Velocity-Field Theory, Boltzmann’s Transport Equation, Geometry and Emergent Time

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Boltzmann Equation, 1872

2nd Law of Thermodynamics
Dynamical Origin: Einstein Theory (Geometry of ”dynamics”) ?

- \( u(x, 't') \): Velocity distribution of Fluid Matter
- Size of fluid-particles: \( L \triangleq \text{Atomic (10}^{-10}\text{m)} \ll L \leq \text{Optical Microscope (10}^{-6}\text{m)} \)
- Temporal development of Distribution Function \( f('t', x, v) \): probability of particle having velocity \( v \) at space \( x \) and time \( 't' \)
Notion of Energy is obscure when Dissipation occurs. Consider the movement of a particle under the influence of the friction force. The emergent heat (energy) during the period \([t_1, t_2]\) can not be written as:

\[
\int_{x_1}^{x_2} F_{\text{friction}} \, dx = [E\{x(t), \dot{x}(t)\}]_{t_1}^{t_2} = E|_{t_2} - E|_{t_1},
\]

where \(x(t)\): Orbit (path) of Particle.
Sec 1. Introduction: c. Discrete Morse Flow

- Time should be re-considered, when dissipation occurs. → Step-Wise approach to time-development.
- Connection between step $n$ and step $n-1$ is determined by the minimal energy principle.
- Time is "emergent" from the principle.
- Direction of flow (arrow of time) is built in from the beginning.

New approach to Statistical Fluctuation
Discrete Morse Flow Method (Kikuchi, ’91)
Holography (AdS/CFT, ’98)
Sec 2. Emergent Time and Diff. Eq. a. Energy Functional

1 dim viscous fluid, \( u(x) \): velocity field (distribution),

Energy Functional

\[
l_n[u(x)] = \int dx \left\{ \frac{\sigma}{2 \tilde{\rho}_0} \left( \frac{du}{dx} \right)^2 + V(u) + u \frac{dV^1(x)}{dx} + \frac{1}{2h} (u - u_{n-1})^2 \right\} + l_n^0, \quad \sigma \equiv 1, \quad \tilde{\rho}_0 \equiv 1, \quad n = 1, 2, \cdots
\]

\[
V(u) = \frac{m^2}{2} u^2 + \frac{\lambda}{4!} u^4, \quad u = u(x), \quad u_{n-1} = u_{n-1}(x). \quad (2)
\]

periodic bound. cond. \( u(x) = u(x + 2l) \). \quad (3)
Sec. 2 Emergent Time and Diffusion Equation

Variation $\delta l_n(u) = 0 (u(x) \rightarrow u(x) + \delta u(x))$ gives Next step $u_n(x)$

$$\frac{1}{h} (u_n(x) - u_{n-1}(x)) = \frac{\sigma}{\tilde{\rho}_0} \frac{d^2 u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u} - \frac{dV^1_n(x)}{dx}, \quad (4)$$

$l_n[u_n] \leq l_n[u_{n-1}]$ but $l_n[u_n] \leq l_{n-1}[u_{n-1}]$ does NOT hold. \quad (5)

Discrete time $t_n = nh = n\tau_0 \times (\frac{h}{\tau_0})$, $\tau_0 \equiv h\sqrt{\lambda \sigma / m}$, $t_0 \equiv 0$. \quad (6)

Noting $u(x, t_n) \equiv u_n(x)$, $t_n = t_{n-1} + h$, as $h \to 0$,

$$\frac{\partial u(x, t)}{\partial t} = \frac{\sigma}{\tilde{\rho}_0} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\delta V(u(x, t))}{\delta u(x, t)} - \frac{\partial V^1(x, t)}{\partial x}, \quad 1 \text{ D diff. eq. (7)}$$
Sec.2 Emer. T and Diff. Eq. : b. Variat. Principle

Figure: The energy functional $I_n[u(x)]$, (2), of the velocity-field $u(x)$. 

\[ I_n[u(x)] \]

\[ \{u(x)\} \]

Un(x)    Un-1(x)
Noting \( u(x) - u_{n-1}(x) \) in (2) should be \( u(x + hu_{n-1}) - u_{n-1}(x) \), (4) is corrected as \( (I_n \rightarrow \tilde{I}_n) \)

\[
\frac{1}{h}(u_n(x) - u_{n-1}(x)) + u_{n-1}(x) \frac{du_n(x)}{dx} = \frac{\sigma}{\tilde{\rho}_0} \frac{d^2 u_n}{dx^2} - \frac{\delta V(u_n)}{\delta u_n} - \frac{dV_1^1(x)}{dx}.
\]

