Electromagnetic self-force on Kerr spacetime

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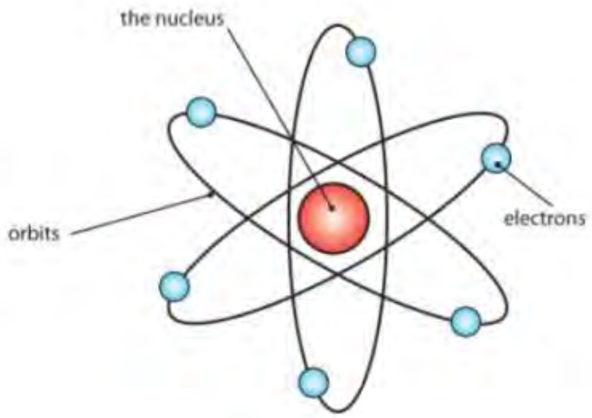
23rd Capra Meeting on Radiation Reaction in General Relativity

Electromagnetic self-force on Kerr spacetime

Rutherford's atomic model



Abraham-Lorentz force $\mathbf{F} = \frac{2}{3}\frac{q^2}{4\pi\epsilon_0 c^3}\dot{\mathbf{a}}$

Dissipative force $\longrightarrow$ Collapse of the atom

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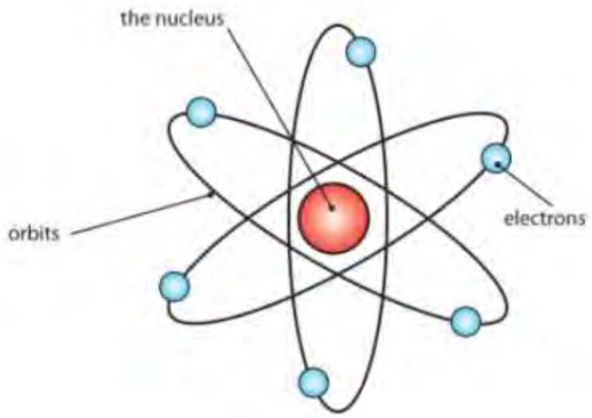

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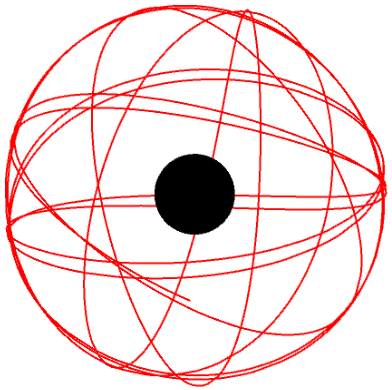
<u>Particle around Kerr Black Hole</u>



Dissipative & Conservative force

$$\mathbf{F} \approx \frac{q^2}{4\pi\epsilon_0 c^3}\left(\frac{2}{3}\frac{d\mathbf{g}}{dt} + \frac{GMc}{r^3}\hat{\mathbf{r}}\right)$$

DeWitt Force

Electromagnetic self-force on Kerr spacetime

Set-up:

Charged particle on equatorial circular orbit

$$J^\alpha = \frac{q}{r_0^2}\delta(r - r_0)\delta(\phi - \phi_p)\delta(\theta - \pi/2)u^\alpha$$

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We are looking for the Faraday tensor $F_{\mu\nu}$
via Maxwell scalars

$$\phi_0 \equiv F_{\mu\nu} l^\mu m^\nu, \quad \phi_2 \equiv F_{\mu\nu} \overline{m}^\mu n^\nu, \; \phi_1 \equiv \frac{1}{2} F_{\mu\nu} \left(l^\mu n^\nu - m^\mu \overline{m}^\nu \right)$$

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Separation of variable:

$$\phi_0 = \sum_{lm} R^{lm}_{+1}(r) S^{lm}_{+1}(\theta) e^{im(\phi - \Omega t)}.$$

$$2(r - ia\cos(\theta))^2 \phi_2 = \sum_{lm} R^{lm}_{-1}(r) S^{lm}_{-1}(\theta) e^{im(\phi-\Omega t)}.$$

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Separation of variable:

$$\phi_0 = \sum_{lm} R_{+1}^{lm}(r)S_{+1}^{lm}(\theta)e^{im(\phi-\Omega t)}.$$

$$2(r-ia\cos(\theta))^2\phi_2 = \sum_{lm} R_{-1}^{lm}(r)S_{-1}^{lm}(\theta)e^{im(\phi-\Omega t)}.$$

Teukolsky equations (in vacuum)

$$\left(\Delta_r\mathcal{D}^\dagger\mathcal{D} - 2i\omega r\right)R_{-1} = \lambda R_{-1},$$
$$\left(\Delta_r\mathcal{D}\mathcal{D}^\dagger + 2i\omega r\right)\Delta_r R_{+1} = \lambda\Delta_r R_{+1},$$
$$\left(\mathcal{L}\mathcal{L}_1^\dagger + 2a\omega\cos\theta\right)S_{-1} = -\lambda S_{-1},$$
$$\left(\mathcal{L}^\dagger\mathcal{L}_1 - 2a\omega\cos\theta\right)S_{+1} = -\lambda S_{+1},$$

