Hidden Yangian symmetry in sigma model on squashed sphere

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I.K. Kentaroh Yoshida, arXiv:1008.0776

I.K. Domenico Orlando, Kentaroh Yoshida, in preparation に基づく

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Introduction

AdS/CFT対応 Maldacena,1997

$AdS_5 \times S^5$上の
type ⅡB super string

等価

4次元N=4
super Yang-Mills

$\lambda \gg 1$

$\lambda \ll 1$

可解スピン鎖模型

対応関係の詳細な検証 ← 可積分性が重要

Minahan,Zarembo,2002 Bena,Polchinski,Roiban,2003

弦理論側の可積分性

Bena,Polchinski,Roiban,2003

$AdS_5 \times S^5$上のtype ⅡB super string

↓ Virasoro条件を無視

coset $PSU(2,2|4)/(SO(1,4)\times SO(5))$上のシグマ模型

semisymmetric supercoset

↓

一般に古典可積分

無限次元対称性が存在する

Symmetric Coset

Symmetric cosetの定義

$$M = G/H \qquad \hat{g} = \hat{h} \oplus \hat{m}$$

$$[\hat{h}, \hat{h}] \subset \hat{h}$$

$$[\hat{h}, \hat{m}] \subset \hat{m}$$ reductive condition

Symmetric coset

$$[\hat{m}, \hat{m}] \subset \hat{h}$$

例

$$S^n = SO(n+1)/SO(n)$$

$$AdS_{n+1} = SO(n,2)/SO(n,1)$$

Symmetric coset上のsigma model

$$M = G/H$$

Maurer-Cartan one-form

$$J_\mu = g^{-1}\partial_\mu g \qquad g \in G$$

① left “global” transformation

$$g \to g'g \qquad g' \in G$$

の下で不変

② right “local” transformation

$$g \to gh \qquad h \in H$$

の下で $J_\mu \to h^{-1}\partial_\mu h + h^{-1}J_\mu h$

$$J_\mu = A_\mu + K_\mu \qquad A_\mu \in \hat{h},\ \ K_\mu \in \hat{m}$$

$$S = \int dt dx \mathrm{tr}[K_\mu K^\mu]$$

global G symmetric　　local H symmetric

local H transformationの下での変換性

$$K_\mu \to h^{-1} K_\mu h$$
$$A_\mu \to h^{-1} \partial_\mu h + h^{-1} A_\mu h$$

運動方程式

$$\partial^\mu k_\mu = 0 \qquad k_\mu = g K_\mu g^{-1}$$

local H不変なGカレント

保存カレントのflatness

Flatness condition

$$\epsilon^{\mu\nu}(\partial_\mu k_\nu - 2k_\mu k_\nu) = 0$$

↓

無限個の保存カレント　　　“local” & “non-local”

non-local charge

BIZZ construction Brezin,Itzykson,Zinn-Justin,Zuber,1979

帰納的に高次の保存カレントを構成する

$$D_\mu = \partial_\mu - \alpha j_\mu$$

$$\partial^\mu D_\mu = D_\mu \partial^\mu \quad \leftarrow \quad \partial^\mu j_\mu = 0$$

$$\epsilon^{\mu\nu} D_\mu D_\nu = 0 \quad \leftarrow \quad \epsilon^{\mu\nu}(\partial_\mu j_\nu - \alpha j_\mu j_\nu) = 0$$

0番目の保存カレント

$$J_{(0)\mu} = j_\mu$$

(n-1)番目までの保存カレントができたとする

$$\partial^\mu J_{(n-1)\mu} = 0 \longrightarrow J_{(n-1)\mu} = \epsilon_{\mu\nu}\partial^\nu \chi_{(n)}$$

$$\chi_{(n)}(x) = \frac{1}{2}\int dy \epsilon(x-y) J_{(n-1)t}(y)$$

$$J_{(n)\mu} = D_\mu \chi_{(n)}$$

$$\begin{aligned}
\partial^\mu J_{(n)\mu} &= \partial^\mu D_\mu \chi_{(n)} \\
&= D_\mu \partial^\mu \chi_{(n)} \\
&= \epsilon^{\mu\nu} D_\mu J_{(n-1)\nu} \\
&= \epsilon^{\mu\nu} D_\mu D_\nu \chi_{(n-1)} \\
&= 0
\end{aligned}$$

