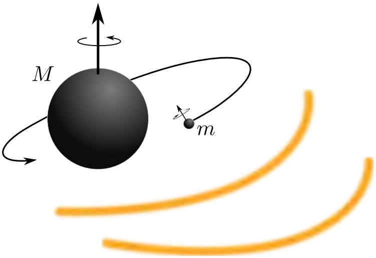
23 June 2020

Why second order?



- m perturbs M's metric $g_{\mu\nu}$:

$$\mathsf{g}_{\mu\nu} = g_{\mu\nu} + \epsilon h^1_{\mu\nu} + \epsilon^2 h^2_{\mu\nu} + \ldots$$

where $\epsilon \sim m/M$

- m's trajectory satsfies

$$\frac{D^2 z^\mu}{d\tau^2} = \epsilon F^\mu_1 + \epsilon^2 F^\mu_2 + \ldots$$

- inspiral occurs slowly, on time scale $\tau \sim 1/\epsilon$
- effect of F^μ_2 on z^μ: $\frac{D^2 \delta z^\mu}{d\tau^2} \sim \epsilon^2$

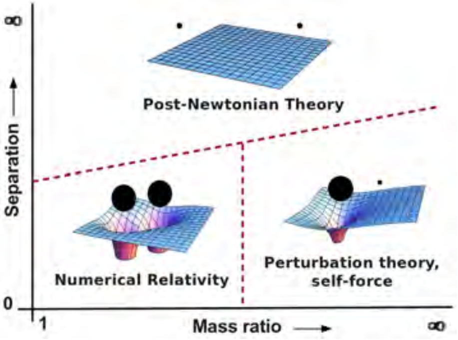
  $\Rightarrow \delta z^\mu \sim \epsilon^2 \tau^2 \sim 1$

  $\Rightarrow$ accurately describing inspiral requires F^μ_2

Improving models of IMRIs and similar-mass binaries

- second-order results will fully fix 5PN dynamics and 6PM dynamics
- also can use SF to *directly* model IMRIs? (see van de Meent's talk)

Binary parameter space



[Image credit: Leor Barack]

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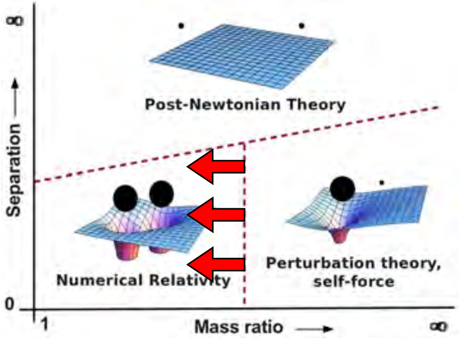

[Image credit: Leor Barack]

Adiabatic and post-adiabatic models

- on inspiral timescale $t \sim 1/\epsilon$, the gravitational wave phase has an expansion

$$\phi = \frac{1}{\epsilon}\phi_0 + \phi_1 + O(\epsilon)$$

- ϕ_0 should be enough to detect most signals
- ϕ_0 and ϕ_1 should be enough for precise parameter extraction

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determined by

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Adiabatic and post-adiabatic models

Post-adiabatic order

determined by

- averaged dissipative piece of F_2^μ
- conservative piece of F_1^μ
- oscillatory dissipative piece of F_1^μ

Adiabatic order

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Second-order self-force theory

- model the small object as a moving puncture in the field equations:

$$
\begin{aligned}
\delta G_{\mu\nu}[h^{\mathcal{R}1}] &= -\delta G_{\mu\nu}[h^{\mathcal{P}1}] \\
\delta G_{\mu\nu}[h^{\mathcal{R}2}] &= \delta^2 G_{\mu\nu}[h^1, h^1] - \delta G_{\mu\nu}[h^{\mathcal{P}2}] \\
\frac{D^2 z^\mu}{d\tau^2} &= -\frac{1}{2}(g^{\mu\nu} + u^\mu u^\nu)(g_\nu{}^\delta - h^{\mathcal{R}\delta}_\nu)(2h^{\mathcal{R}}_{\delta\beta;\gamma} - h^{\mathcal{R}}_{\beta\gamma;\delta})u^\beta u^\gamma
\end{aligned}
$$

