2d $\mathcal{N} = (2, 2)$ SYM on computer

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Dec. 22, 2008 @ RIKEN

- I. Kanamori, H.S., arXiv:0811.2851
It is widely believed that SUperSYmmetry plays an important role in particle physics beyond SM
- hierarchy problem
- superstring theory (gauge/gravity correspondence)

Nonperturbative phenomena?
- color confinement, bound states, spontaneous chiral symmetry breaking, quantum tunneling, ...
- dynamical spontaneous SUSY breaking

Nonperturbative formulation? Lattice?
Nonperturbative Formulation of SUSY Theories

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SUSY on the Lattice?

- Manifest SUSY would be *impossible*, because

\[
\{ Q^A_\alpha, (Q^B_\beta)^\dagger \} = 2\delta^{AB} \sigma^m_{\alpha\beta} P_m
\]

but *no* infinitesimal translations \( P_m \) defined for lattice fields

- However, at least a linear combination \( Q \) of \( Q^A_\alpha \) and \( (Q^B_\beta)^\dagger \)

such that

\[
\{ Q, Q \} = 2Q^2 = 0
\]

could be realized even on the lattice

- Moreover, *if* the continuum action \( S \) can be written as

\[
S = QX
\]

\( Q \)-invariance of \( S \) could be promoted to lattice symmetry!
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SUSY on the Lattice? (cont’d)

(Partial) list of continuum theories with $S = QX$

- $4d \mathcal{N} = 4$ SYM
- $3d \mathcal{N} = 8$ SYM
- $3d \mathcal{N} = 4$ SYM
- $2d \mathcal{N} = (8, 8)$ SYM
- $2d \mathcal{N} = (4, 4)$ SYM
- $2d \mathcal{N} = (2, 2)$ SYM (matter multiplet)
2d $\mathcal{N} = (2, 2)$ Supersymmetric Yang-Mills Theory

- Dimensional reduction of 4d $\mathcal{N} = 1$ SYM from 4 to 2

$$S_{\text{continuum}} = \frac{1}{g^2} \int d^2x \, \text{tr} \left\{ \frac{1}{2} F_{MN} F_{MN} + \Psi^T C \Gamma_M D_M \Psi + \tilde{H}^2 \right\},$$

where $M, N = 0, 1, 2, 3$, $(\mu, \nu = 0, 1)$ and

$$F_{01} = \partial_0 A_1 - \partial_1 A_0 + i[A_0, A_1] \equiv \Phi / 2$$
$$F_{02} = \partial_0 A_2 + i[A_0, A_2] \equiv D_0 A_2,$$
$$F_{23} = i[A_2, A_3],$$

$$\phi \equiv A_2 + i A_3, \quad \bar{\phi} = A_2 - i A_3$$
$$\tilde{H} = H - i \Phi / 2$$

- We will use a particular representation

$$\Gamma_0 = \begin{pmatrix} -i \sigma_1 & 0 \\ 0 & i \sigma_1 \end{pmatrix}, \quad \Gamma_1 = \begin{pmatrix} i \sigma_3 & 0 \\ 0 & -i \sigma_3 \end{pmatrix}, \quad \Gamma_2 = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, \quad \Gamma_3 = C = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}$$

$$\Psi^T \equiv (\psi_0, \psi_1, \chi, \eta / 2)$$
Supersymmetry

\[ \delta A_M = i \epsilon^T C \Gamma_M \psi, \quad \delta \psi = \frac{i}{2} F_{MN} \Gamma_M \Gamma_N \epsilon + i \tilde{H} \Gamma_5 \epsilon \]

\[ \delta \tilde{H} = -i \epsilon^T C \Gamma_5 \Gamma_M D_M \psi \]

Setting

\[ \epsilon^T \equiv -(\epsilon^{(0)}, \epsilon^{(1)}, \tilde{\epsilon}, \epsilon), \quad \delta \equiv \epsilon^{(0)} Q^{(0)} + \epsilon^{(1)} Q^{(1)} + \tilde{\epsilon} \tilde{Q} + \epsilon Q \]

we have

\[ QA_{\mu} = \psi_{\mu} \quad Q \psi_{\mu} = i D_{\mu} \phi \]

\[ Q \phi = 0 \]

\[ Q \chi = H \quad QH = [\phi, \chi] \]

\[ Q \bar{\phi} = \eta \quad Q \eta = [\phi, \bar{\phi}] \]
We see

\[ Q^2 = \delta \phi, \]

where \( \delta \phi \) is an infinitesimal gauge transformation by the parameter \( \phi \), and thus

\[ Q^2 = 0 \] on gauge invariant combinations

The action is moreover \( Q \)-exact

\[
S_{\text{continuum}} = Q \frac{1}{g^2} \int d^2 x \, \text{tr} \left\{ \frac{1}{4} \eta [\phi, \bar{\phi}] - i \chi \Phi + \chi H - i \psi_\mu D_\mu \bar{\phi} \right\}
\]
Global symmetries