Continuous time limit \( (h \rightarrow 0) \) gives Burgers’s equation (1D Navier-Stokes eq.)

\[
\frac{\partial u(x, t)}{\partial t} + u(x, t) \frac{\partial u(x, t)}{\partial x} = \frac{\sigma}{\tilde{\rho}_0} \frac{\partial^2 u(x, t)}{\partial x^2} - \frac{\delta V(u(x, t))}{\delta u(x, t)} - \frac{\partial V_1^1(x, t)}{\partial x}.
\]

Eq. (9), for \( m = 0 \), is inv. under global Weyl transformation.

\[
V^1(x, t) \rightarrow e^{-2\varepsilon} V^1(e^\varepsilon x, e^{2\varepsilon} t), \quad u(x, t) \rightarrow e^{-\varepsilon} u(e^\varepsilon x, e^{2\varepsilon} t),
\]

\[
\partial_x \rightarrow e^{-\varepsilon} \partial_x, \quad \partial_t \rightarrow e^{-2\varepsilon} \partial_t, \quad t \rightarrow e^{2\varepsilon} t, \quad x \rightarrow e^\varepsilon x \quad \text{(10)}
\]
Sec 3. Statistical Fluctuation Effect: Uncertainty

Large # of Particles $\rightarrow$ Statistical Average

Inevitable uncertainty of Present Approach

1. The finite time-increment gives uncertainty to the minimal solution $u_n(x)$.
2. The existence of the characteristic particle size gives uncertainty to the minimal solution
3. The system energy generally changes step by step.

Claim: the fluctuation comes not from the quantum effect but from the statistics caused by above points.

The statistics is taken into account by newly defining the n-th energy functional $\Gamma[u(x); u_{n-1}(x)]$ using the path-integral.

$$e^{-\frac{1}{\alpha} \Gamma[u(x); u_{n-1}(x)]] = \int \mathcal{D}u(x) e^{-\frac{1}{\alpha} \tilde{I}_n[u(x)]}$$ \hspace{1cm} (11)

Let us evaluate it perturbatively around the minimal path $u_n(x)$.

$$u(x) = u_n(x) + \sqrt{\alpha} q(x), \quad |\sqrt{\alpha} q| \ll |u_n|, \quad \left. \frac{\delta \tilde{I}_n[u]}{\delta u} \right|_{u=u_n} = 0$$ \hspace{1cm} (12)

new expansion parameter $\alpha$ is introduced. $[\alpha]=\left[\tilde{I}_n\right]=ML^2T^{-2}$
Sec 3. Stat. Fluct. Effect: \( \alpha \) Not \( \hbar \) But \( \alpha \)

Claim: \( \alpha \) should be small and should be chosen as

1) dimension is consistent
2) proportional to the small scale parameter which characterizes the relevant physical phenomena (ex. the mean-free path of the fluid particle). NOT include Planck constant, \( \hbar \), because fluctuation does not come from the quantum effect
3) the precise value should be best-fitted with the experimental data
Sec 3. Statistical Fluctuation Effect

The background-field method gives, at the Gaussian (quadratic, 1-loop) approximation,

\[
e^{-\frac{1}{\alpha} \Gamma[u_n(x); u_{n-1}(x)]} = e^{-\frac{1}{\alpha} \tilde{l}_n[u_n(x)]} \times (\det D)^{-1/2},
\]

\[
D \equiv -\frac{\sigma(=1)}{\tilde{\rho}_0(=1)} \frac{d^2}{dx^2} + \lambda u_n^2 + m^2 + \frac{1}{h} - \frac{du_{n-1}}{dx},
\]

\[
(\det D)^{-1/2} = \exp \left\{ \frac{1}{2} \text{Tr} \int_0^\infty \frac{e^{-\tau D}}{\tau} d\tau + \text{const} \right\}, \quad (13)
\]

([\tau] = [D^{-1}] = L/M.)

