Black Holes and the Fluctuation Theorem

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+ work in progress

1. Introduction [1/3]

Backgrounds

- Black hole thermodynamics $\begin{cases} \text{1st law}: \frac{\kappa}{8\pi G} dA_{BH} = dM \ , \ \kappa = \frac{1}{4GM} \\ \text{2nd law}: \Delta A_{BH} \geq 0 \end{cases}$

- Hawking radiation – classical gravity + quantum field

  – Planck distribution with temperature $T_H = \frac{\kappa}{2\pi} = \frac{1}{8\pi GM}$

  ⇨ $\begin{cases} \text{1st law}: T_H dS_{BH} = dM \ , \ S_{BH} = \frac{A_{BH}}{4G} \\ \text{generalized 2nd law}: \Delta S_{BH} + \Delta S_{\text{matter}} \geq 0 \end{cases}$

Motivations

- We want to investigate non-equilibrium nature of black holes

– cf. Einstein's theory of Brownian motion, fluctuations are important

– Analyzing an asymptotically flat BH as a thermodynamically unstable system

– Information loss problem

– Refining "The Einstein Equation of State" [Jacobson '95]

– Applications for gauge/gravity duality

1. Introduction [2/3]

Method

- The fluctuation theorem [Evans, Cohen & Morris '93]
  - Roughly speaking, non-equilibrium fluctuations satisfy

$$\frac{\mathrm{Prob}(\text{entropy difference} = \Delta S)}{\mathrm{Prob}(\text{entropy difference} = -\Delta S)} = e^{\Delta S}$$

  - violation of the second law of thermodynamics
  - A bridge between microscopic theory with time reversal symmetry and the second law of thermodynamics

Advantages

- The theorem includes fluctuation around equilibrium (linear response theory)
- The theorem is applicable to (almost) arbitrary non-equilibrium process
  - In particular unstable systems and steady states
  - Evaporating BHs can be treated

1. Introduction [3/3]

Plan

2. The Langevin eq. and the Fokker-Planck eq. [2]

- fundamental tools to study non-equilibrium physics

3. The Fluctuation Theorem [3]

- give a proof and review a first experimental evidence

4. Black Hole with Matter [3]

- show an effective EOM for scalar

5. The Fluctuation Theorem for BH [5] ← Main Topic

- show our results, the fluctuation thm for BH w/ matter, Green-Kubo relation for thermal current

6. Summary and Discussions [3]

2. The Langevin eq. and The Fokker-Planck eq. [1/2]

The Langevin eq. : EOM of particle with friction and noise.

The Fokker-Planck eq. : EOM of particle's probability distribution.

The Langevin equation is a phenomenological (or effective) equation of motion with friction and thermal noise.

$$m\ddot{x} = -\gamma\dot{x} - \frac{\partial V}{\partial x} + \xi$$

$$, \ \langle \xi(t) \rangle = 0 \ , \ \langle \xi(t)\xi(t') \rangle = 2\gamma T \delta(t-t')$$

The Fokker-Planck equation can be obtained from the Langevin equation.

$$\gamma\dot{x} = -\frac{\partial V}{\partial x} + \xi \quad \Rightarrow \quad \partial_t P(x,t|x_0,0) = \partial_x \left[\frac{1}{\gamma}\frac{\partial V}{\partial x} P(x,t|x_0,0) \right] + \frac{T}{\gamma}\partial_x^2 P(x,t|x_0,0).$$

$P(x,t|x_0,0)$: conditional probability to see a event $x(t) = x$
which starts from the value $x(0) = x_0$, $P(x,t=0|x_0,0) = \delta(x-x_0)$.

2. The Langevin eq. and The Fokker-Planck eq. [2/2]

$$\gamma\dot{x} = -\frac{\partial V}{\partial x} + \xi \quad \Rightarrow \quad \partial_t P(x,t|x_0,0) = \partial_x \left[\frac{1}{\gamma}\frac{\partial V}{\partial x}P(x,t|x_0,0) + \frac{T}{\gamma}\partial_x P(x,t|x_0,0)\right]$$

This eq. has the stationary distribution : $\partial_t P^{st} = 0 \Rightarrow P^{st}(x) = e^{-\frac{1}{T}V(x)}/Z$.