Teukolsky-Starobinsky identities

$$\Delta_r\mathcal{D}\mathcal{D}R_{-1} = \mathcal{B}\,\Delta_r R_{+1},$$
$$\Delta_r\mathcal{D}^\dagger\mathcal{D}^\dagger\Delta_r R_{+1} = \mathcal{B}\,R_{-1},$$
$$\mathcal{L}^\dagger\mathcal{L}_1^\dagger S_{-1} = \mathcal{B}\,S_{+1},$$
$$\mathcal{L}\mathcal{L}_1 S_{+1} = \mathcal{B}\,S_{-1},$$

Electromagnetic self-force on Kerr spacetime

<u>Solving the sourced Teukolsky equation:</u>

$$\left[\Delta\left(\partial_r - \frac{isK}{\Delta}\right)\left(\partial_r + \frac{isK}{\Delta}\right) + 2ism\Omega - \lambda\right] P_s = A_s \delta(r - r0) + B_s \delta'(r - r0)$$

$$P_{+1} = \Delta R_{+1} \text{ and } P_{-1} = R_{-1}$$

Electromagnetic self-force on Kerr spacetime

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Solution:

$$P_s = \alpha_s^{\infty} P_s^{\infty}(r)\Theta(r - r0) + \alpha_s^{h} P_s^{h}(r)\Theta(r0 - r).$$

$$\begin{pmatrix} \alpha_s^{\infty} \\ \alpha_s^{h} \end{pmatrix} = \frac{1}{W_s} \begin{bmatrix} -\left(\frac{P_s^h}{\Delta}\right)' & \frac{P_s^h(r_0)}{\Delta_0} \\ -\left(\frac{P_s^{\infty}}{\Delta}\right)' & \frac{P_s^{\infty}(r_0)}{\Delta_0} \end{bmatrix} \begin{pmatrix} B_s \\ A_s \end{pmatrix},$$

Wronskian

Evaluated at the particle
$r = r_0$

Electromagnetic self-force on Kerr spacetime

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Black Hole Perturbation Toolkit

$$\begin{pmatrix} \alpha_s^\infty \\ \alpha_s^h \end{pmatrix} = \frac{1}{W_s}\begin{bmatrix} -\left(\frac{P_s^h}{\Delta}\right)', & \frac{P_s^h(r_0)}{\Delta_0} \\ -\left(\frac{P_s^\infty}{\Delta}\right)', & \frac{P_s^\infty(r_0)}{\Delta_0} \end{bmatrix}\begin{pmatrix} B_s \\ A_s \end{pmatrix},$$

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Wronskian

Evaluated at the particle $r = r_0$

$$P_s \to \phi_0, \phi_1, \phi_2 \to F_{\alpha\beta}$$

Compute the force

$$\mathcal{F}_\alpha = qF_{\alpha\beta}u^\beta \qquad u^\beta = u^t(1, 0, 0, \Omega)$$

Dissipative part

$$\mathcal{F}_t = qF_{t\phi}u^\phi.$$

$$\mathcal{F}_\phi = qF_{\phi t}u^t = -\frac{\mathcal{F}_t}{\Omega}.$$

Conservative part

$$\mathcal{F}_r = qF_{rb}u^b = qu^t\left(F_{rt} + \Omega F_{r\phi}\right)$$

Electromagnetic self-force on Kerr spacetime

Dissipative component: $\mathcal{F}_t = qF_{t\phi}u^{\phi}.$

Electromagnetic self-force on Kerr spacetime

<u>Dissipative component:</u> $\mathcal{F}_t = qF_{t\phi}u^\phi.$

Linked to energy (angular momentum) fluxes

Current *Killing vector*

$$Y^a = T_{ab}K^b$$

Energy-momentum tensor

$$Y^a{}_{;a} = F_{ab}K^a J^b$$

Conservation law

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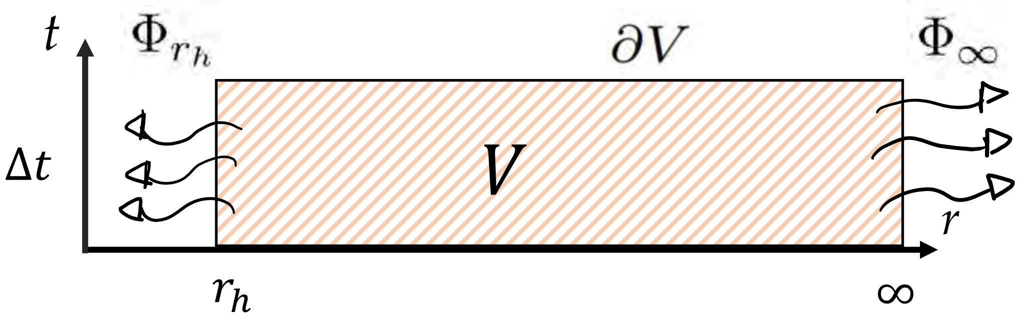

