Energy and angular momentum balance laws in second-order self-force theory

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History of Flux Balance Laws

Orbits in Kerr characterized by three slowly evolving constants of motion during inspiral: energy $\boldsymbol{E}$, angular momentum $\boldsymbol{L}$, Carter constant $\boldsymbol{C}$

- Leading-order balance laws for E, L derived [Gal'tsov '82]
  - can evolve generic orbits in Schwarzschild
  - equatorial and spherical orbits in Kerr
- Leading-order evolution for C derived in terms of Teukolsky modes
  - Complete, practical adiabatic waveform-generation scheme for Kerr [Sago et al '06, Hughes et al '05, '17], following a breakthrough in [Mino '03]
- **Goal**: extend these results to post-adiabatic order!
  - currently calculate more than is required, in Schwarzschild [Miller, Pound, Warburton, Wardell '19]
  - seek something more efficient and streamlined

See Sec 6.1 of [Barack and Pound, '19] for details

Second-Order Self-Force Theory

Expansion of metric

$$g_{\mu\nu} = g^{(0)}_{\mu\nu} + \epsilon h^{(1)}_{\mu\nu} + \epsilon^2 h^{(2)}_{\mu\nu} + O(\epsilon^3)$$

Expansion of phase

$$\phi = \frac{1}{\epsilon}[\phi^{(0)} + \epsilon^{(1/2)}\phi^{(1/2)} + \epsilon\phi^{(1)} + ...]$$

$\phi^{(0)}$: adiabatic order
$\phi^{(\frac{1}{2})}$: resonances (ignored in our analysis)
$\phi^{(1)}$: post-adiabatic order

- To compute $\phi^{(1/2)}$ and $\phi^{(1)}$, we need the time-averaged dissipative part of the second-order self-force $\langle F^{\alpha}_{2,diss} \rangle$

Two-Timescale Approximation Method

Orbital motion in terms of action-angle variables

$$\frac{d\mathcal{J}_A}{d\tilde{t}} = f_A(\mathcal{J}_B, \epsilon) \qquad \text{and} \qquad \frac{d\psi_A}{dt} = \omega_A(\mathcal{J}_B, \epsilon)$$

where $\mathcal{J}_A = \{E, L, C, \delta M, \delta a\}$

Metric perturbation in two-timescale form

$$h^n_{\alpha\beta} = \sum_{mpq} \tilde{h}^{n,\omega_{mpq}}_{\alpha\beta}(\mathcal{J}_A(\tilde{t}), r, \theta, \varphi) e^{-i(p\psi_r + q\psi_\theta + m\psi_\varphi)}$$

- $\tilde{h}^{n,\omega_{mpq}}_{\alpha\beta}$ purely slow-time evolving quantity
- Compute amplitudes and phases through post-adiabatic order
- Ultimately, for a practical evolution scheme, want useful perturbative expressions for $d\mathcal{J}_A/d\tilde{t}$

Local Evolution Equations for E and L

- Focus on just two of the $\mathcal{J}_A$ variables: $E = -Q_{\xi_{(t)}}, \;\; L = Q_{\xi_{(\phi)}}$
- Define *effective* metric

$$\breve{g}_{\alpha\beta} = g^{(0)}_{\alpha\beta} + h^{\mathcal{R}}_{\alpha\beta}$$

- More physically intuitive expressions for conserved charges

$$\breve{Q}_\xi = \breve{g}_{\alpha\beta}\breve{u}^\alpha \xi^\beta$$

- Rate of change

$$\frac{d\breve{Q}_\xi}{d\breve{\tau}} = \breve{u}^\alpha \breve{u}^\gamma \breve{\nabla}_\gamma(\breve{g}_{\alpha\beta}\xi^\beta) = \frac{1}{2}\breve{u}^\alpha \breve{u}^\beta \mathcal{L}_\xi h^{\mathcal{R}}_{\alpha\beta}$$

- Goal: express in terms of asymptotic fluxes in t

$$\left\langle \frac{d\breve{Q}_\xi}{dt} \right\rangle = \left\langle \frac{1}{2\breve{u}^t}\breve{u}^\alpha \breve{u}^\beta \mathcal{L}_\xi h^{\mathcal{R}}_{\alpha\beta} \right\rangle$$

Expressions in terms of the Radiative Field

- Relate to asymptotic fluxes by first rewriting in terms of the *radiative* fields $h^{(n)\mathrm{Rad}}_{\alpha\beta}$

$$h^{\mathrm{Rad}(1)}_{\alpha\beta} = \int_V dV' \ G^{\mathrm{Rad}}_{\alpha\beta\alpha'\beta'} T^{\alpha'\beta'}$$
$$h^{\mathrm{Rad}(2)}_{\alpha\beta} = \int_V dV' \ G^{\mathrm{Rad}}_{\alpha\beta\alpha'\beta'} S^{\alpha'\beta'}_{\mathrm{eff}}$$

- G^{Rad} is the antisymmetric piece of G^{ret}
- Express balance laws in terms of these fields by appealing to symmetries of Green's functions

Expressions in terms of the Radiative Field cont.

