Doubled D-branes in generalized geometry

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Outline

Motivation

Doubled D-branes in doubled geometry

Dirac structures and Courant algebroids in Generalized geometry

Doubled D-branes as Courant algebroids

Conclusion
Geometrically, T-duality arises from compactifying a theory on a circle with radius $R$, and such a theory describes the same physics as a theory compactified on a circle with radius $1/R$ with the winding mode and momentum mode exchanged.
Doubled geometry:

A doubled torus with coordinates \((x, \tilde{x})\)

Physical space

T dual

Generalized geometry:

\[(TE \oplus T^*E)_{S^1} \xrightarrow{\mathbb{R}} (T\hat{E} \oplus T^*\hat{E})_{\hat{S}^1} \]

\(\pi \quad \tilde{\pi} \quad TM\)
In doubled formalism, half of the components obey the Dirichlet boundary condition while the other half obey the Neumann boundary condition. I.e. A D-brane and it’s T-dual can be described simultaneously in doubled geometry–A doubled D-brane.

In generalized geometry, D-branes are described by objects called Dirac structures.

We conjecture that a doubled D-brane in doubled geometry is equivalent to a Courant algebroid in generalized geometry.
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In doubled geometry (Hull et al.), the key component is a \(2n \times 2n\)-matrix \(H\) called a generalized metric which transform as an \(O(n, n)\)-tensor:

\[
H = \begin{pmatrix}
G - BG^{-1}B & BG^{-1} \\
-G^{-1}B & G^{-1}
\end{pmatrix},
\]

and also an \(O(n, n)\)-invariant constant matrix \(L\) conveniently chosen as

\[
L = \begin{pmatrix}
0 & \mathbb{I}_{n \times n} \\
\mathbb{I}_{n \times n} & 0
\end{pmatrix}.
\]

Doubled coordinate is defined by \(\tilde{X}^I = (X^i, \tilde{X}_i)\) where \(\tilde{X}^i = (\tilde{X}_a, X^\nu)\).
Neumann boundary condition:

\[ \partial_1 X^a |_{\partial \Sigma} = 0. \]

Dirichlet boundary condition:

\[ \delta X^\mu |_{\partial \Sigma} = 0. \]

T-duality exchanges Dirichlet and Neumann boundary conditions.
Doubled D-brane

On the doubled space, we can define the corresponding projectors (Albertsson, Kimura and Reid-Edwards, 2009):

- Dirichlet projector: $\Pi$,
  Neumann projectors: $\bar{\Pi}$,
- Projectors by definition: $\Pi + \bar{\Pi} = \mathbb{I}$,
- The projectors need to satisfy the following conditions
  1. Normal condition: $\Pi^2 = \Pi$, and $\bar{\Pi}^2 = \bar{\Pi}$.
  2. Orthoganality condition: $\bar{\Pi} \mathbb{I} \Pi = 0$.
  3. Integrability condition: $\bar{\Pi}^K \bar{\Pi}^L J \partial_{[K} \bar{\Pi}^M_{L]} = 0$. 
The Dirichlet projector is used to express the Dirichlet boundary conditions in a covariant way:

$$\Pi^t \partial_0 X|_{\partial \Sigma} = 0,$$

While the Neumann projectors give rise to Neumann boundary conditions:

$$\tilde{\Pi}^t \mathbb{H} \partial_1 X|_{\partial \Sigma} = 0.$$

Here the self-duality condition is used to eliminate half of the degrees of freedom:

$$\partial_{\alpha} X^I = \epsilon_{\alpha \beta} L^{IJ} \mathbb{H}_{JK} \partial_{\beta} X^K.$$
Let $h \in O(n, n; \mathbb{Z})$.