$$\int_V Y^a{}_{;a}\sqrt{-g}\,d^4x = \int_{\partial V} Y^a d\Sigma_a \longrightarrow \mathcal{F}_a K^a \Delta t = \int_{\partial V} Y^a d\Sigma_a.$$

$$K^a = (1,0,0,0) \quad \text{or} \quad (0,0,0,1)$$

$$\boxed{\mathcal{F}_a K^a = \Phi_\infty + \Phi_{r_h}}$$

Electromagnetic self-force on Kerr spacetime

<u>Dissipative component:</u> $\mathcal{F}_t = qF_{t\phi}u^\phi = \Phi_\infty + \Phi_h$ with $\Phi_{\infty,h} = \lim_{r\to\infty,r_h} \int T^r{}_t\sqrt{-g}d\theta d\phi$

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At infinity: $T^r{}_t = \frac{|\phi_2|^2}{2\pi}$

- $2(r - ia\cos(\theta))^2 \phi_2 = \sum_{l,m} \left({}_{-1}\alpha^{\infty}_{lm}\right) {}_{-1}P^{\infty}_{lm}(r) S^{lm}_{-1}(\theta) e^{-i(\omega t - m\phi)}$
- $\int \bar{S}^{lm}_{-1}(\theta) S^{l'm}_{-1}(\theta) \sin(\theta) d\theta = \frac{\delta_{ll'}}{2\pi}$
- $\lim_{r\to\infty} \frac{|{}_{-1}P^{\infty}_{lm}(r)|^2}{r^2} = 1$

$$\longrightarrow \boxed{\Phi_\infty = \sum_{lm} \frac{|{}_{-1}\alpha^{\infty}_{lm}|^2}{4}}$$

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$$\Phi_\infty = \sum_{lm} \frac{|{}_{-1}\alpha^\infty_{lm}|^2}{4}$$

$$\Phi_h = \sum_{lm} \frac{\omega}{8Mr_+(\omega - m\Omega_h)}|{}_{+1}\alpha^h_{lm}|^2$$

$$\Omega_h = \frac{a}{2Mr_+}$$

$\omega - m\Omega_h < 0$ $\longleftrightarrow$ Superradiance

If $\Phi_\infty = -\Phi_h$ $\rightarrow$ "Floating orbit"

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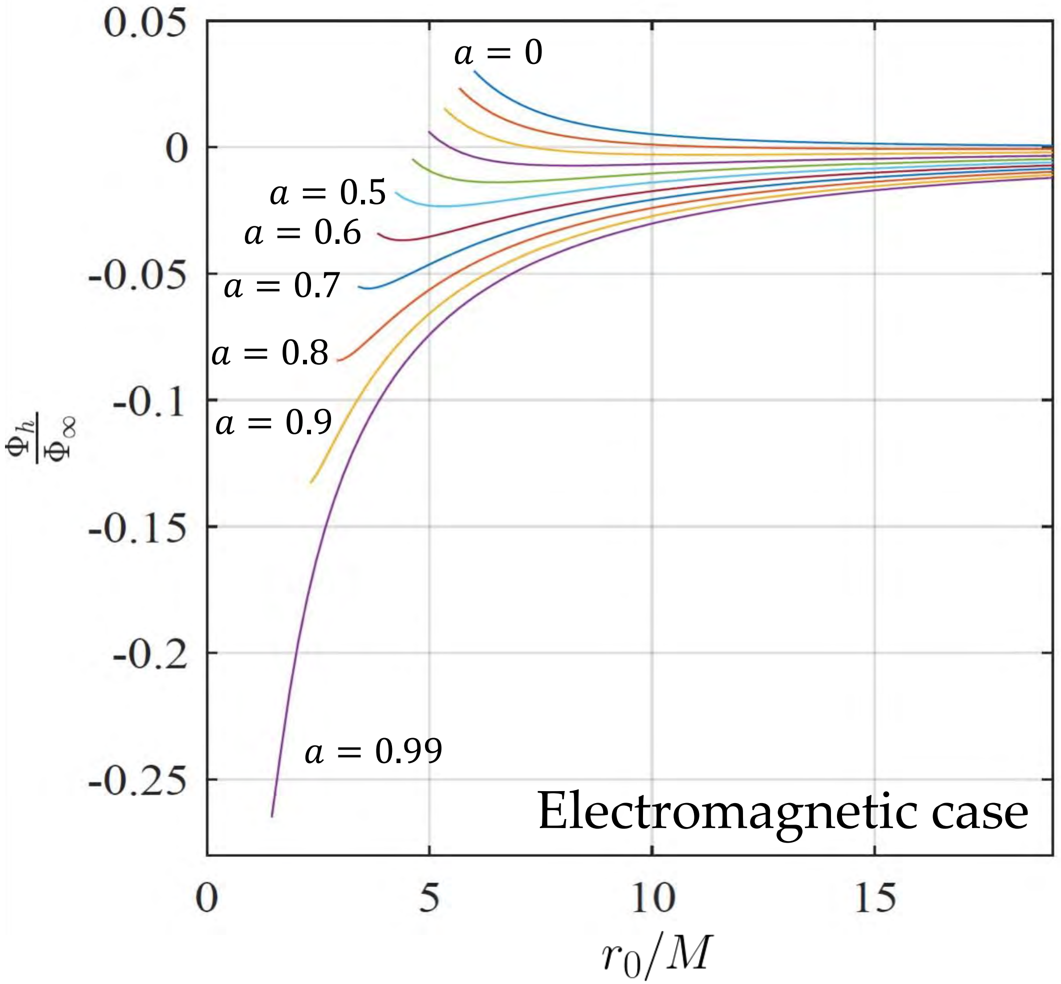