- $U(1)_A$ symmetry ($\iff$ 2-3 plane rotation in 4d)
  \[ \psi \rightarrow \exp\{\alpha \Gamma_2 \Gamma_3\} \psi, \quad \phi \rightarrow \exp\{2i\alpha\} \phi, \quad \overline{\phi} \rightarrow \exp\{-2i\alpha\} \overline{\phi} \]

- $U(1)_V$ symmetry ($\iff$ $U(1)_R$ symmetry in 4d SYM)
  \[ \psi \rightarrow \exp\{i\alpha \Gamma_5\} \psi \]

- a $Z_2$ symmetry ($\iff$ reflection in 2-direction in 4d)
  \[ \psi \rightarrow i\Gamma_2 \psi, \quad \phi \rightarrow -\overline{\phi}, \quad \overline{\phi} \rightarrow -\phi \]
This is a “toy” field theory, but no obvious low-energy description
In 2d, no SSB of bosonic global symmetries (no chiral lagrangian)
no controllable parameter except $N_c$ (large $N_c$ limit is non-trivial) gauge coupling $g$ simply provides a mass scale, like $\Lambda_{QCD}$
flat directions $[\phi, \bar{\phi}] = 0$, but (probably) no vacuum modulus in 2d, Witten index is unknown (SSUSYB?, Hori-Tong)
LATTICE FORMULATION
Recent Developments in Lattice Formulation

- lattice formulations with exact fermionic symmetries $Q$ of various gauge theories
  - Cohen, Kaplan, Katz, Ünsal, Endres
  - Sugino
  - Catterall
  - D’Adda, Kanamori, Kawamoto, Nagata
  - Damgaard, Matsuura
  - Kikukawa, Sugino

- cf. 2d $\mathcal{N} = (2, 2)$ WZ model, Sakai-Sakamoto (’83 !)
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Sugino’s Lattice Formulation of 2d $\mathcal{N} = (2, 2)$ SYM

- 2d Lattice

$$\Lambda = \{ x \in a\mathbb{Z}^2 \mid 0 \leq x_0 < \beta, \; 0 \leq x_1 < L \}$$

- Lattice $Q$-transformation

\[
QU(x, \mu) = i\psi_\mu(x)U(x, \mu) \quad \text{Link variables}
\]
\[
Q\psi_\mu(x) = i\psi_\mu(x)\psi_\mu(x) - i\left(\phi(x) - U(x, \mu)\phi(x + a\hat{\mu})U(x, \mu)^{-1}\right)
\]
\[
Q\phi(x) = 0
\]
\[
Q\chi(x) = H(x) \quad QH(x) = [\phi(x), \chi(x)]
\]
\[
Q\overline{\phi}(x) = \eta(x) \quad Q\eta(x) = [\phi(x), \overline{\phi}(x)]
\]

is nilpotent on the lattice

$$Q^2 = \delta_\phi \simeq 0$$
Lattice Action

- Imitating the continuum action, we adopt

\[
S = Q \frac{1}{a^2 g^2} \sum_{x \in \Lambda} \text{tr}\left\{ \frac{1}{4} \eta(x)[\phi(x), \bar{\phi}(x)] - i \chi(x) \hat{\Phi}(x) \right\} \\
+ \chi(x) H(x) - i \sum_{\mu=0}^{1} \psi_\mu(x) \left( U(x, \mu) \bar{\phi}(x + a\hat{\mu}) U(x, \mu)^{-1} - \bar{\phi}(x) \right) \right\},
\]

where the lattice field strength \( \hat{\Phi} \) is

\[
\hat{\Phi}(x) \simeq -i U(x, 0) U(x + a\hat{0}, 1) U(x + a\hat{1}, 0)^{-1} U(x, 1)^{-1} + \text{h.c.}
\]

with some important modification
The above lattice formulation possesses a manifest lattice symmetry $Q$ (and $U(1)_A$)

But how about other $Q^{(0)}$, $Q^{(1)}$, $\tilde{Q}$? (and $U(1)_V$, $Z_2$)?