- punctures diverge at worldline z^μ:

$$
\begin{aligned}
h^{\mathcal{P}1}_{\mu\nu} &\sim \frac{m}{|x-z|} + \ldots \\
h^{\mathcal{P}2}_{\mu\nu} &\sim \frac{m^2}{|x-z|^2} + \frac{m h^{\mathrm{R}1}}{|x-z|} + \ldots
\end{aligned}
$$

- solve for residual fields $h^{\mathcal{R}n}_{\mu\nu} = h^n_{\mu\nu} - h^{\mathcal{P}n}_{\mu\nu}$

Typical approach at first order

- system parameters $J_A = \{\mathcal{E}, \mathcal{L}, \mathcal{C}\}$
- ordinary Fourier series

$$h^1_{\mu\nu} = \sum_{k^A} h^{n,\omega_k}_{\mu\nu}(J_A, r, \theta, \phi) e^{-i(k^r\Omega_r + k^\theta\Omega_\theta + k^\phi\Omega_\phi)t}$$

- evolution

$$\frac{d\psi_A}{dt} = \Omega_A(J_B) \qquad \text{and} \qquad \frac{dJ_A}{dt} = 0$$

Two-timescale expansion

- system parameters $J_A = \{\mathcal{E}, \mathcal{L}, \mathcal{C}, \delta M, \delta S\}$
- two-timescale expansion

$$h^n_{\mu\nu} = \sum_{k^A} h^{n,\omega_k}_{\mu\nu}(J_A, r, \theta, \phi) e^{-i(k^r\psi_r + k^\theta\psi_\theta + k^\phi\psi_\phi)}$$

- evolution

$$\frac{d\psi_A}{dt} = \Omega_A(J_B, \epsilon) \quad \text{and} \quad \frac{dJ_A}{d\tilde{t}} = f_A(J_B, \epsilon)$$

Field equations

- idea: $\partial_t h^n \to (\omega_k + \epsilon \dot{J}^B \partial_B) h^{n,\omega_k}_{\mu\nu}(J_A, r, \theta, \phi)$
- phases factor out of equations

$$\delta G[h^1] = T^1$$

$$\delta G[h^{\mathcal{R}2}] = \delta^2 G[h^1, h^1] - \delta G[h^{\mathcal{P}2}]$$

- we need NITs as *preprocessing* for second order. Otherwise the phases don't factor out because the amplitudes oscillate on fast time

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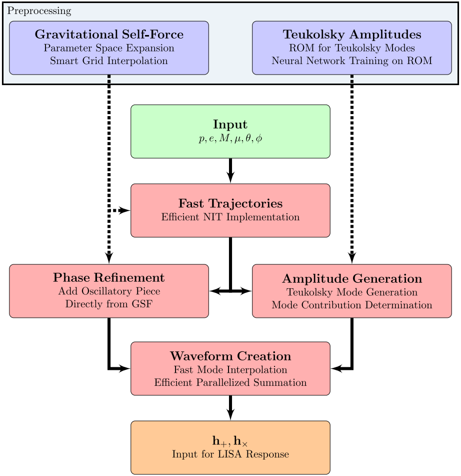
Wave generation

- compute amplitudes $h^{1,\omega_k}(J_A)$, frequencies $\omega_A(J_B)$, and driving forces $f_A(J_B)$ across J_A space
- generate waveform $\sum_{k^A} h^{1,\omega_k}(J_A) e^{-ik^A\psi_A}$ by solving

$$\frac{d\psi_A}{d\tilde{t}} = \frac{1}{\epsilon}\Omega_A(J_B,\epsilon)$$
$$\frac{dJ_A}{d\tilde{t}} = f_A(J_B,\epsilon)$$