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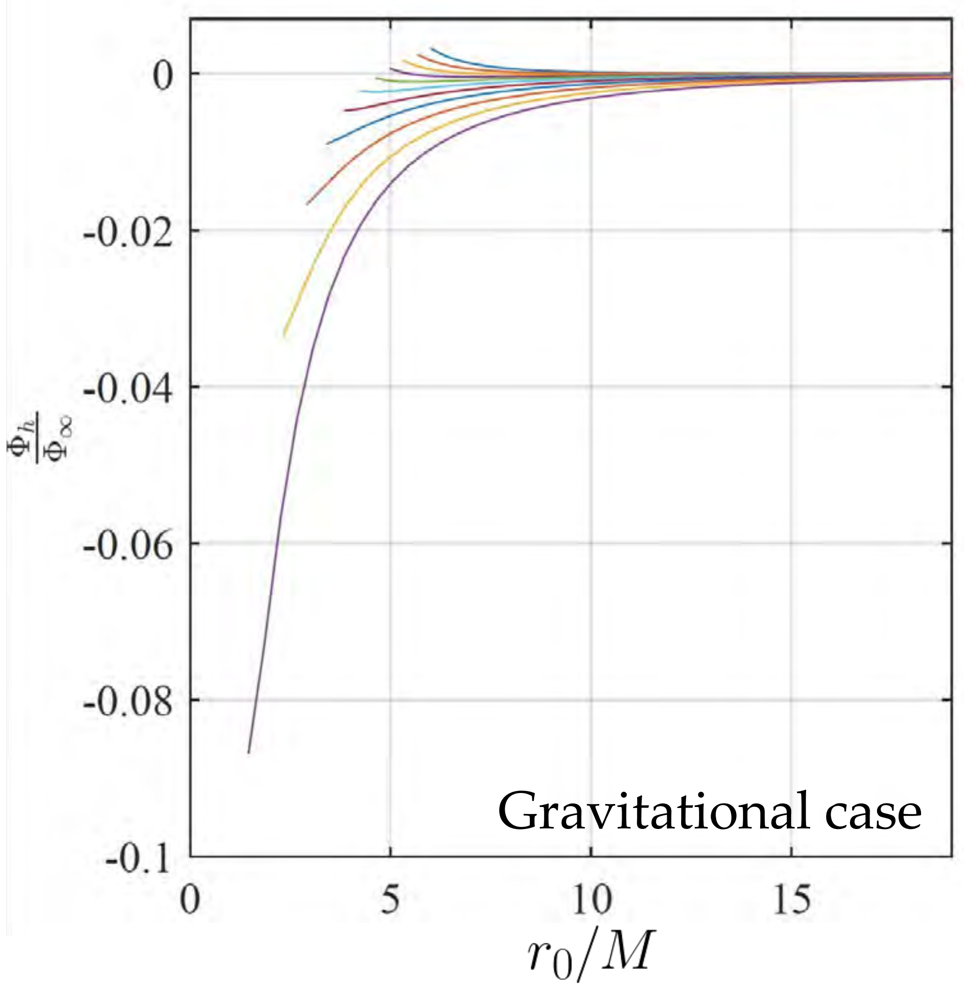
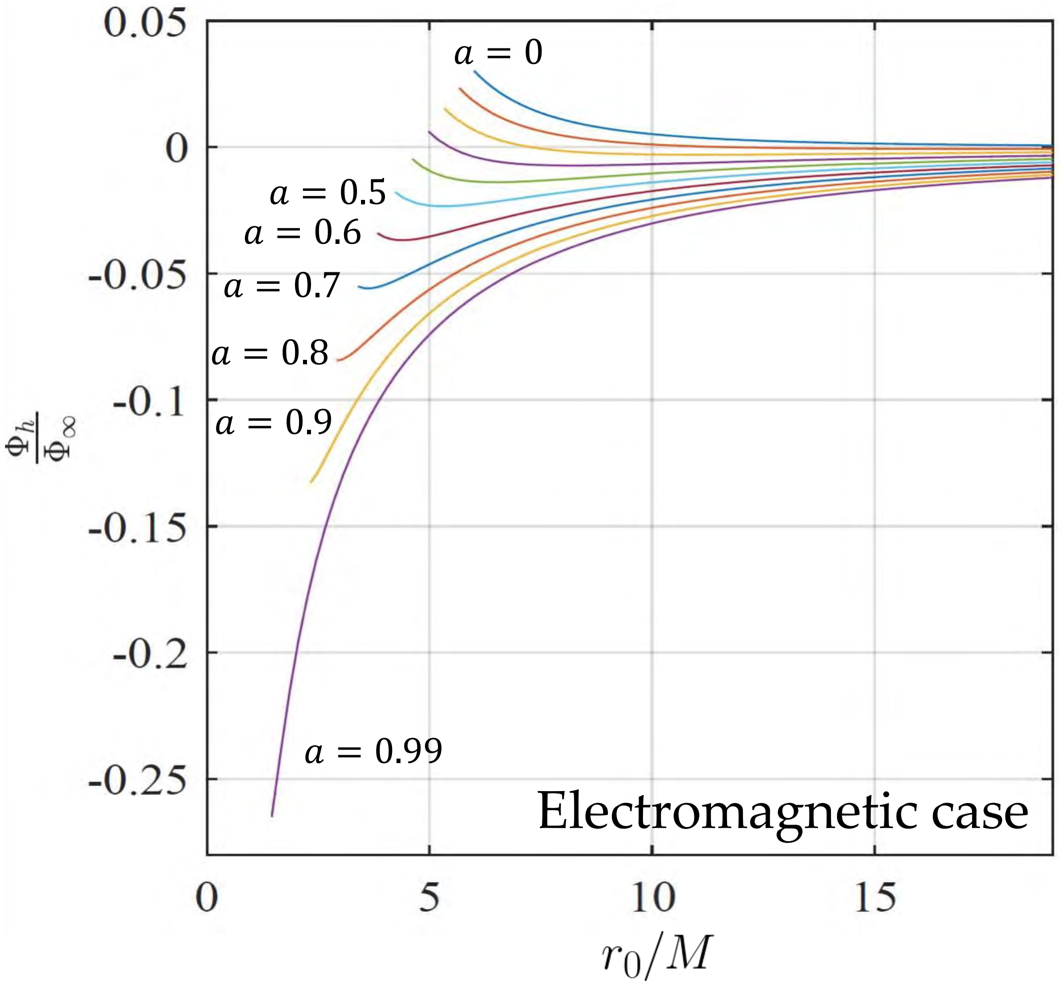