The best thing we can hope is that these are restored in the continuum limit $a \to 0$

Is this really the case?

This is our main objective here!

In what follows, the gauge group is $SU(N_c)$: Our numerical results are for $SU(2)$ only
RESTORATION OF SUSY?
How SUSY (Other than Q) Is Restored?

- Perturbative argument (Kaplan et al.):
  - SUSY breaking (owing to the lattice regularization) can be removed by \textit{local} counterterms in the continuum limit
  - Possible local term in the effective action in the $\ell$-loop

$$a^{p+2\ell-4}(g^2)^{\ell-1}\int d^2 x \varphi^a \partial^b \psi^{2c}, \quad p \equiv a + b + 3c \geq 0$$

(up to some powers of $\ln a$)

- Operators with $p + 2\ell - 4 \leq 0$ survive in the continuum limit $a \to 0$. It is enough to consider $\ell = 0, 1, 2$

- For $\ell = 0$, the continuum limit coincides with the target theory
How SUSY (Other than $Q$) Is Restored? (cont’d)

- For $\ell = 1$, only $p = 0, 1, 2$ are dangerous
  - $p = 0 \Rightarrow$ identity operator, no dynamical effect
  - $p = 1 \Rightarrow \varphi$, but $\text{tr}\{\varphi\} \equiv 0$
  - $p = 2 \Rightarrow \varphi \varphi \leftarrow$ prohibited by gauge, $U(1)_A$, $Q$ symmetries

One-loop scalar self-energy

- Each of these is logarithmically divergent
- If SUSY, the sum vanishes at zero external momentum
- For $\ell = 2$, only $p = 0$ is marginal (i.e., the identity)
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Motivation for Direct Confirmation

- Although the above argument is highly plausible, it is not completely free from question (at least for me)
- There is a “hidden” dimensionful parameter \( L \), the physical size of the system. If this is relevant to the argument, \[ L^{p+2\ell-4}(g^2)^{(\ell-1)} \int d^2x \phi^a \partial^b \psi^c, \quad p \equiv a + b + 3c, \]
  for example, does survive in all loops
- Another concern: Is the non-linear symmetry \( Q \) realized as it stands in the 1PI effective action? (Probably OK in 1 loop level)
- In any case, direct (nonperturbative) confirmation of SUSY restoration by numerical means is certainly desirable
But How?

- It is not so straightforward
  - We cannot directly measure $\langle \phi(x)\bar{\phi}(0) \rangle$, because it is not gauge invariant
  - We must consider something gauge invariant (that is necessarily composite field)
  - The above argument however refers to the effective action of elementary fields
- (After many trial and fails) we finally decided to observe the conservation law of the supercurrent

$$s_\mu \equiv -\frac{1}{g^2} C \Gamma_M \Gamma_N \Gamma_\mu \text{ tr} \left\{ F_{MN} \Psi \right\}$$

- The 4 spinor components of $s_\mu$ correspond to

  $$(s_\mu)_1 \rightarrow Q^{(0)}, \quad (s_\mu)_2 \rightarrow Q^{(1)}, \quad (s_\mu)_3 \rightarrow \tilde{Q}, \quad (s_\mu)_4 \rightarrow Q$$
SUSY Ward-Takahashi (WT) identities

- More definitely, we take the fermionic operator

\[ f_\mu \equiv \frac{1}{g^2} \Gamma_\mu \left( \Gamma_2 \text{tr}\{A_2 \psi\} + \Gamma_3 \text{tr}\{A_3 \psi\} \right) \]

and examine SUSY Ward-Takahashi identities

\[
\begin{align*}
\partial_\mu \langle (s_\mu)_1(x)(f_\nu)_1(0) \rangle &= -i\delta^2(x) \left\langle Q^{(0)}(f_\nu)_1(0) \right\rangle \\
\partial_\mu \langle (s_\mu)_2(x)(f_\nu)_2(0) \rangle &= -i\delta^2(x) \left\langle Q^{(1)}(f_\nu)_2(0) \right\rangle \\
\partial_\mu \langle (s_\mu)_3(x)(f_\nu)_3(0) \rangle &= -i\delta^2(x) \left\langle \tilde{Q}(f_\nu)_3(0) \right\rangle \\
\partial_\mu \langle (s_\mu)_4(x)(f_\nu)_4(0) \rangle &= -i\delta^2(x) \left\langle Q(f_\nu)_4(0) \right\rangle 
\end{align*}
\]

