Progress on Second-Order Self-Force in Kerr.

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Overview

1. **Extending** fist-order methods (the **Teukolsky equation**) to second order
2. Why **gauge at second order** is more complicated
3. **Gauge fixing** a second-order quantity (to achieve gauge invariance)

Current Second-Order GSF Methods

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These methods use the Lorenz gauge, where the linearised Einstein field equation is **non-separable in Kerr** spacetime.

Let's return to first-order SF for inspiration on how to formalise a second-order method in Kerr...

Notation: ε expansion

$$
\begin{aligned}
g_{ab} =& g^{(0)}_{ab} + \varepsilon^1 h^{(1)}_{ab} + \varepsilon^2 h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3),\\
\Rightarrow g_{ab} =& g^{(0)}_{ab} + h^{(1)}_{ab} + h^{(2)}_{ab} + \mathcal{O}(\varepsilon^3).
\end{aligned}
$$

$$
\psi_4 = \psi_4^{(0)} + \psi_4^{(1)} + \psi_4^{(2)} + \mathcal{O}(\varepsilon^3).
$$

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We **want to extend these methods to second order** so that we can repeat this procedure to calculate the second-order GSF.

The Second-Order Teukolsky Equation

Campanelli & Lousto derived a second-order Teukolsky equation:

$$\mathcal{O}\psi_4^{(2)} = \mathcal{S}[T_{ab}^{(2)}] + S_{LC}[\psi_2^{(1)}, \psi_3^{(1)}, \psi_4^{(1)}, \alpha^{(1)}, \beta^{(1)}, ..., \tau^{(1)}].$$

[Campanelli & Lousto. *Phys. Rev. D*, 59(12):124022, 1999.]

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However, for GSF calculations this **appears to be non-integrable**:

$$S_{LC}[h_{ab}^{(1)}, h_{ab}^{(1)}] \sim (\partial_r \partial_r h_{ab}^{(1)})(\partial_r \partial_r h_{ab}^{(1)}) \sim r^{-6},$$

as $h_{ab}^{(1)} \sim r^{-1}$, where r is distance from the worldline; hence,

$$\int_0^R S_{LC}[h_{ab}^{(1)}, h_{ab}^{(1)}] r^2 d\Omega dr \sim \int_0^R r^{-4} dr = \left[-\frac{1}{3r^3} \right]_0^R,$$

A “New” Second-Order Teukolsky Equation

The Teukolsky equation gives the (spin -2) Wald identity:

$$\mathcal{O}\psi_4^{(1)} = \mathcal{S}[T_{ab}^{(1)}]$$
$$\mathcal{O}\mathcal{T}(h_{ab}^{(1)}) = \mathcal{S}\mathcal{E}(h_{ab}^{(1)}),$$

where $\mathcal{T}(h_{ab}^{(1)}) := \psi_4^{(1)}$ (and $\mathcal{E}(h_{ab}^{(1)}) = T_{ab}^{(1)}$).

[Wald. *PRL*, 41(4):243, 1978.] [Green, Hollands & Zimmerman. *CLASSICAL QUANT GRAV*, 37(7):075001, 2020.]

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Acting on $h_{ab}^{(2)}$ instead gives

$$\mathcal{O}\mathcal{T}(h_{ab}^{(2)}) = \mathcal{S}\mathcal{E}(h_{ab}^{(2)})$$
$$\mathcal{O}(\psi_{4L}^{(2)}) = \mathcal{S}\Big(T_{ab}^{(2)} - \delta^2 G(h_{ab}^{(1)}, h_{ab}^{(1)})\Big),$$

where $\psi_{4L}^{(2)} := \mathcal{T}(h_{ab}^{(2)})$ (and $\mathcal{E}(h_{ab}^{(2)}) + \delta^2 G(h_{ab}^{(1)}, h_{ab}^{(1)}) = T_{ab}^{(2)}$).

What is $\psi_{4L}^{(2)}$?

[Campanelli & Lousto. *Phys. Rev. D*, 59(12):124022, 1999.]

$$\begin{aligned}\psi_4^{(2)} &= \mathcal{T}[h_{ab}^{(2)}] + \delta^2\psi_4[h_{ab}^{(1)} h_{ab}^{(1)}] \\ &= \psi_{4L}^{(2)} + \psi_{4Q}^{(2)};\end{aligned}$$

i.e. $\psi_{4L}^{(2)}$ is the linear in $h_{ab}^{(2)}$ part of $\psi_4^{(2)}$.

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And (unlike $\psi_4^{(2)}$) $\psi_{4L}^{(2)}$ is infinitesimal tetrad rotation invariant.

Making the Source Regular

Implement a puncture scheme to handle $T^{(2)}_{ab}$:

$$\mathcal{O}[\psi^{(2)}_{4L}] = \mathcal{S}\Big[-\delta^2 G[h^{(1)}_{ab}, h^{(1)}_{ab}]\Big] \qquad \forall\ x^\alpha \notin \gamma,$$

$$\mathcal{O}[\psi^{\mathcal{R}(2)}_{4L}] = \mathcal{S}\Big[-\delta^2 G[h^{(1)}_{ab}, h^{(1)}_{ab}]\Big] - \mathcal{O}[\psi^{\mathcal{P}(2)}_{4L}] \qquad \forall\ x^\alpha,$$

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The source is still very singular

$$\mathcal{S}\Big[-\delta^2 G[h_{ab}^{(1)}, h_{ab}^{(1)}]\Big] \sim \partial_r \partial_r \delta^2 G[h_{ab}^{(1)}, h_{ab}^{(1)}] \sim \frac{1}{r^5}.$$

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However, $\mathcal{S}$ is a linear operator and in the **highly regular gauge** (Sam Upton's talk) $\delta^2 G[h^{(1)}_{ab}, h^{(1)}_{ab}]$ is, at leading order, a linear operator.