"the Boltzmann distribution"

The solution of the Fokker-Planck equation can be represented by a path integral.

$$\partial_t P(x,t|x_0,0) = \hat{L}_{FP} P(x,t|x_0,0)$$

$$\Rightarrow P(x,t|x_0,0) = e^{t\hat{L}_{FP}}\delta(x-x_0) = \int_{x(0)=x_0}^{x(t)=x} \mathcal{D}x \; e^{-\frac{1}{4\gamma T}\int_0^t dt' [\gamma\dot{x}(t') + V'(x(t'))]^2}$$

The "Lagrangian" $L = \frac{1}{4\gamma T}[\gamma\dot{x} + V']^2$ is called "the Onsager-Machlup function".

[Onsager, Machlup '53]

$$L = \left[\frac{\gamma}{4T}\dot{x}^2\right] + \left[\frac{1}{4\gamma T}V'^2\right] - \left[-\frac{1}{2T}\dot{x}V'\right] \equiv \Phi(\dot{x}) + \Psi(x) - \dot{S}(x,\dot{x})$$

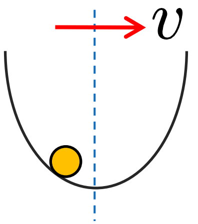
"dissipation function" , "entropy production" time ~~reversed~~ sym.

Most probable path: $\gamma\dot{x}_* = -V'(x_*) \Leftrightarrow \Phi(\dot{x}_*) + \Psi(x_*) = \dot{S}(x_*, \dot{x}_*)$

"Onsager's principle of minimum energy dissipation"

3. The Fluctuation Theorem [1/3]

We consider a externally controlled potential $V(x;\lambda_t)$.



Simple example: $V(x;\lambda_t) = \frac{1}{2}k(x-\lambda_t)^2$, $\lambda_t^F = vt.$

The probability to see a trajectory $\Gamma_\tau = \{x(0) = x_i \to x(\tau) = x_f\}$,

$$P^F[\Gamma_\tau | x_i] \propto e^{-\frac{1}{4\gamma T}\int^{\Gamma_\tau} dt' \left[\gamma \dot{x}(t') + k(x(t') - \lambda_{t'}^F)\right]^2}.$$

Reversed trajectory $\Gamma_\tau^\dagger = \{x^\dagger(0) = x_f \to x^\dagger(\tau) = x_i\}$ with the reversely controlled potential.

$$P^R[\Gamma_\tau^\dagger | x_f] \propto e^{-\frac{1}{4\gamma T}\int^{\Gamma_\tau^\dagger} dt' \left[\gamma \dot{x}(t') + k(x(t') - \lambda_{\tau-t'}^F)\right]^2}.$$

We will call λ_t^F "forward protocol" and $\lambda_t^R \equiv \lambda_{\tau-t}^F$ "reversed protocol" .

$$\frac{P^F[\Gamma_\tau|x_i]}{P^R[\Gamma_\tau^\dagger|x_f]} = e^{-\frac{1}{4\gamma T}\int^{\Gamma_\tau} dt' [\gamma\dot{x}(t') + k(x(t') - vt')]^2 + \frac{1}{4\gamma T}\int^{\Gamma} dt' [-\gamma\dot{x}(t') + k(x(t') - vt')]^2}$$

$$= e^{-\frac{k}{T}\int^{\Gamma_\tau} dt' \dot{x}(t')(x(t') - vt')}.$$

Cancellation occur except for time reversed sym. violated term.