Taking the infrared cut-off parameter $\mu \equiv \sqrt{\sigma} / l$ and the ultraviolet cut-off parameter $\Lambda \equiv h^{-1}$ the mass parameter $m^2$ shifts under the influence of the fluctuation.

$$m^2 \rightarrow m^2 + \frac{\alpha}{\sqrt{\pi \epsilon \mu}} \epsilon \lambda = m^2 + \alpha \lambda \sqrt{\frac{l \tilde{\rho}_0}{\pi \sigma \sqrt{\sigma}}}$$  \hspace{1cm} (14)

When the functional (2) (effectively) works well, all effects of the statistical fluctuation reduces to the simple shift of the original parameters. This corresponds to the renormalizability condition in the field theory.
Sec 4. Boltzmann’s Transport Equation: 

a. Step-Wise Approach

The step-wise development equation (8) with \( V_1^n = 0 \), is written as

\[
\frac{1}{h} (u_n(x) - u_{n-1}(x)) = \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!} u_n^3 - u_{n-1} \frac{du_n}{dx} 
\]

or

\[
u_{n-1}(x) = \frac{u_n(x) - h\left\{ \frac{d^2 u_n}{dx^2} - m^2 u_n - \frac{\lambda}{3!} u_n^3 \right\}}{1 - h \frac{du_n}{dx}}. \tag{15}
\]

The equilibrium state \( u^\infty(x) \), after sufficient recursive computation \((n \gg 1)\), satisfies

\[
\frac{d^2 u^\infty}{dx^2} - m^2 u^\infty - \frac{\lambda}{3!} u^\infty^3 - u^\infty \frac{du^\infty}{dx} = 0 \quad , \tag{16}
\]
Sec 4. Boltzmann’s Trans. Eq.: Distribution

The probability for the particle in the interval \( x \sim x + dx \) and \( v \sim v + dv \), at the step \( n \), is given by

\[
\frac{1}{N_n} f_n(x, v) dx dv, \quad f_n(x, v): \text{distribution function} \quad (17)
\]

Then the \( n \)-th distribution \( f_n(x, v) \) and the equilibrium distribution \( f^\infty(x, v) \) can be introduced as

\[
\begin{align*}
  u^\infty(x) &= \frac{1}{\rho^\infty(x)} \int v f^\infty(x, v) dv, \\
  u_n(x) &= \frac{1}{\rho_n(x)} \int v f_n(x, v) dv,
\end{align*}
\]

\[
u_n(x) \to u^\infty(x) \text{ and } f_n(x, v) \to f^\infty(x, v) \text{ as } n \to \infty, (18)
\]

where \( u^\infty(x) \) is the equilibrium velocity distribution. \( \rho_n(x) \) is the particle number density. The continuity equation is given by

\[
\frac{1}{h} \left( \rho_n(x) - \rho_{n-1}(x) \right) + \frac{d}{dx} \left( \rho_n(x) u_n(x) \right) = 0 \quad . \quad (19)
\]
(15) is expressed, in terms of the distribution function, as

\[
\frac{1}{\hbar} \left[ f_n(x + hu_{n-1}(x), v) - f_{n-1}(x, v) \right] = \\
\frac{\partial^2 f_n(x, v)}{\partial x^2} - m^2 f_n(x, v) - \frac{\lambda}{3!} f_n(x, v) u_n(x)^2,
\]

where \( u_n(x) = \frac{1}{\rho_n(x)} \int v f_n(x, v) dv \), \( \lambda \) is a collision integral.

This is Boltzmann’s transport equation. Physical quantities are

Entropy: \( S_n \equiv -k_B \int dv \int dx \ f_n(x, v) \ln f_n(x, v) \)

Total particle #: \( \bar{N}_n = \int dx \ rho_n(x) = \int dx \int dv \ f_n(x, v) \)

Particle # density: \( \rho_n(x) = \int dv \ f_n(x, v), \quad (21) \)
the momentum conservation at each point, \( x \), requires

\[
0 = \tilde{\rho}_n(x) \int dv (v - u_n(x)) f_n(x, v), \quad u_n(x) = \frac{1}{\rho_n(x)} \int dv \ v \ f_n(x, v) \tag{22}
\]

Some distributions are given by

\[
\text{Temperature :} \quad \frac{1}{2} k_B T_n(x) \equiv \frac{1}{\rho_n(x)} \int dv \frac{m_1}{2} (v - u_n(x))^2 f_n(x, v),
\]

\[
\text{Heat Current :} \quad q_n(x) \equiv \int dv \frac{m_1}{2} (v - u_n(x))^3 f_n(x, v),
\]

\[
\text{Pressure :} \quad P_n(x) \equiv m_1 \int dv (v - u_n(x))^2 f_n(x, v), \tag{23}
\]
Figure: The harmonic oscillator with friction.
The $n$-th energy function

$$K_n(x) = V(x) + \frac{\eta}{2h}(x - x_{n-1})^2 + \frac{m}{2h^2}(x - 2x_{n-1} + x_{n-2})^2 + K_0^n,$$

Harmonic oscillator: $V(x) = kx^2/2$, Constant: $K_0^n$,
friction coefficient: $\eta$, mass: $m$  \hspace{1cm} (24)

minimal principle: $\delta K_n = 0$, $x \rightarrow x + \delta x$.