The doubled coordinate, generalized metric and Dirichlet/Neumann projectors transform under T-duality via

$$
\begin{align*}
H_{IJ} &\mapsto \tilde{H}_{IJ} = (h^{-1}Hh)_{IJ}, \\
X^I &\mapsto \tilde{X}^I = (h^{-1})^I_J X^J, \\
\Pi &\mapsto \tilde{\Pi} = h^{-1}\Pi h, \\
\bar{\Pi} &\mapsto \tilde{\bar{\Pi}} = h^{-1}\bar{\Pi} h.
\end{align*}
$$
Consider a 2-dimensional model with

\[
\mathbb{H} = \begin{pmatrix} R^2 & 0 \\ 0 & R^{-2} \end{pmatrix}, \quad \mathbb{L} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}
\]

and double coordinates \( X = (x, \tilde{x})^t \).

The possible allowed Dirichlet projectors are

\[
\Pi^{(1)} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad \Pi^{(2)} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} = \bar{\Pi}^{(1)}.
\]

The self-duality condition \( \partial_\alpha x^I = \epsilon_\alpha^\beta \mathbb{L}^{IJ} \mathbb{H}^{JK} \partial^\beta \tilde{x}^K \) gives

\[
\partial_0 x = R^{-2} \partial_1 \tilde{x}, \quad \partial_0 \tilde{x} = R^2 \partial_1 x.
\]
Case I: Solving the boundary conditions along with the self-duality condition for $\Pi^{(1)}$, we find

\[ \partial_0 x = 0, \quad \partial_1 \tilde{x} = 0, \]
Case I: Solving the boundary conditions along with the self-duality condition for $\Pi^{(1)}$, we find

$$\partial_0 x = 0, \quad \partial_1 \tilde{x} = 0,$$
Case II: Solving the boundary conditions along with the self-duality condition for $\Pi^{(2)}$, we find

$$\partial_1 x = 0, \quad \partial_0 \tilde{x} = 0,$$
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Generalized geometry was first developed by Hitchin to unify both symplectic geometry and complex geometry. Generalized geometry has been of great interest due to emerging connections with areas of mathematical physics, for instance:

- Relation with string theory, ex. $B$-field symmetries,
- Connection with Mirror symmetry
- Adaptation of T-duality to generalized geometry

Reference:
Hitchin math.DG/0209099
Gualtieri math.DG/0401221
Cavalcanti math.DG/0501406
Let $M$ be a manifold. $TM \oplus T^*M$ is called the Generalized tangent space of $M$.
There are two natural operations on $TM \oplus T^*M$:
(1) $TM \oplus T^*M$ has a natural symmetric non-degenerate bilinear form defined by
\[ \langle X + \xi, Y + \eta \rangle = \frac{1}{2} (\iota_Y \xi + \iota_X \eta) \]
where $X, Y \in \Gamma(TM)$, and $\xi, \eta \in \Gamma(T^*M)$.
(2) Courant bracket:
The canonical bracket originally introduced by Courant is:
\[ \llbracket x + \xi, y + \eta \rrbracket = [x, y] + \mathcal{L}_x \eta - \mathcal{L}_y \xi + \frac{1}{2} d(\iota_y \xi - \iota_x \eta). \]
Properties of the Courant bracket

- A Courant bracket is skew-symmetric
- It does not satisfy the Jacobi-identity:
  Let \( A, B, C \in \Gamma(TM) \oplus \Gamma(T^*M) \), and \( f \in C^\infty M \), define
  \[
  \text{Jac}(A, B, C) = [[[A, B]], C] + cycl = dNij(A, B, C),
  \]
  here
  \[
  Nij(A, B, C) = \frac{1}{3}(\langle [A, B], C \rangle + cycl).
  \]
- It does not in general satisfy the Leibnitz rule: Let \( \rho : TM \oplus T^*M \to TM \) be an anchor, then
  \[
  [A, fB] = f[A, B] + (\rho(A)f)B - \langle A, B \rangle df.
  \]
- \( \exists \) an automorphism defined by \( B \in \wedge^2 T^*M, dB = 0. \)
Let $B$ be a smooth 2-form which maps $TM \to T^*M$ via the interior product $x \mapsto \iota_x B$. There is an infinitesimal transformation given by

$$e^B = \begin{pmatrix} 1 & 0 \\ B & 1 \end{pmatrix} : x + \xi \mapsto x + \xi + \iota_x B$$