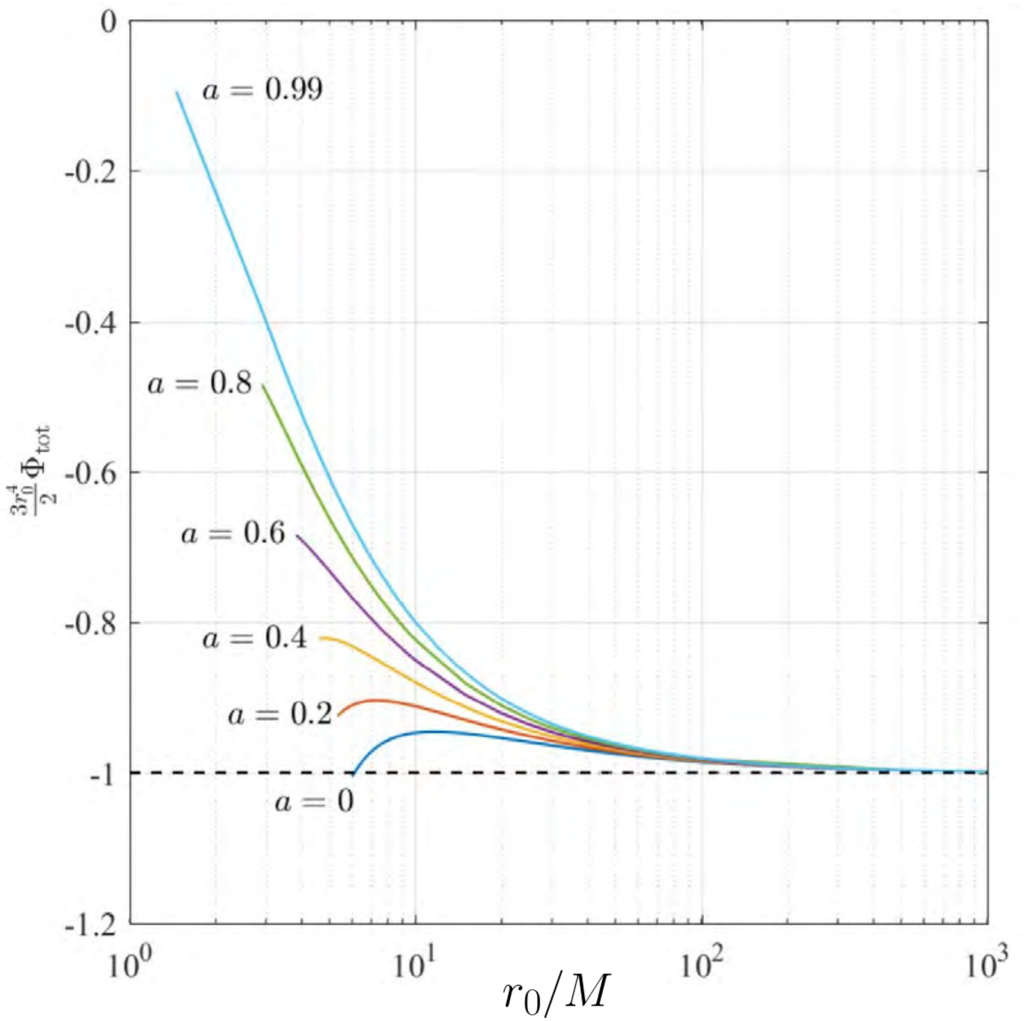


Electromagnetic self-force on Kerr spacetime

Dissipative component: $\mathcal{F}_t = qF_{t\phi}u^{\phi} = \Phi_\infty + \Phi_h$

Comparison with the weak field limit

$$\mathbf{F}_{\text{self}} \approx \frac{q^2}{4\pi\epsilon_0 c^3}\left(\frac{2}{3}\frac{d\mathbf{g}}{dt} + \frac{GMc}{r^3}\hat{\mathbf{r}}\right)$$



Electromagnetic self-force on Kerr spacetime

<u>Conservative component:</u> $$\mathcal{F}_r = qF_{rb}u^b = qu^t\left(F_{rt} + \Omega F_{r\phi}\right)$$

$\mathcal{F}_r$ depends on ϕ_0, ϕ_2 and ϕ_1

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$$\mathcal{F}_r = qu^t \sum_{l,m} \frac{\sqrt{2}\left((a^2+r^2)\Omega - a\right)}{(r - ia\cos\theta)} \sin(\theta)\left[-\frac{iR^{lm}_{+1}S^{lm}_{+1}}{4} + \frac{iR^{lm}_{-1}S^{lm}_{-1}}{4\Delta_r}\right] + (1-a\Omega)\frac{g^{lm}_{+1}\mathcal{L}_1 S^{lm}_{+1} - iaf^{lm}_{-1}\mathcal{D}P^{lm}_{-1}}{\sqrt{2}(r-ia\cos\theta)^2} + c.c.$$

divergent sum!

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Regularisation:

- Expand into scalar spherical harmonics
- Subtract "regularisation parameters" to get a convergent sum

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$S^{lm}_s \to Y^{lm}_s \to Y^{lm}$

Expand in $\cos(\theta)$ (only necessary in Kerr)

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Expand in $\cos(\theta)$ (only necessary in Kerr)

Add monopole: $\mathcal{F}^0_r = u^t \frac{\left(1 - a\Omega\sin^2\theta\right)\left(r^2 - a^2\cos^2\theta\right)}{\left(r^2 + a^2\cos^2\theta\right)^2}$