Non-local charge

$$Q_{(n)} = \int dx J_{(n)t}(x)$$

$$Q^A_{(0)} = \int dx j^A_t(x)$$
$$Q^A_{(1)} = \int dx j^A_x(x) - \frac{\alpha}{4} \int \int dx dy \epsilon(x-y) \varepsilon^A_{\ BC} j^B_t(x) j^C_t(y)$$

→ これらのchargeのなす代数？

カレント代数

$$\{j^A_t(x), j^B_t(y)\}_{\mathrm{P}} = f^{AB}_{\ \ C} j^C_t(x) \delta(x-y)$$
$$\{j^A_t(x), j^B_x(y)\}_{\mathrm{P}} = f^{AB}_{\ \ C} j^C_x(x) \delta(x-y) + \delta^{AB} \partial_x \delta(x-y)$$
$$\{j^A_x(x), j^B_x(y)\}_{\mathrm{P}} = 0$$

Yangian代数 Drinfel'd,1988

$$\{Q^A_{(0)}, Q^B_{(0)}\}_\mathsf{P} = f^{AB}{}_C Q^C_{(0)}$$

$$\{Q^A_{(0)}, Q^B_{(1)}\}_\mathsf{P} = f^{AB}{}_C Q^C_{(1)}$$

$$\{Q^A_{(n)}, Q^B_{(1)}\}_\mathsf{P} = f^{AB}{}_C Q^C_{(n+1)} + \cdots$$

Serre関係式

$$\{Q^A_{(1)}, \{Q^B_{(1)}, Q^C_{(0)}\}_\mathsf{P}\}_\mathsf{P} + \text{cyclic}$$
$$= \alpha^{ABC}{}_{DEF}\{Q^D_{(0)}, Q^E_{(0)}, Q^F_{(0)}\}$$

$$\{\{Q^A_{(1)}, Q^B_{(1)}\}_\mathsf{P}, \{Q^C_{(0)}, Q^D_{(1)}\}_\mathsf{P}\}_\mathsf{P} + (A,B) \leftrightarrow (C,D)$$
$$= (\alpha^{ABH}{}_{EFG} f^{CD}{}_H + (A,B) \leftrightarrow (C,D))\{Q^E_{(0)}, Q^F_{(0)}, Q^G_{(1)}\}$$

$$\alpha^{ABC}{}_{DEF} = \frac{1}{24} f^{AI}{}_D f^{BJ}{}_E f^{CK}{}_F f_{IJK}$$

local charge

$$j_\pm = j_t \pm j_x \quad \partial_\pm = \partial_t \pm \partial_x$$

$$\partial^\mu j_\mu = 0 \qquad \epsilon^{\mu\nu}(\partial_\mu j_\nu - \alpha j_\mu j_\nu) = 0$$

$$\partial_- j_+ + \partial_+ j_- = 0 \qquad \partial_- j_+ - \partial_+ j_- = 2\alpha[j_-, j_+]$$

$$\therefore \quad \partial_- j_+ = \alpha[j_-, j_+] \quad \partial_+ j_- = \alpha[j_+, j_-]$$

$$\partial_- \mathrm{tr}(j_+^m) = 0 \qquad \partial_+ \mathrm{tr}(j_-^m) = 0$$

local conservedカレント

m=2はenergy-momentum tensor

$$q_{\pm m} = \int dx \mathrm{tr}(j_\pm^m)$$

local chargeはnon-local chargeと交換する

Non Symmetric Coset

(semi)symmetric (super)coset → 無限次元対称性

必ずAdS×(internal manifold)の形　Zarembo,2010

AdS/NRCFT対応

non symmetric cosetへの拡張

squashe S^3

$$ds^2 = \frac{L^2}{4}[d\theta^2 + \cos^2\theta d\phi^2 + (1+C)(d\psi + \sin\theta d\phi)^2]$$