- NB: These should hold irrespective of boundary conditions
Lattice Artifacts in WT Identities

- Composite operator $s_\mu(x)$, for example, has $O(a)$ discretization ambiguity
- We must be sure that this ambiguity, when combined with UV divergence arising from the composite operator, does not modify the WT identities

General rule

UV finite functions are safe
Possible UV Divergences in WT Identities

Supercurrent itself is UV finite in 2d

\[ \propto C \Gamma_\mu \text{tr}\{\psi\} = 0 \]

\[ = \text{finite! and no mixing} \]

The one-loop scalar self-energy in sub-diagrams

\[ = \text{finite for SUSY field content} \]
Possible UV Divergences in WT Identities (cont’d)

- Lowest 1-loop diagram

\[ x \otimes \quad 0 \quad = \quad \text{UV diverging if } x = 0 \]

- In fact, the divergence at \( x = 0 \) may modify the WT identities, as

\[
\partial_\mu \left\langle (s_\mu)_1(x)(f_\nu)_1(0) \right\rangle \\
= -i\delta^2(x) \left\langle Q^{(0)}(f_\nu)_1(0) \right\rangle + \frac{1}{4\pi} (N_c^2 - 1)(c - 1) \partial_\nu \delta^2(x)
\]

- Conclusion:
  - \( \langle s_\mu(x)f_\nu(0) \rangle \) is UV convergent for \( x \neq 0 \)
  - We should examine the WT identities for \( x \neq 0 \)!
We need to introduce a scalar mass term

\[ S_{\text{mass}} = \frac{\mu^2}{g^2} \int d^2 x \, \text{tr} \{ \bar{\phi} \phi \} \implies \frac{\mu^2}{g^2} \sum_{x \in \Lambda} \text{tr} \{ \bar{\phi}(x) \phi(x) \} \]

This (softly) breaks SUSY and the WT identifies become

\[ \partial_\mu \langle (s_\mu)_i(x)(f_\nu)_i(0) \rangle - \frac{\mu^2}{g^2} \langle (f)_i(x)(f_\nu)_i(0) \rangle = 0 \quad \text{no sum over } i, \]

where

\[ f \equiv -2C (\Gamma_2 \text{tr}\{A_2 \psi\} + \Gamma_3 \text{tr}\{A_3 \psi\}) \]

The reason for \( S_{\text{mass}} \) will be elucidated later.
MONTE CARLO RESULTS
Before Going to That...

• Simulation with dynamical fermions (tough task. . .)
  • Partition function
    \[ Z = \mathcal{N} \int d\mu \, e^{-S} = \mathcal{N}' \int d\mu_B \, e^{-S_B} \text{Pf}\{D\} \]
  
• Pseudo-fermion
    \[ \text{Pf}\{D\} = e^{i \text{Arg Pf}\{D\}} (\det D^\dagger D)^{1/4} \]
    \[ = e^{i \text{Arg Pf}\{D\}} \int d\varphi \, d\bar{\varphi} \, e^{-\bar{\varphi}(D^\dagger D)^{-1/4}\varphi} \]

• Rational approximation (RHMC ’04)
    \[ x^{-1/4} \simeq \alpha_0 + \sum_{i=1}^{N} \frac{\alpha_i}{x + \beta_i} \]
    Remez algorithm, multi-shift solver, . . .
Simulation Parameters \((\sim 20,000 \text{ CPU} \cdot \text{hour})\)

- 2d rectangular lattice

\[ \Lambda \equiv \left\{ x \in aZ^2 \mid 0 \leq x_0 < 2L, \ 0 \leq x_1 < L \right\}, \quad Lg = 1.414 \]

- Lattice sizes

\[ 12 \times 6, \ 16 \times 8, \ 20 \times 10 \]

- Lattice spacings

\[ ag = 0.2357, \ 0.1768, \ 0.1414 \]

- Scalar masses

\[ \mu^2/g^2 = 0.04, \ 0.25, \ 0.49, \ 1.0, \ 1.69 \]