- At first order, $T^{\alpha'\beta'}$ has same structure as $\left\langle u^\alpha u^\beta \right\rangle$

$$\left\langle \frac{1}{u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{\mathcal{R}(1)}_{\alpha\beta} \right\rangle = \left\langle \frac{1}{u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{\mathrm{Rad}(1)}_{\alpha\beta} \right\rangle$$

- Second-order source has no such convenience, but a combination does (by symmetry)

$$\left\langle \frac{1}{u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{\mathcal{R}(2)}_{\alpha\beta} \right\rangle = \left(\left\langle \frac{1}{u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{\mathcal{R}(2)}_{\alpha\beta} \right\rangle + \frac{1}{16\pi\mu} \int dV' S^{\alpha'\beta'}_{(2)\mathrm{eff}} \mathcal{L}_{\xi'} h^{(1)}_{\alpha'\beta'} \right) - \frac{1}{16\pi\mu} \int dV' S^{\alpha'\beta'}_{(2)\mathrm{eff}} \mathcal{L}_{\xi'} h^{(1)}_{\alpha'\beta'}$$

- In terms of the radiative field + volume term

$$\left\langle \frac{d\check{Q}_\xi}{dt} \right\rangle = \left\langle \frac{1}{2u^t} u^\alpha u^\beta \mathcal{L}_\xi \left(\epsilon h^{\mathrm{Rad}(1)}_{\alpha\beta} + 2\epsilon^2 h^{\mathrm{Rad}(2)}_{\alpha\beta} \right) \right\rangle - \frac{\epsilon^2}{16\pi\mu} \int dV' S^{\alpha'\beta'}_{(2)\mathrm{eff}} \mathcal{L}_{\xi'} h^{(1)}_{\alpha'\beta'}$$

absorbed two $h^{\mathcal{R}(1)}$ terms from the RHS into a re-definition of Q and t

Contribution from h^{Rad}

- Convert h^{Rad} contribution to asymptotic fluxes
  - $\langle uu\mathcal{L}h^{\mathrm{Rad}}\rangle$ in terms of boundary integrals - as done historically at first order [Gal'tsov and others]

$$\left\langle \frac{1}{2u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{(1)\mathrm{Rad}}_{\alpha\beta} \right\rangle = \frac{1}{8\pi} \int_{\partial V} d\Sigma_\alpha \delta^2 G^{\alpha\beta}[h^{(1)}, h^{(1)}] \xi_\beta$$

$$\left\langle \frac{1}{u^t} u^\alpha u^\beta \mathcal{L}_\xi h^{(2)\mathrm{Rad}}_{\alpha\beta} \right\rangle = \frac{1}{8\pi} \int_{\partial V} d\Sigma_\alpha (\delta^2 G^{\alpha\beta}[h^{(1)}, h^{(2)}] + \delta^2 G^{\alpha\beta}[h^{(2)}, h^{(1)}]) \xi_\beta$$

- This is the only place that $h^{(2)}$ appears in the final balance law

Contribution from Volume Integral

- Previously unresolved [J. Moxon previous Capras]
- Volume term derived previously is a fairly simple geometric quantity in a carefully defined metric $\mathfrak{g} \sim g^{(0)} + h^{(1)}$ (with no $h^{(2)}$)
- Using Stoke's Theorem, we reduce volume integral to a boundary integral at ∂V

$$\begin{aligned}
\int_V dV[g^{(0)}] S_{\text{eff}}^{(2)\alpha\beta} \mathcal{L}_\xi h^{(1)}_{\alpha\beta} &= -2 \left[\int_V dV[\mathfrak{g}] \mathfrak{G}^{\alpha\beta} \mathcal{L}_\xi \mathfrak{g}_{\alpha\beta} \right]^{(3,0)} \\
&= -2 \left[\int_{\partial V} d\Sigma[\mathfrak{g}]_\alpha \mathfrak{G}^{\alpha\beta} \xi_\beta \right]^{(3,0)}
\end{aligned}$$

Final Flux Balance Laws

- Adding up the contributions, we get physical flux across ∂V

$$\left\langle \frac{d\check{Q}_\xi}{dt} \right\rangle \sim \int_{\partial V} d\Sigma_\alpha[\mathrm{g}]\xi_\beta \left(G^{\alpha\beta}[g] - \delta G^{\alpha\beta}[h] \right) + \text{local terms in } h^{\mathcal{R}(1)}$$

neglecting slow-time derivatives in first term

Required Inputs from a Two-Timescale Expansion

- What do we need to calculate as input into the balance law to drive the evolution?
- Calculate inputs from a two-timescale expansion of the field equations
- Can get everything we need from
  (a) first-order: the full first-order field
  (b) second-order: mode amplitudes of the second-order Weyl scalar ψ_4

Summary and Outlook

- First pertubative post-adiabatic balance laws for E and L
  - key step: converting $h^{\mathcal{R}}$ into h^{Rad}
  - key step: converting $\int_V$ into $\int_{\partial V}$
- Directly relate dE/dt and dL/dt to quantities calculable from field values at $\mathcal{H}^+$ and $\mathcal{I}^+$
- These balance laws enable efficient two-timescale evolution of equatorial orbits using a second-order Teukolsky equation - avoid full metric reconstruction
- Highly general post-adiabatic derivation, but...
  - need Carter constant for generic orbits

Thank you for listening!