S^2 $\quad$ S^1

アイソメトリー $\quad SU(2) \times U(1)$

C=0でS^3

アイソメトリー $\quad SU(2) \times SU(2)$

group elementによる表現

$$g \in SU(2) = S^3$$

$$J = g^{-1}dg = J_1 T_1 + J_2 T_2 + J_3 T_3$$

$$[T_A, T_B] = \varepsilon_{AB}{}^{C} T_C \qquad \mathrm{tr}(T_A T_B) = -\frac{1}{2}\delta_{AB}$$

$$ds^2 = \frac{L^2}{4}[(J_1)^2 + (J_2)^2 + (1+C)(J_3)^2]$$

アイソメトリー $SU(2) \times U(1)$

$$g \to g'g \qquad g \to g e^{\alpha T_3}$$

$$g = e^{\phi T_1} e^{\theta T_2} e^{\psi T_3}$$

$$ds^2 = \frac{L^2}{4}[d\theta^2 + \cos^2\theta d\phi^2 + (1+C)(d\psi + \sin\theta d\phi)^2]$$

warped AdS_3

$$S^3 \to AdS_3 \quad \text{or} \quad SU(2) \to SL(2,R)$$

squashed S^3 のdouble Wick rotation

$$ds^2 = \frac{L^2}{4}[d\theta^2 + \cos^2\theta d\phi^2 + (1+C)(d\psi + \sin\theta d\phi)^2]$$

spacelike warped AdS_3

$$\theta \to i\sigma, \quad \phi \to iu, \quad \psi \to \tau$$

$$ds^2 = \frac{L^2}{4}[d\sigma^2 - \cosh^2\sigma d\tau^2 + (1+C)(du + \sinh\sigma d\tau)^2]$$

timelike warped AdS_3

$$\theta \to i\sigma, \quad \phi \to \tau, \quad \psi \to iu$$

$$ds^2 = \frac{L^2}{4}[d\sigma^2 + \cosh^2\sigma du^2 - (1+C)(d\tau - \sinh\sigma du)^2]$$

Sigma model

$$S = \int dtdx\{\mathrm{tr}(J_\mu J^\mu) - 2C\mathrm{tr}(T_3 J_\mu)\mathrm{tr}(T_3 J^\mu)\}$$

$SU(2) \times U(1)$ global symmetry

↓

無限次元対称性に拡大

運動方程式

$$\partial^\mu j_\mu^{con} = 0$$

$$j_\mu^{con} = \partial_\mu g \cdot g^{-1} - 2C\mathrm{tr}(T_3 J_\mu) g T_3 g^{-1}$$

note:U(1)カレントの保存則はSU(2)カレント保存則から従う

Flatness

素朴に計算

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu^{con} - j_\mu^{con} j_\nu^{con}) = -C\epsilon^{\mu\nu}\partial_{\mu(}gT_3g^{-1})\partial_{\nu(}gT_3g^{-1})$$

カレントの不定性

$$j_\mu^{imp} = j_\mu^{con} + \underline{A\epsilon_{\mu\nu}\partial^\nu(gT_3g^{-1})}$$

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu^{imp} - j_\mu^{imp} j_\nu^{imp})$$
$$= (A^2 - C)\epsilon^{\mu\nu}\partial_\mu(gT_3g^{-1})\partial_\nu(gT_3g^{-1})$$

$$A = \pm\sqrt{C}$$

action levelでのimprovement

improvement term

局所的には理論を変えない

↓

topologicalな寄与を拾う可能性あり

実は問題ない

improvement term : $A\epsilon_{\mu\nu}\partial^{\nu}(gT_3g^{-1})$

$$
\begin{aligned}
\delta S &= 2A\int dtdx\epsilon_{\mu\nu}\mathrm{tr}(T_3J^{\mu}J^{\nu}) \\
&= -2A\int dtdx\epsilon_{\mu\nu}\mathrm{tr}[T_3(\partial^{\mu}g^{-1})(\partial^{\nu}g)]
\end{aligned}
$$