- Number of uncorrelated configurations

\[ 800–1800 \]
Following 4 \( (i = 1, 2, 3, 4) \) coincide in the continuum theory

\[
\langle (s_0)_i(x)(f_0)_i(0) \rangle / g^2 \quad i = 1, 2, 3, 4
\]

owing to the \( U(1)_V \) and the \( Z_2 \) symmetries

Figure: 12 x 6, \( ag = 0.2357 \), \( \mu^2 / g^2 = 1.0 \). Along the line \( x_1 = L/2 \). \( i = 1 \) (+), \( i = 2 \) (x), \( i = 3 \) (□), \( i = 4 \) (■)
SUSY WT identity (aPBC)

- The left-hand side of the WT identity

\[ \partial_\mu \langle (s_\mu)_1(x)(f_0)_1(0) \rangle / g^3 - \frac{\mu^2}{g^2} \langle (f)_1(x)(f_0)_1(0) \rangle / g^3 \]

Figure: 20 \times 10, \( ag = 0.1414, \mu^2/g^2 = 1.0 \)
SUSY WT identities (PCSC relation) (aPBC)

The ratio

\[ \frac{\partial \mu}{\mu} \langle (s_\mu)_1(x)(f_0)_1(0) \rangle \left( \Rightarrow \frac{\mu^2}{g^2} \right) \]

Figure: $\mu^2/g^2 = 1.0$. Along the line $x_1 = L/2$. $ag = 0.2357 (+)$, $ag = 0.1768 (\times)$, $ag = 0.1414 (\Box)$
\[ \chi^2 \text{-fit for the Plateau Region (aPBC)} \]

- The ratio

\[ \frac{\partial_{\mu} \langle (s_\mu)_1(x)(f_0)_1(0) \rangle}{\langle (f)_1(x)(f_0)_1(0) \rangle} \]

Figure: \( \mu^2/g^2 = 0.04 (+), \mu^2/g^2 = 0.25 (\times), \mu^2/g^2 = 0.49 (\square), \mu^2/g^2 = 1.0 (\blacksquare), \mu^2/g^2 = 1.69 (\bigcirc) \)
We Observe PCSC! (aPBC)

- The continuum limit of the ratio

\[ \frac{\partial \mu \langle (s_\mu)_i(x)(f_0)_i(0) \rangle}{\langle (f)_i(x)(f_0)_i(0) \rangle} \left( \Rightarrow \frac{\mu^2}{g^2} \right) \]

Figure: \( i = 1 (+), i = 2 (\times), i = 3 (\square), i = 4 (\blacksquare) \)
Summary at This Stage

- For $\mu^2/g^2 > 0$, with aPBC, PCSC is observed in the continuum limit
  - Breaking of SUSY (and other symmetries) owing to lattice regularization in fact disappears
  - The target ($2d \mathcal{N} = (2, 2)$ SYM with SUSY breaking scalar mass) seems to be obtained in the continuum limit

- This is the first example in lattice gauge theory in which the restoration of SUSY was clearly confirmed!
How about the Periodic BC (PBC) Case?

- Following 4 ($i = 1, 2, 3, 4$) coincide in the continuum theory

$$\langle (s_0)_i(x)(f_0)_i(0) \rangle / g^2 \quad i = 1, 2, 3, 4$$

**Figure:** PBC. $12 \times 6$, $ag = 0.2357$, $\mu^2 / g^2 = 1.0$. Along the line $x_1 = L/2$. $i = 1 (+)$, $i = 2 (\times)$, $i = 3 (\Box)$, $i = 4 (\blacksquare)$
How about the Periodic BC (PBC) Case? (cont’d)

For $\mu^2/g^2 > 0$, PBC case is the subject of further study
Without the Scalar Mass? $\mu^2/g^2 = 0$

- We could not achieve the thermalization...  

**Figure:** Monte Carlo evolution of $a^2 \text{tr}\{\bar{\phi}\phi\}$ with aPBC. $12 \times 12$, $ag = 0.1179$
Without the Scalar Mass? $\mu^2 / g^2 = 0$ (cont’d)

- and, generally, scalar fields acquire **very large** value along the flat directions

$$\phi, \bar{\phi} \gtrsim \frac{1}{a}$$

**Figure:** Monte Carlo evolution of $a^2 \text{tr}\{\phi \bar{\phi}\}$ with aPBC. $18 \times 12$, $ag = 0.1179$
Very Large Value of Scalars may Cause Trouble