Electromagnetic self-force on Kerr spacetime

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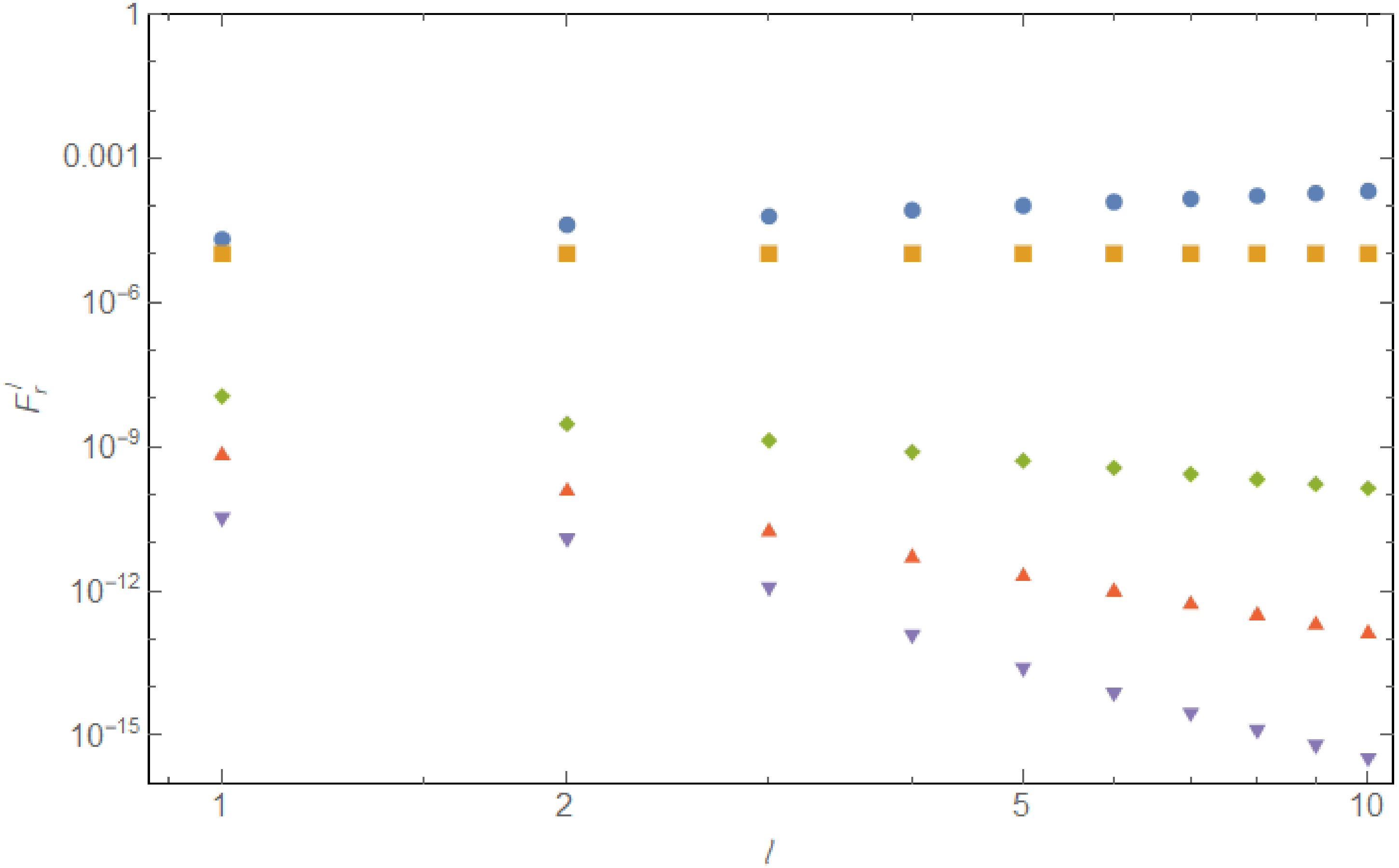
Schwarzschild case a=0



$\mathcal{F}_r^\ell$

$\mathcal{F}_r^\ell - A(2\ell+1)$

$\mathcal{F}_r^\ell - A(2\ell+1) - B$