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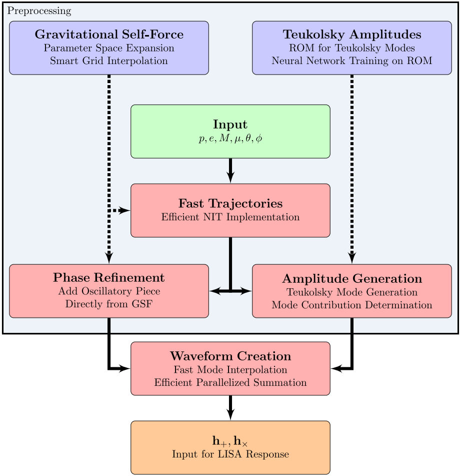
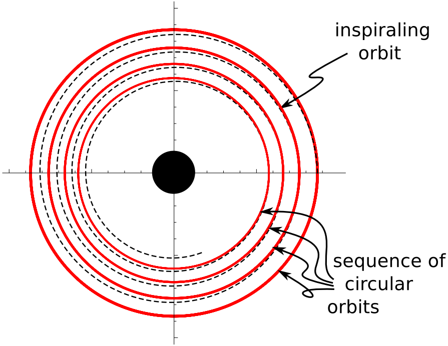


Example: quasicircular orbits

- parameters: $J_A = \{r_p, \delta M, \delta S\}$
- field: $h \sim \sum_{nlm} \epsilon^n h^n_{\omega_m lm}(J_A, r) e^{-im\phi_p} Y_{lm}$
- phase: $\phi_p = \int \Omega\, dt = \frac{1}{\epsilon} \int (\Omega_0 + \epsilon\Omega_1 + \ldots) d\tilde{t}$
- frequencies: $\omega_m = m\Omega_0$

- the simple test case
- 8 years and still kicking...



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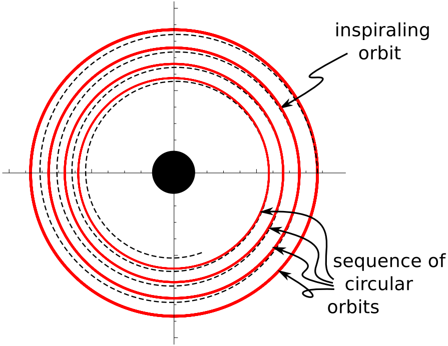

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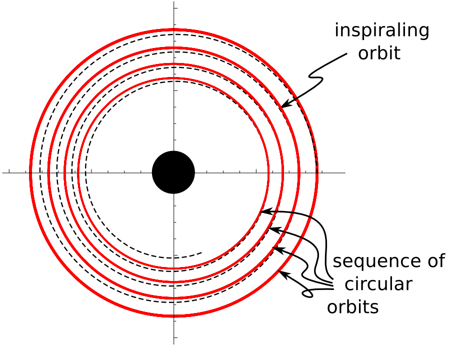