$\beta$-transform is another important symmetry given by $\beta \in \Lambda^2(TM)$:

$$e^\beta = \begin{pmatrix} 1 & \beta \\ 0 & 1 \end{pmatrix} : x + \xi \mapsto x + \xi + \iota_\xi \beta.$$

$e^B$ and $e^\beta$ are both elements of the special orthogonal group $SO(TM \oplus T^*M) \cong SO(n, n)$ which preserves the natural pairing $\langle , \rangle$. 
A **Courant algebroid** over a manifold $M$ is a vector bundle $E \rightarrow M$ equipped with
- a nondegenerate symmetric bilinear form $\langle \cdot, \cdot \rangle$,
- a Courant bracket,
- an anchor $\rho : E \rightarrow TM$.

Example:
$TE \oplus T^*E$ with the natural pairing, trivial anchor map and the original Courant bracket is a Courant algebroid.
Dirac structure

$L \in TM \oplus T^*M$ is a Dirac structure if (1) $L$ is maximally isotropic, (2) $L$ is involutive, i.e. $[[\Gamma(L), \Gamma(L)], \Gamma(L)] \in \Gamma(L)$.

Examples of Dirac structures are:

- $TM$ and $T^*M$.
- $\bigoplus_p T_p \bigoplus_q T_q^*$, where $p + q = \dim(M)$.
- $e^B(TM)$.
- $e^\beta(T^*M)$.

In generalized geometry, D-branes are described by Dirac structures.
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Doubled D-branes in $TM \oplus T^*M$
Procedures: (1) The Neumann boundary condition

\[ \bar{\Pi}^t \partial_1 X |_{\partial \Sigma} = 0 \]

gives us a basis on the tangent part of the doubled space, i.e.

\[ X := x_l (\bar{\Pi}^t \partial X)^l. \]

while the Dirichlet boundary condition

\[ \Pi^t \partial_0 X |_{\partial \Sigma} = 0 \]

gives us a basis on the cotangent part of the doubled space, i.e.

\[ \Xi := \xi^l (\Pi^t dX)_l. \]

(2) \( \{ X + \Xi \} \) requiring

\[ \partial \tilde{x}_i \mapsto dx_i, \quad d\tilde{x}_i \mapsto \partial x_i \]

becomes a basis of \( TM \oplus T^*M \).
Let us consider a 4-dimensional example. We start with flat metric and constant $B$-field, i.e.

$$G = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & b \\ -b & 0 \end{pmatrix},$$

it follows that the generalized metric is given by

$$\mathcal{H} = \begin{pmatrix} 1 + b^2 & 0 & 0 & b \\ 0 & 1 + b^2 & -b & 0 \\ 0 & -b & 1 & 0 \\ b & 0 & 0 & 1 \end{pmatrix}.$$
A 4-dimensional Example