Disc-Time Evol: \[\frac{\delta V}{\delta x} \bigg|_{x=x_n} + \frac{\eta}{h}(x_n - x_{n-1}) + \frac{m}{h^2}(x_n - 2x_{n-1} + x_{n-2}) = 0\]

Diff Eq of HO with Friction: \[\frac{dV(x)}{dx} + \frac{\eta}{dt} + \frac{m}{dt^2} = 0\]

See Fig.2. This is a simple dissipative system.
Fluctuation of Path comes from uncertainty principle of quantum mechanics in this case. (1 degree of freedom. No statistical procedure.)

Classical value $x_n : x = x_n + \sqrt{\hbar} q$ where $\hbar$ is Planck constant.

$$e^{-\frac{1}{\hbar} \hbar \Gamma(x_n;x_{n-1},x_{n-2})} = \int dx \ e^{-\frac{1}{\hbar} \hbar K_n(x)} = \int dq \ e^{-\frac{1}{\hbar} \hbar K_n(x_n+\hbar q)},$$

$$\Gamma_n \equiv \Gamma(x_n;x_{n-1},x_{n-2}) = K_n(x_n) + \frac{\hbar}{2h} \ln(k + \frac{\eta}{h} + \frac{m}{h^2}) \quad (26)$$

The quantum effect does not depend on the step number $n$. 
Sec 5. Trajectory Geometry

\[ x_n - x_{n-1} \equiv \Delta x_n \text{ and } (x_n - 2x_{n-1} + x_{n-2})/\hbar \equiv v_n - v_{n-1} \equiv \Delta v_n \]

We find the **metric** in the energy at \( n \)-step.

\[
K_n(x_n) = V(x_n) + \frac{\eta}{2\hbar}(x_n - x_{n-1})^2 + \frac{m}{2\hbar^2}(x_n - 2x_{n-1} + x_{n-2})^2 + K_n^0 \\
= \frac{1}{\hbar^2} \left\{ V(x_n)(\Delta t)^2 + \frac{\eta\hbar}{2}(\Delta x_n)^2 + \frac{m\hbar^2}{2}(\Delta v_n)^2 \right\} + K_n^0 , (27)
\]

\[
(\Delta s_n)^2 \equiv 2\hbar^2 K_n(x_n) = 2V(x_n'/\sqrt{\eta\hbar})(\Delta t)^2 + (\Delta x_n')^2 + (\Delta v_n')^2 , \\
x_n' \equiv \sqrt{\eta\hbar}x_n , \quad v_n' \equiv \sqrt{m\hbar^2}v_n , (28)
\]

\[ V(x_n'/\sqrt{\eta\hbar}) = (k'/2)x_n'^2 , \quad k' \equiv k/\eta\hbar .\]

Energy line-element \( \Delta s^2 \) in the \((t, x_n', v_n')\) space.

→ the geometrical basis for fixing the statistical ensemble.
Sec 5. Traj. Geom.: Choice of $K_0^n$

Taking the value $K_0^n$ as

$$K_0^n = -V(x_n) - \frac{m}{2h^2}(x_n - 2x_{n-1} + x_{n-2})^2 + V(x_0) + \frac{m}{2h^2}(x_1 - x_0)^2, \quad (29)$$

the graphs of movement and energy change, for various viscosities, are shown in Fig.3-9.

**no friction case**: the oscillator keeps the initial energy (Fig.4).

**viscous case**: the energy changes step by step, and finally reaches a constant *nonzero* value (Fig.6, Fig.7, Fig.9).

**finally-remaining energy** (constant): dissipative one. Physically, the pressure and the temperature of the particle’s ”environment”.