$\mathcal{F}_r^\ell - A(2\ell+1) - B - \dfrac{C}{(2\ell-1)(2\ell+3)}$

$\mathcal{F}_r^\ell - A(2\ell+1) - B - \dfrac{C}{(2\ell-1)(2\ell+3)} - \dfrac{D}{(2\ell-3)(2\ell-1)(2\ell+3)(2\ell+5)}$

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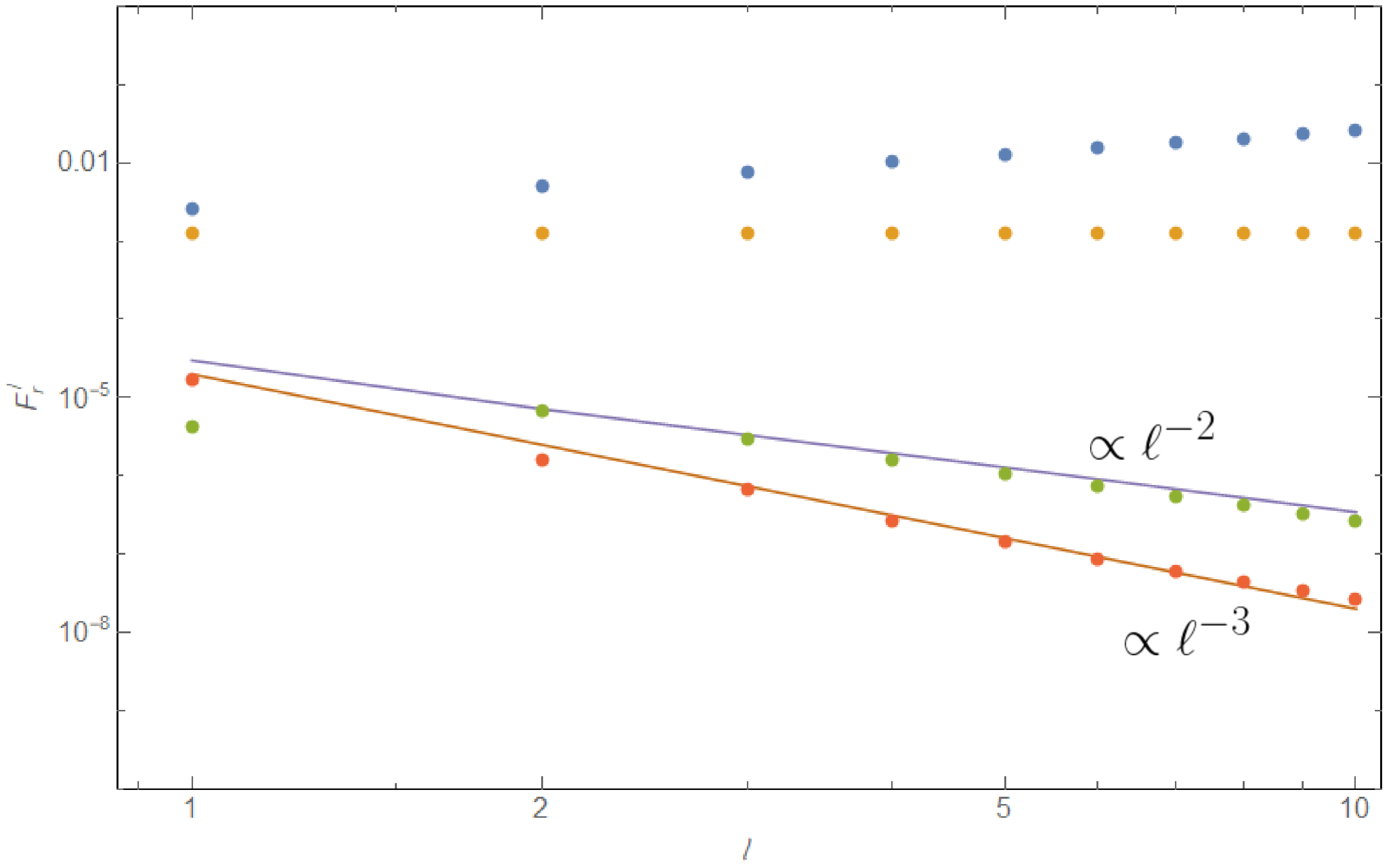
Kerr case a=1/2