カレント代数

squashing parameterが入る

$$\{j_t^A(x), j_t^B(y)\}_{\mathrm{P}} = \varepsilon^{AB}{}_C j_t^C(x)\delta(x-y)$$

$$\{j_t^A(x), j_x^B(y)\}_{\mathrm{P}}$$
$$= \varepsilon^{AB}{}_C j_x^C(x)\delta(x-y) + (1+C)\delta^{AB}\partial_x\delta(x-y)$$

$$\{j_x^A(x), j_x^B(y)\}_{\mathrm{P}} = -C\varepsilon^{AB}{}_C j_t^C(x)\delta(x-y)$$

Serre関係式は保たれているか？

SU(2)Yangian代数

$$\{Q^A_{(0)}, Q^B_{(0)}\}_\mathsf{P} = \varepsilon^{AB}{}_C Q^C_{(0)}$$

$$\{Q^A_{(0)}, Q^B_{(1)}\}_\mathsf{P} = \varepsilon^{AB}{}_C Q^C_{(1)}$$

$$\{Q^A_{(1)}, Q^B_{(1)}\}_\mathsf{P}$$

$$= \varepsilon^{AB}{}_C [Q^C_{(2)} + \underline{\tfrac{1}{12} Q^C_{(0)} Q^D_{(0)} Q_{(0)D}} - C Q^C_{(0)}]$$

Serre関係式は破れていない

local charge

$$j_\pm = \partial_\pm g \cdot g^{-1} - 2C\mathrm{tr}(T_3 J_\pm) g T_3 g^{-1} \pm A \partial_\pm (g T_3 g^{-1})$$

$$\mathrm{tr}(j_\pm^2) = (1+A^2)\mathrm{tr}(J_\pm^2) - 2(2C+C^2-A^2)[\mathrm{tr}(T_3 J_\pm)]^2$$

$$T_{\pm\pm} = -2\mathrm{tr}(J_\pm^2) + 4C[\mathrm{tr}(T_3 J_\pm)]^2$$

$$\boxed{A = \pm\sqrt{C}}$$

energy-momentum tensor

$$T_{\pm\pm} = -\frac{2}{1+C}\mathrm{tr}(j_\pm^2)$$

non-local chrgeと可換

Conclusion

$C = 0$

$S^3 \; NLSM$

→

$C \neq 0$

$sq \; S^3 \; NLSM$

$SU(2)_{\mathrm{L}} \times SU(2)_{\mathrm{R}}$ → Yangian代数

$SU(2)_{\mathrm{L}} \times U(1)_{\mathrm{R}}$ → Yangian代数

WZNWモデル

単純なシグマ模型 → string theoryへの埋め込み？

理論の拡張

- supersymmetricにする
- Wess-Zumino termを加える

今回はこちらについて

Wess-Zumino term

$$S_{WZ} = \frac{2}{3} K \int_0^1 ds \int dt dx \epsilon_{ijk} \mathrm{tr}(J_s^i J_s^j J_s^k)$$

SU(2)カレント

$$j_\mu^{con} = \partial_\mu g \cdot g^{-1} - 2C \mathrm{tr}(T_3 J_\mu) g T_3 g^{-1} - \underline{K \epsilon_{\mu\nu} j^\nu}$$

flatnessを満たさない：

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu^{con} - j_\mu^{con} j_\nu^{con})$$
$$= -(C - \frac{CK^2}{1+C}) \epsilon^{\mu\nu} \partial_\mu (g T_3 g^{-1}) \partial_\nu (g T_3 g^{-1})$$

⟶ $j_\mu^{imp} = j_\mu^{con} + A \epsilon_{\mu\nu} \partial^\nu (g T_3 g^{-1})$

$$\epsilon^{\mu\nu}(\partial_\mu j_\nu^{imp} - j_\mu^{imp} j_\nu^{imp})$$
$$= (A^2 - C + \frac{CK^2}{1+C})\epsilon^{\mu\nu}\partial_\mu(gT_3g^{-1})\partial_\nu(gT_3g^{-1})$$