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Velocity-Field Theory, Boltzmann's Transport
Sec 6. Movement & Energy Change of HO

Sec 6. Mov & Ene Change: a. No Friction, Move

Figure: Haromic oscillator with no friction, Movement

Xn Position at n-Step
n(Step No) x h(Step Intvl) x om(Freq)
NO FRICTION
h=0.1, om=0.01, eta/m=0.0
"outfrHO.dat"
Sec 6. Movement & Energy Change of HO

Sec 6. Mov & Ene Change: b. No Friction, Energy

Figure: Harmonic oscillator with no friction, Energy change

![Graph showing harmonic oscillator with no friction and energy change]
Sec 6. Mov & Ene Change: c. Friction, Move

**Figure:** Harmonic oscillator with friction, Movement, (1) Elasticity dominate and (2) Viscosity dominate
Figure: Harmonic oscillator with friction, Energy change, Elasticity dominate

Figure: Harmonic oscillator with friction, Energy change, Viscosity dominate

Viscosity Dominate

h=0.1, om=0.01, eta/m=0.03

"EnefrHO(2).dat"
Figure: Harmonic oscillator with friction, Movement, Resonant

RESONANT CASE

h=0.1, $\omega_0=0.01$, $\eta/m=0.02$

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"outfrHO(3).dat"
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Figure: Harmonic oscillator with friction, Energy change, Resonant
Sec 7. Statistical Ensemble: a. Dirac-type Metric

Take \( \mathbf{N} \) 'copies' of previous model. Consider the 'macro' system: \( \mathbf{N} \gg 1 \). They interact each other and exchange energy, but we assume the interaction is so moderate that every particle obeys the common field equation (25). They form a statistical ensemble caused by the arbitrariness of initial condition, Taking "Dirac-type" metric\[SI,2010Apr\].

\[
(ds^2)_D \equiv 2V(X)dt^2 + dX^2 + dP^2 \quad \text{on-path} \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)dt^2,
\]

\[
L_D = \int_0^\beta ds|_{\text{on-path}} = \int_0^\beta \sqrt{2V(y) + \dot{y}^2 + \dot{w}^2} dt,
\]

\[
d\mu = e^{-\frac{1}{\alpha}L_D}DyDw \quad , \quad e^{-\beta F} = \int \prod_n dy_n dw_n e^{-\frac{1}{\alpha}L_D} \quad , \quad (30)
\]

\( \alpha \): a parameter with dimension of length (\([\alpha]=L\)). See Fig.10.
Figure: The path of line in 3D bulk space (X,P,t).

\[ (y(0), w(0)) \]

\[ (y(\beta), w(\beta)) \]
Taking "**Standard-type**" metric,

\[(ds^2)_S \equiv \frac{1}{dt^2}[(ds^2)_D]^2 - \text{on-path} \rightarrow (2V(y) + \dot{y}^2 + \dot{w}^2)^2 dt^2,\]

\[L_S = \int^\beta_0 ds|_{\text{on-path}} = \int^\beta_0 (2V(y) + \dot{y}^2 + \dot{w}^2) dt,\]

\[d\mu = e^{-\frac{1}{\alpha}L_S} Dy Dw, \quad e^{-\beta F} = \int \prod_n dy_n dw_n e^{-\frac{1}{\alpha}L_S}. \quad (31)\]

**Exactly** the same expression as the free energy expression in the Feynman’s textbook.
Another choice: **surfaces** instead of **lines**.

\[ X^2 + P^2 = r^2(t), \quad 0 \leq t \leq \beta \]  \hspace{1cm} (32)

We respect here the **isotropy** of the 2 dim phase space \((X, P)\). See Fig.11.
Sec 7. Statistical Ensemble

Sec 7. Stat. Ensemble: Path(surface) in 3D Bulk

**Figure:** Two dimensional surface in 3D bulk space \((X,P,t)\).

\[(X=x(t), P=p(t))\]
Induced metric $g_{ij}$ on the surface (32)

$$(ds^2)_D\big|_{\text{on-path}} = 2V(X)dt^2 + dX^2 + dP^2\big|_{\text{on-path}} = \sum_{i,j=1}^{2} g_{ij} dX^i dX^j, \quad (g_{ij}) = \begin{pmatrix} 1 + \frac{2V}{r^2} X^2 & \frac{2V}{r^2} X P \\ \frac{2V}{r^2} P X & 1 + \frac{2V}{r^2} P^2 \end{pmatrix} \quad (33)$$

where $(X^1, X^2) = (X, P)$. Area is given by

$$A = \int \sqrt{\det g_{ij}} d^2X = \int \sqrt{1 + \frac{2V}{r^2}} dXdP \quad , \quad (34)$$
Consider all possible surfaces. Statistical distribution is

\[ e^{-\beta F} = \int_0^\infty d\rho \int r(0) = \rho \prod_t DX(t) DP(t) e^{-\frac{1}{\alpha} A}, \quad (35) \]

We have directly defined the distribution function \( f(t, x, v) \) using geometry of the 3 dim bulk space.