$\Rightarrow$ Integrable source (using distribution theory).

A Second-order Gauge Invariant?

At first order life is easy:

$$\psi_4'^{(1)} = \psi_4^{(1)} + \mathcal{L}_{\vec{\xi}^{(1)}} \psi_4^{(0)} = \psi_4^{(1)},$$

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At second order life is terrible:

$$\begin{aligned}\psi_4'^{(2)} =& \psi_4^{(2)} + \mathcal{L}_{\vec{\xi}^{(1)}}\psi_4^{(1)} + \mathcal{L}_{\vec{\xi}^{(2)}}\psi_4^{(0)} + \frac{1}{2}\mathcal{L}_{\vec{\xi}^{(1)}}\mathcal{L}_{\vec{\xi}^{(1)}}\psi_4^{(0)} \\ =& \psi_4^{(2)} + \mathcal{L}_{\vec{\xi}^{(1)}}\psi_4^{(1)} \\ \neq& \psi_4^{(2)}.\end{aligned}$$

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I.e. $\psi_4^{(2)}$ is not gauge invariant (similar story for $\psi_{4L}^{(2)}$).
But gauge fixing requires only knowing $\vec{\xi}^{(1)}$.

Gauge Fixing Region by Region

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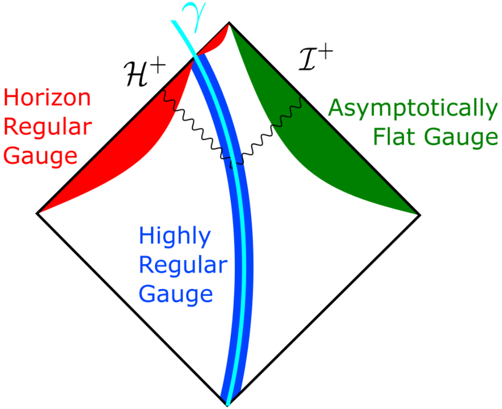
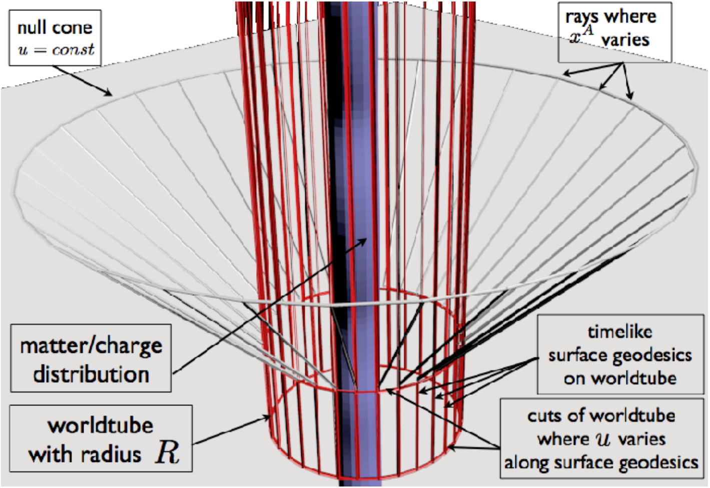
- For each physically significant region we want a **fully fixed** *good* gauge:



The Bondi–Sachs Gauge



Source: [T. Mädler J. Winicour (2016). Bondi-Sachs Formalism. Scholarpedia, 11, 33528.]

- Gauge conditions: $g_{rr} = g_{rA} = 0$ and $Det[f_{AB}] = Det[q_{AB}]$

Fully fixed Good Gauge at $\mathcal{I}^+$: Bondi–Sachs Gauge

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$\Rightarrow$ Gauge fix the second-order Teukolsky equation

$\Rightarrow$ Gauge invariant results ($\psi'^{(2)}_{4L}$)!

Finding a regular $h_{ab}^{(1)}$ *(ADVERTISEMENTS)*

Current $h_{ab}^{(1)}$ calculations use CCK metric reconstruction which produces **pathological singularities** in $h_{ab}^{(1)}$.

[M Van De Meent. *Physics Review D*, 94(19):104033,2018]

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Tomorrow 7:40 CDT, Stephen Green (with Hollands & Zimmerman):

- Teukolsky formalism for nonlinear Kerr perturbations (an extension of metric reconstruction to non-vacuum spacetime)

Tomorrow 8:00 CDT, Peter Zimmerman (with Green, Spiers & Pound):

- Implementing the non-linear Teukolsky formalism for a point particle in flat space (**to ameliorate these singularities** for second-order calculations)

Summary

- **a “new” second-order Teukolsky equation**, whose source is easier to regularise
- A plan for **gauge fixing** the second-order Teukolsky equation in **three regions** with good gauges
- A practical method for fixing to the Bondi–Sachs gauge and fixing the BMS frame.

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Thank you for listening

Please feel free to contact me (A.R.C.Spiers@soton.ac.uk)

and look out for our forthcoming paper on this work (*Spiers, Moxon & Pound*)

Get hyped for Peter Zimmerman's talk tomorrow at 8:00 CDT