improvementでflatカレントに：

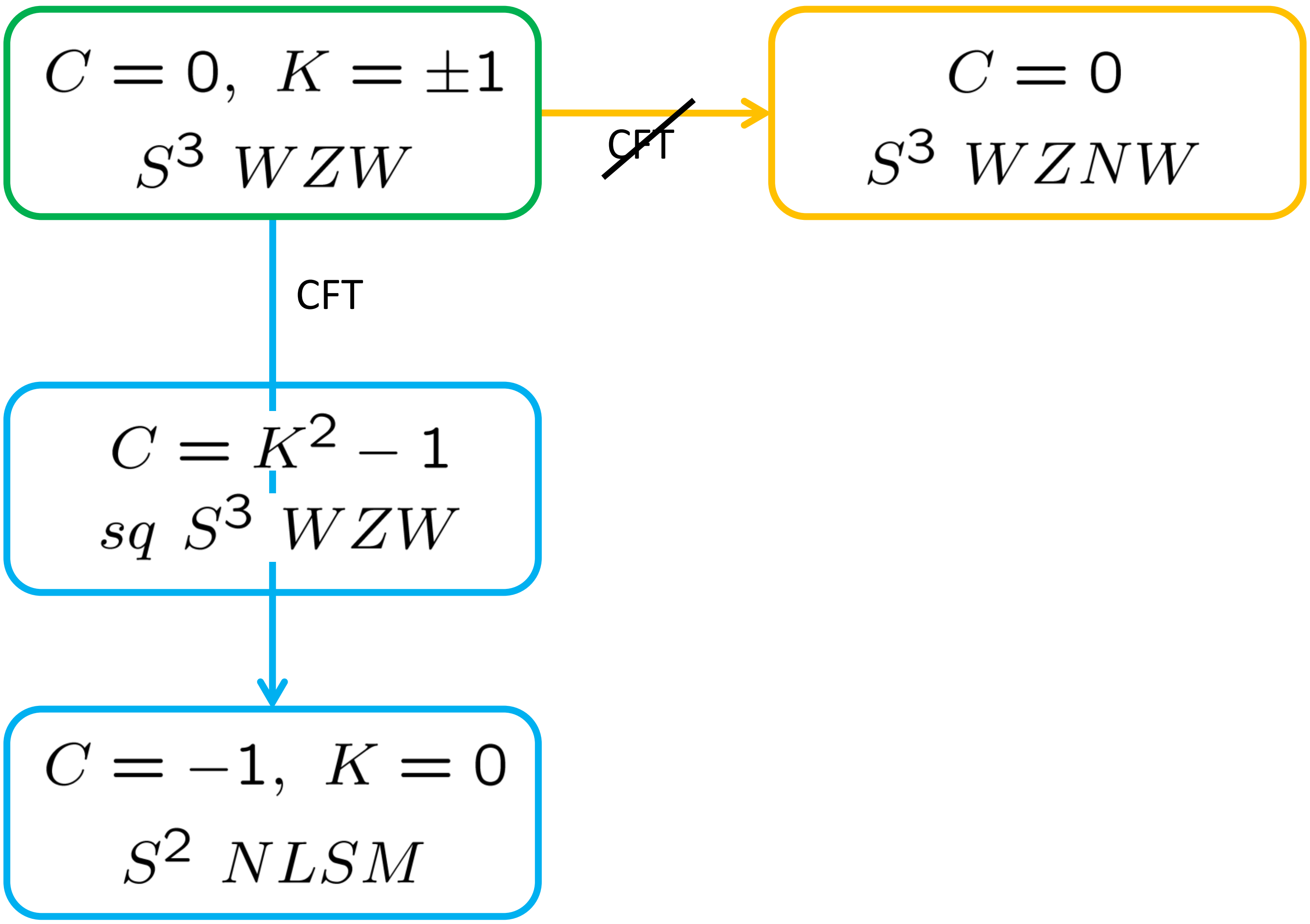
$$A = \pm\sqrt{C(1 - \frac{K^2}{1+C})}$$

$$K = \pm\sqrt{1+C}$$

であればimprovementなしでflatカレント

※）このKの値でCFTになる

Israel,Kounnas,Orlando,Petropoulos,2004


$C = 0, \;\; K = \pm 1$
$S^3 \; WZW$
CFT
$C = 0$
$S^3 \; WZNW$
CFT
$C = K^2 - 1$
$sq \; S^3 \; WZW$
$C = -1, \;\; K = 0$
$S^2 \; NLSM$