Reminder: governing equations

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$$h^{\mathcal{P}1} \sim \frac{m}{|x^\alpha - z^\alpha|}$$
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$$\begin{aligned}
h^{\mathcal{P}1}_{\omega_m lm} &\sim \text{const.} + |r - r_0| \\
h^{\mathcal{P}2}_{\omega_m lm} &\sim m^2 \log|r - r_0| + \text{const.} + (mh^{\mathcal{R}1} + r_1 + \dot{r}_0)|r - r_0|
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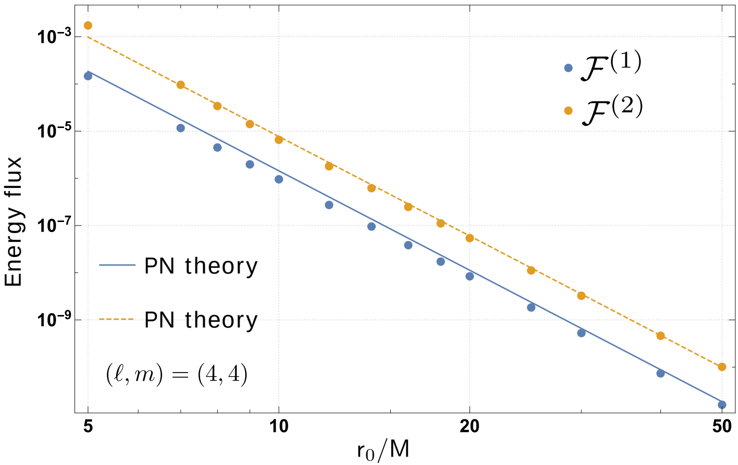
Second-order energy fluxes at infinity

Good agreement with PN



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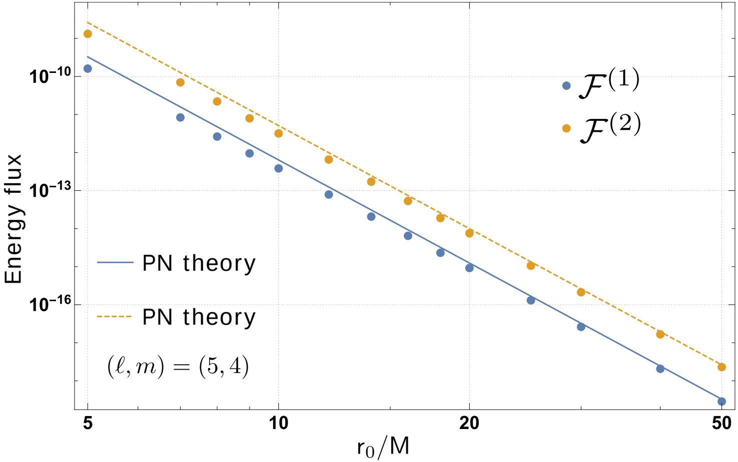

Summary

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- native outputs already in form suitable for accelerated waveform production
- still work to do for quasicircular orbits in Schwarzschild, but waveforms should be available soon(ish). See paper by Miller and Pound; more to come
- extension to eccentric orbits also looks feasible. See talk by B. Leathers
- **problem**: calculating second-order source is extremely expensive. See talks by S. Upton and C. Kavanagh.
- **problem**: Lorenz gauge not good in Kerr. See talks by A. Spiers, S. Green, and P. Zimmerman
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Topics of discussion

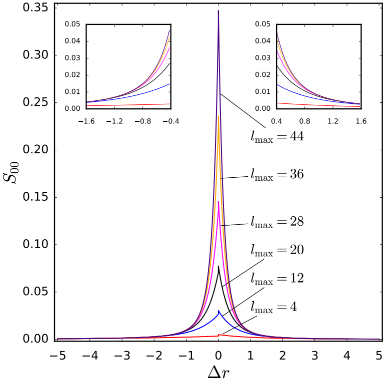
- current status
- 2SF in Kerr: starting with a good first-order metric
- post-adiabatic evolution of the Carter constant
- filling the parameter space, continued

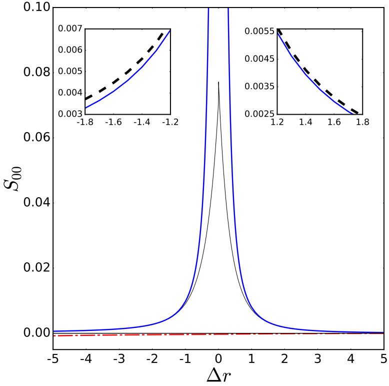
Status

- We can now calculate any mode of $h^2_{\alpha\beta}$ for quasicircular inspirals in Schwarzschild
- still missing some ingredients (more $\partial_{r_0} h^1$ data, puncture at horizon)
- and some quantities are misbehaving