<table>
<thead>
<tr>
<th>Solutions</th>
<th>D-brane</th>
<th>T-dual</th>
<th>Generalized space</th>
</tr>
</thead>
<tbody>
<tr>
<td>$\Pi_1$</td>
<td>$D_1 ({X, Y})$</td>
<td>$\tilde{\Pi}_1$</td>
<td>$e^B(TM) \oplus e^\beta(T^*M)$</td>
</tr>
<tr>
<td>$\Pi_2$</td>
<td>$D_2$</td>
<td>$\Pi_3$</td>
<td>$TM \oplus e^\beta(T^*M)$</td>
</tr>
<tr>
<td>$\Pi_3$</td>
<td>$D_0$</td>
<td>$\Pi_2$</td>
<td>$e^B(TM) \oplus T^*M$</td>
</tr>
<tr>
<td>$\Pi_4$</td>
<td>$D_2$</td>
<td>$\Pi_5$</td>
<td>$TM \oplus e^\beta'(T^*M)$</td>
</tr>
<tr>
<td>$\Pi_5$</td>
<td>$D_0$</td>
<td>$\Pi_4$</td>
<td>$e^{B'}(TM) \oplus T^*M$</td>
</tr>
<tr>
<td>$\Pi_6$</td>
<td>$D_1 (X)$</td>
<td>$\Pi_7$</td>
<td>$e^B(T_Y) \oplus T_X \oplus e^\beta(T^<em>_X) \oplus T^</em>_Y$</td>
</tr>
<tr>
<td>$\Pi_7$</td>
<td>$D_1 (Y)$</td>
<td>$\Pi_6$</td>
<td>$T_X \oplus e^\beta(T^<em>_Y) \oplus T^</em>_X \oplus e^B(T_X) \oplus T_Y$</td>
</tr>
<tr>
<td>$\Pi_8$</td>
<td>$D_1 ({X, Y})$</td>
<td>$\tilde{\Pi}_1$</td>
<td>$e^\theta(TM) \oplus e^\Theta(T^*M)$</td>
</tr>
</tbody>
</table>
Similarly we consider a 6-dimensional example. We start with flat metric and constant $B$-field, i.e.

\[
G = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad B = \begin{pmatrix}
0 & bz & -by \\
-bz & 0 & bx \\
by & -bx & 0
\end{pmatrix},
\]

it follows that the generalized metric is given by

\[
\mathbb{H} = \\
\begin{pmatrix}
1 + b^2 y^2 & -b^2 xy & -b^2 xz & 0 & bz & -by \\
-b^2 xy & 1 + b^2 z^2 + b^2 x^2 & -b^2 yz & -bz & 0 & bx \\
-b^2 xz & -b^2 yz & 1 + b^2 x^2 + b^2 y^2 & by & -bx & 0 \\
0 & -bz & by & 1 & 0 & 0 \\
bz & 0 & -bx & 0 & 1 & 0 \\
-by & bx & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
## A 6-dimensional example

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<td>$D_3$</td>
<td>$\Pi_1$</td>
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</tr>
<tr>
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<td>$D_1(X)$</td>
<td>$\Pi_4$</td>
<td>$e^B(T^<em>_X) \oplus T^</em><em>{Y,Z} \oplus e^B(T</em>{Y,Z}) \oplus T_X$</td>
</tr>
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<td>$\Pi_3$</td>
<td>$e^B(T_X) \oplus T_{Y,Z} \oplus e^B(T^<em>_{Y,Z}) \oplus T^</em>_X$</td>
</tr>
<tr>
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<td>$\Pi_6$</td>
<td>$e^B(T_{X,Y}) \oplus T_Z \oplus e^B(T^<em>_Z) \oplus T^</em>_{X,Y}$</td>
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<td>$\Pi_5$</td>
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</table>
We observe that:

- Doubled D-branes in doubled geometry are equivalent to a Courant algebroid composed of a pair of Dirac structures in generalized geometry, which can be further classified into the following categories:
  - $TM \oplus T^*M$
  - $e^B(TM) \oplus T^*M$
  - $TM \oplus e^\beta(T^*M)$
  - $L \oplus \tilde{L}$ where $L = \bigoplus_p T_p \oplus q T_q^*$ and $\tilde{L} = e^B(\bigoplus_p T_p^*) \oplus e^\beta(\bigoplus_q T_q)$ requiring $p + q = \text{dim}(M)$ and $dB = d\beta = 0$.

- $\Pi$ and $\tilde{\Pi} \equiv \bar{\Pi}$ in doubled geometry corresponds to $TM \leftrightarrow T^*M$ in generalized geometry.

- $B$-transform reduces the dimension of a $D$-brane while $\beta$-transform increases the dimension of a $D$-brane.
Thank you!