カレント代数

$$\{j_t^A(x), j_t^B(y)\}_{\mathsf{P}}$$
$$= \varepsilon^{AB}{}_C j_t^C(x)\delta(x-y) - 2K\delta^{AB}\partial_x\delta(x-y)$$

$$\{j_t^A(x), j_x^B(y)\}_{\mathsf{P}}$$
$$= \varepsilon^{AB}{}_C j_x^C(x)\delta(x-y) + (1+C+\frac{K^2}{1+C})\delta^{AB}\partial_x\delta(x-y)$$

$$\{j_x^A(x), j_x^B(y)\}_{\mathsf{P}}$$
$$= -(C+\frac{K^2}{1+C})\varepsilon^{AB}{}_C j_t^C(x)\delta(x-y)$$
$$-2K\varepsilon^{AB}{}_C j_x^C(x)\delta(x-y) - 2K\delta^{AB}\partial_x\delta(x-y)$$

$C = K^2 - 1$ の場合には

$$\{j_t^A(x), j_t^B(y)\}_{\mathsf{P}}$$
$$= \varepsilon^{AB}{}_C j_t^C(x)\delta(x-y) - 2K\delta^{AB}\partial_x\delta(x-y)$$

$$\{j_t^A(x), j_x^B(y)\}_{\mathsf{P}}$$
$$= \varepsilon^{AB}{}_C j_x^C(x)\delta(x-y) + (1+K^2)\delta^{AB}\partial_x\delta(x-y)$$

$$\{j_x^A(x), j_x^B(y)\}_{\mathsf{P}}$$
$$= -K^2\varepsilon^{AB}{}_C j_t^C(x)\delta(x-y)$$
$$-2K\varepsilon^{AB}{}_C j_x^C(x)\delta(x-y) - 2K\delta^{AB}\partial_x\delta(x-y)$$

Yangian代数

$$\{Q^A_{(0)}, Q^B_{(0)}\}_\mathsf{P} = \varepsilon^{AB}{}_C Q^C_{(0)}$$

$$\{Q^A_{(0)}, Q^B_{(1)}\}_\mathsf{P} = \varepsilon^{AB}{}_C Q^C_{(1)}$$

$$\{Q^A_{(1)}, Q^B_{(1)}\}_\mathsf{P}$$
$$= \varepsilon^{AB}{}_C [Q^C_{(2)} + \tfrac{1}{12} Q^C_{(0)} Q^D_{(0)} Q_{(0)D}$$
$$+ 2K Q^C_{(1)} - (C + \tfrac{K^2}{1+C}) Q^C_{(0)}]$$

この場合もSerre関係式は成り立つ

local charge

$$j_\pm = \partial_\pm g \cdot g^{-1} - 2C\mathrm{tr}(T_3 J_\pm) g T_3 g^{-1}$$
$$\mp K \partial_\pm g \cdot g^{-1} \pm A \partial_\pm (g T_3 g^{-1})$$

$$\mathrm{tr}(j_\pm^2) = ((1 \mp K)^2 + A^2)\mathrm{tr}(J_\pm^2)$$
$$- 2(2C(1 \mp K) + C^2 - A^2)[\mathrm{tr}(T_3 J_\pm)]^2$$