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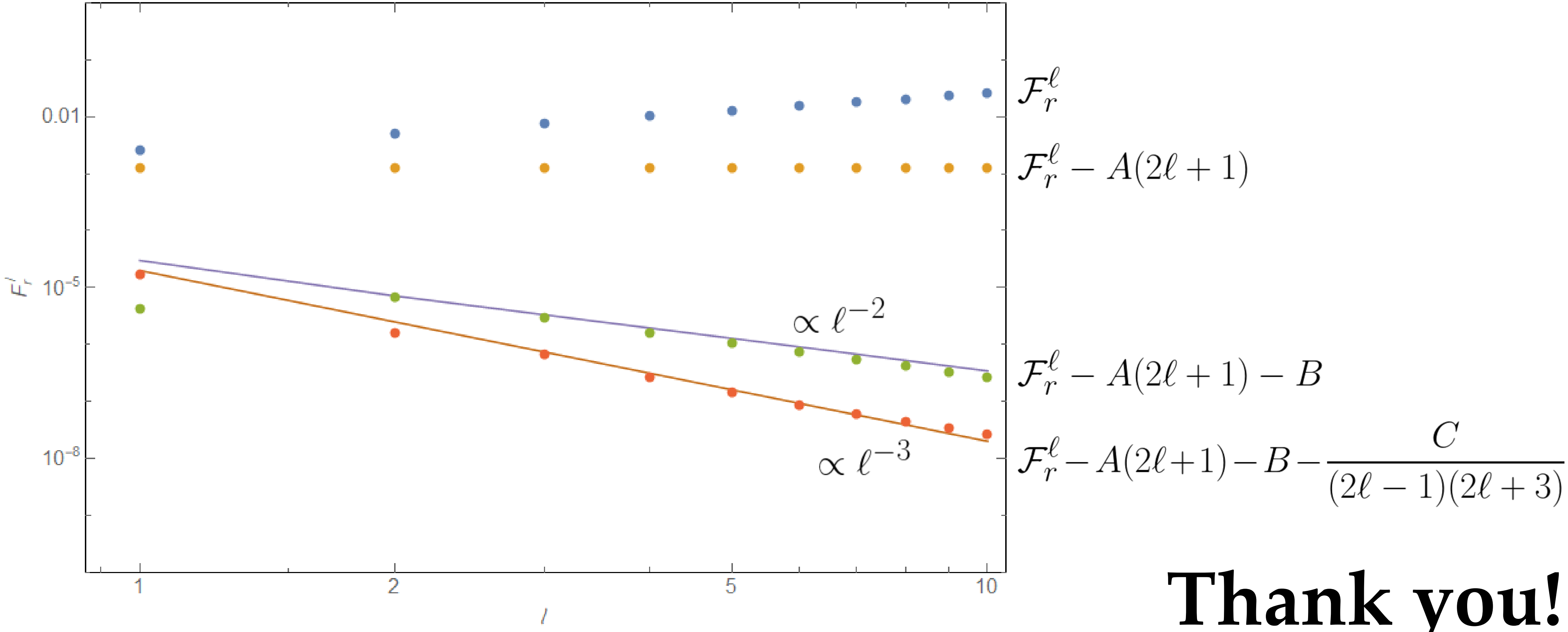

Thank you!