Challenges: infinite sums of modes

- impossibility of computing source by summing first-order modes
- solution: split $\delta^2 G[h^1, h^1]$ into $\delta^2 G[h^{\mathcal{P}1}, h^{\mathcal{P}1}] + \delta^2 G[h^{\mathcal{P}1}, h^{\mathcal{R}1}] + \delta^2 G[h^{\mathcal{R}1}, h^{\mathcal{P}1}] + \delta^2 G[h^{\mathcal{R}1}, h^{\mathcal{R}1}]$. Compute $\delta^2 G_{lm}[h^{\mathcal{P}1}, h^{\mathcal{P}1}]$ from 3D $h^{\mathcal{P}1}$. Compute other pieces from modes of $h^{\mathcal{P}1}$ and $h^{\mathcal{R}1}$
- this problem will be ameliorated by using the highly regular gauge. Also see Chris Kavanagh's talk tomorrow.





Challenge: breakdowns at the boundaries

- the source doesn't decay quickly enough at the boundaries, causing retarded integral to diverge
- solution: find local time-domain approximations near horizon and infinity, use them as punctures, solve for the remainder
- this might be helped further by working with the Bondi-Sachs gauge. Remains to be assessed.

Challenge: lots of moving parts

- from first order: h^1_{lm}, $h^{\mathcal{P}1}_{lm}$, $\partial_{r_0} h^1_{lm}$ all on a radial grid; $h^{\mathcal{R}1}_{\alpha\beta}$ and $\partial_\gamma h^{\mathcal{R}1}_{\alpha\beta}$; contributions from δM and δS to everything; F^α_1 and $\partial_{r_0} F^\alpha_1$; coefficients in large-$r$ and near-horizon expansions of $h^1_{\alpha\beta}$
- in second-order source: punctures at both boundaries; modes of many types of punctures at the particle; $\delta^2 G_{lm}$ computed as numerical integral, sum of first-order modes, analytical expansions in different regions.
- improvements at first order help second order!

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2SF in Kerr: starting with a good first-order metric

- in Kerr, the only $h^1_{\alpha\beta}$ we've got (courtesy of Maarten van de Meent in full relativity, or Chris Kavanagh and Bini et al in a PN series) is singular on a breathing Boyer-Lindquist sphere that intersects the particle at each instant.
- $h^1 \sim \delta(r - r_p(t)) + \delta(r - r_p(t)) \Rightarrow S^2_{Teukolsky} \sim \partial^2_r[\delta'(r - r_p(t))]^2$ — badly ill defined
- How do we get a better metric to start with?
  - regularize with a local gauge transformation?
  - get everything in the Lorenz gauge?
  - somehow rigorously regularize the source without transforming the gauge?

Post-adiabatic evolution of the Carter constant

- We now have balance laws for energy and angular momentum at second order, allowing us to calculate $\langle\dot{\mathcal{E}}\rangle$ and $\langle\dot{\mathcal{L}}\rangle$ from modes of $\psi_4^{(2)}$
- Can we express $\langle\dot{\mathcal{C}}\rangle$ in a similar way? i.e., can we extend Mino's result (and the subsequent work by Sago et al and Hughes et al) to second order?

Filling the parameter space, continued

- second-order calculations are very big and very slow
- how can we fill the parameter space?
  - faster computational methods
  - clever sampling and fitting. e.g., through EOB
  - time domain for highly eccentric orbits. See Thursday morning session.
  - scattering orbits $\rightarrow$ bound orbits. See Olly Long's talk tomorrow
  - Green's function methods. See Abe Harte's talk earlier today, Isoyama et al.'s Hamiltonian formulation in past Capra's.
  - MST/PN analytical approximations to 2SF theory. See Chris Kavanagh's talk tomorrow.
  - taking information directly from PN theory, EOB, extrapolated NR data