$$T_{\pm\pm} = -2\mathrm{tr}(J_\pm^2) + 4C[\mathrm{tr}(T_3 J_\pm)]^2$$

$$A = \pm\sqrt{C(1 - \frac{K^2}{1+C})}$$

energy-momentum tensor

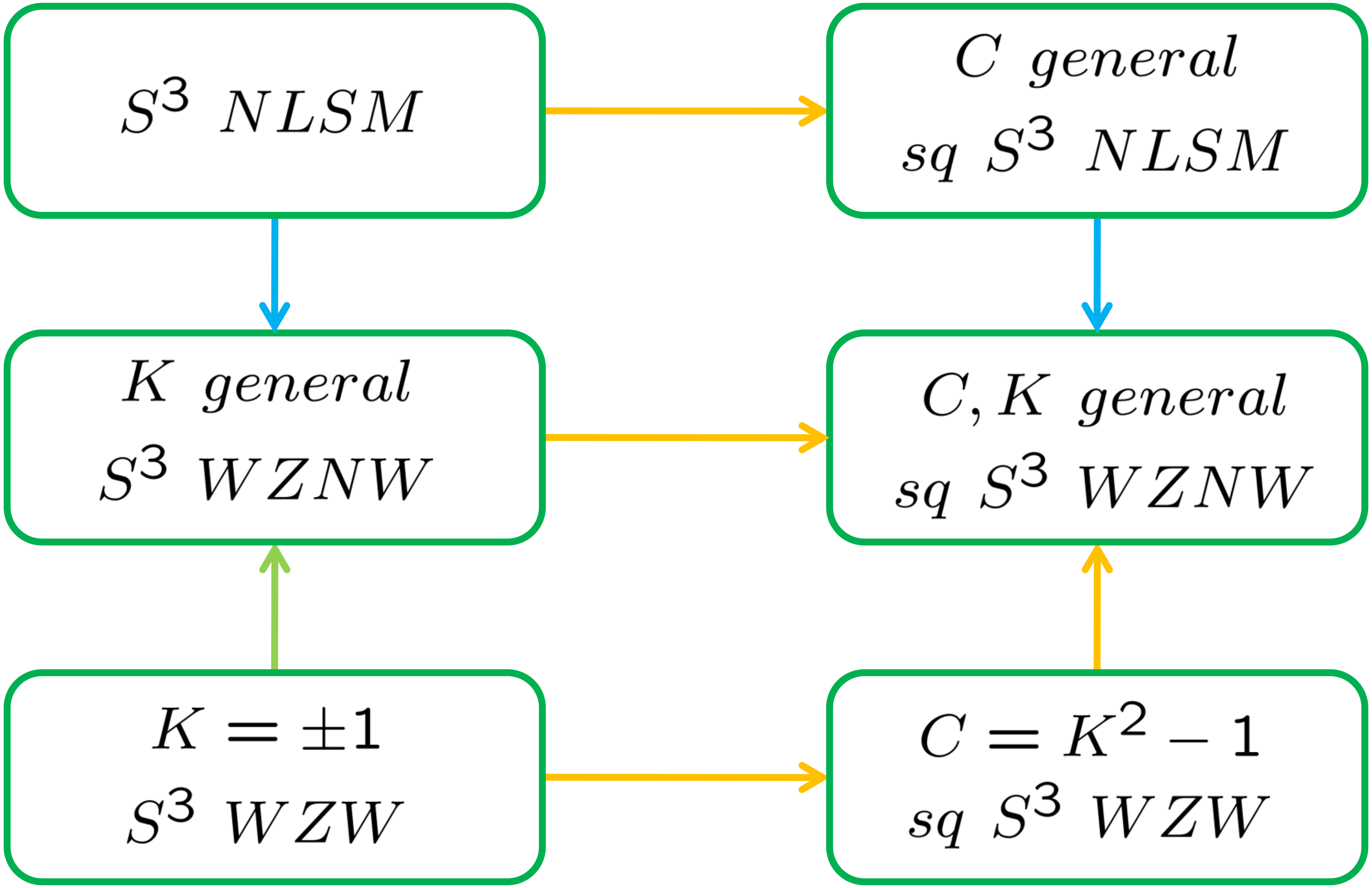
$$T_{\pm\pm} = -\frac{2(1+C)}{(1+C\mp K)^2}\mathrm{tr}(j_\pm^2)$$

$C = K^2 - 1$ の場合には

$$T_{\pm\pm} = -\frac{2}{(1\mp K)^2}\mathrm{tr}(j_\pm^2)$$

non-local chrgeと可換

Conclusion



S^3 $NLSM$
C $general$
sq S^3 $NLSM$
K $general$
S^3 $WZNW$
C, K $general$
sq S^3 $WZNW$
$K = \pm 1$
S^3 WZW
$C = K^2 - 1$
sq S^3 WZW