Such very large value could amplify $O(a)$ quantities to $O(1)$,

$$a\phi \sim O(1), \text{ instead of } O(a)$$

and could ruin the power counting. For example, the combination

$$Q(a \text{tr}\{\bar{\phi}\psi_\mu\}) = a \text{tr}\{\eta\psi_\mu\} + a \text{tr}\{\bar{\phi}iD_\mu\phi\},$$

might be $O(1)$. This is invariant under gauge, $U(1)_A$, $Q$ transformations, but is not invariant under $Q^{(0)}$, $Q^{(1)}$, $\tilde{Q}^{(0)}$
SOME PHYSICS

$2d \mathcal{N} = (2, 2)$ SYM with (small) SUSY breaking scalar mass
Correlation Functions with Power-like Behavior

- This system has no mass gap (Witten) ⇐ ’t Hooft anomaly matching condition
- More definitely, on $\mathbb{R}^2$ (Fukaya, Kanamori, H.S., Hayakawa, Takimi)

\[
- \frac{i}{2} \langle j_\mu(x) \epsilon_{\nu\rho} j_{5\rho}(0) \rangle = \frac{1}{4\pi} (N_c^2 - 1) \int \frac{d^2p}{(2\pi)^2} e^{ipx} \left\{ -\frac{1}{p^2} (p_\mu p_\nu - \epsilon_{\mu\rho} \epsilon_{\nu\sigma} p_\rho p_\sigma) + \tilde{c} \delta_{\mu\nu} \right\}
= \frac{1}{4\pi} (N_c^2 - 1) \left\{ \frac{1}{\pi} \frac{1}{(x^2)^2} (x_\mu x_\nu - \epsilon_{\mu\rho} \epsilon_{\nu\sigma} x_\rho x_\sigma) + \tilde{c} \delta_{\mu\nu} \delta^2(x) \right\} ,
\]

where $j_\mu$ and $j_{5\rho}$ are $U(1)_V$ and $U(1)_A$ currents, respectively ($\tilde{c}$ is ambiguity in operator definition)
Can We See This Massless Bosonic State?

- Power-like behavior on $\mathbb{R}^2$

\[-\frac{i}{2} \langle j_0(x) \epsilon_{0\rho} j_{5\rho}(0) \rangle = \frac{3}{4\pi^2} \frac{1}{(x_0)^2}, \quad \text{for } N_c = 2 \text{ along } x_1 = 0\]

- If so, the $U(1)_V$ symmetry is restored

Figure: IV: $\mu^2 / g^2 = 0.25$. 20 × 16, $ag = 0.1414$. aPBC
Almost Degenerated Fermionic State

- SUSY WT identity

\[ \langle (s_0)_i(x)(f_0)_i(0) \rangle = -\frac{i}{2} \langle j_0(x)\epsilon_{0\rho}\bar{f}_{5\rho}(0) \rangle \]

\[ O(g^2); \text{ no massless singularity} \]

\[ -\left\langle j_0(x)\epsilon_{0\rho} \frac{1}{g^2} \text{tr} \{ A_3(0)F_{\rho2}(0) - A_2(0)F_{\rho3}(0) \} \right\rangle \]

(This follows from \( \delta\langle j_\mu(x)f_\nu^T(0) \rangle = 0 \), neglecting \( \mu^2 \) and aPBC)
Static Potential between Charges in Fund. Reps.

- Static potential between charges in the fundamental representation $V(R)/g$

$$- \ln \{ W(T, R) \} = V(R)T + c(R)$$

This confining behavior appears distinct with a conjecture in the ’90s by Armoni, Frishman and Sonnenschein.
**Static Potential (cont’d)**

- Static potential between charges in the fundamental representation $V(R)/g$ for various scalar masses

![Graph showing static potential between charges](image)

- The broken line: Gross, Klebanov, Matytsin, Smilga for $\mu^2/g^2 \to \infty$
SUMMARY
Summary

- SUSY breaking owing to lattice regularization certainly disappears in the continuum limit (this is the first firm demonstration!)

- It appears that $2d \mathcal{N} = (2, 2)$ SYM with a (small) SUSY breaking scalar mass is realized in the machine

- We illustrated some physical application

Outlook

- Physical questions: Further study of the static potential, spectrum of excited states, etc.
- SUSY theory by $\mu^2/g^2 \to 0$ limit
- Spontaneous SUSY breaking in this limit (Kanamori, Sugino, H.S.)
- Issue of the vacuum modulus
- Other theories, other formulation on the basis of similar idea.