3. The Fluctuation Theorem [2/3]

We assume the initial distribution is in a equilibrium (i.e. Boltzmann distribution) .

$$\Rightarrow \frac{P^F[\Gamma_\tau|x_i]P^{eq}(x_i)}{P^R[\Gamma_\tau^\dagger|x_f]P^{eq}(x_f)} = e^{-\frac{k}{T}\int^{\Gamma_\tau} dt'\dot{x}(t')(x(t')-vt') - \frac{1}{T}(V(x_i,\lambda_0^F)-V(x_f,\lambda_\tau^F)) + \frac{1}{T}(F(\lambda_0^F)-F(\lambda_\tau^F))} = e^{W_d[\Gamma]}.$$

We defined $W^d[\Gamma_\tau] \equiv \beta\int^{\Gamma_\tau} dt'\left[-\dot{x}k(x-vt') + \dot{x}\frac{\partial V}{\partial x} + \dot{\lambda}\frac{\partial V}{\partial \lambda_t^F}\right] - \beta\Delta F = \beta\int^{\Gamma_\tau} dt' vk(x-vt') - \beta\Delta F = \beta(W[\Gamma_\tau]-\Delta F)$
which is called "dissipative work" or "entropy production".

We establish the fluctuation theorem.

$$\rho^F(W_\tau^d) \equiv \int \mathcal{D}\Gamma_\tau P^F[\Gamma_\tau|x_i]P^{eq}(x_i)\delta(W_\tau^d - W^d[\Gamma_\tau]) = \int \mathcal{D}\Gamma_\tau P^R[\Gamma_\tau^\dagger|x_f]P^{eq}(x_f)e^{W^d[\Gamma_\tau]}\delta(W_\tau^d - W^d[\Gamma_\tau])$$

$$= e^{W_\tau^d}\int \mathcal{D}\Gamma_\tau^\dagger P^R[\Gamma_\tau^\dagger|x_f]P^{eq}(x_f)\delta(W_\tau^d + W^d[\Gamma_\tau^\dagger]) = e^{W_\tau^d}\rho^R(-W_\tau^d)$$

Negative entropy production always exists.

The Jarzynski equality [Jarzynski '97]: $\langle e^{-\beta(W_\tau-\Delta F)}\rangle = 1$

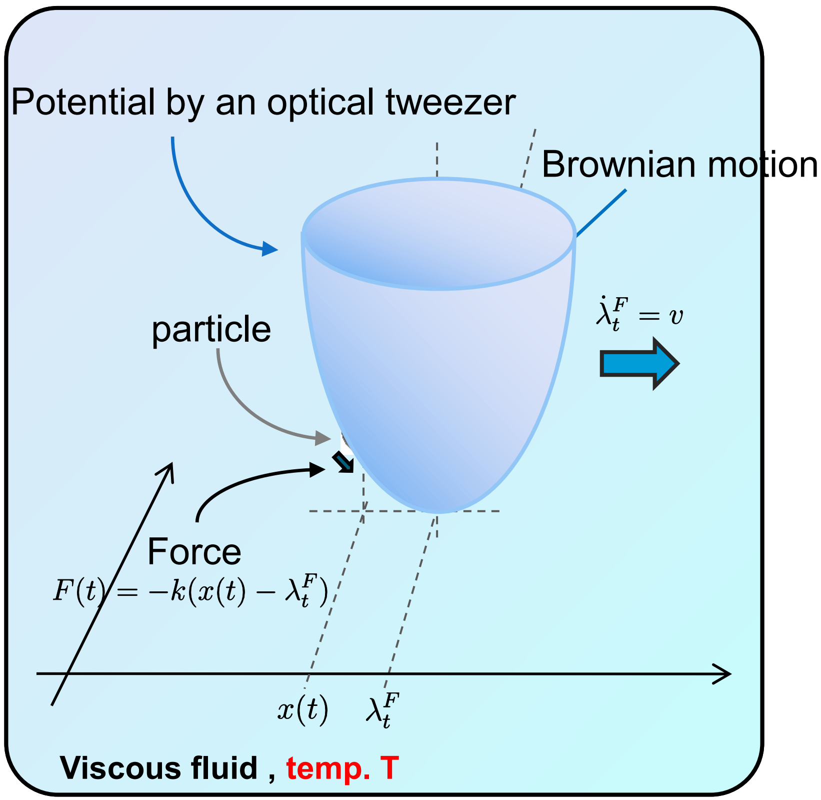
By Using $\langle e^x\rangle \geq e^{\langle x\rangle}$, we obtaine "the second law of thermodynamics" as a corollary.

$$\Rightarrow \langle W_\tau\rangle - \Delta F \geq 0$$

There exists a trajectory that violates the second law.

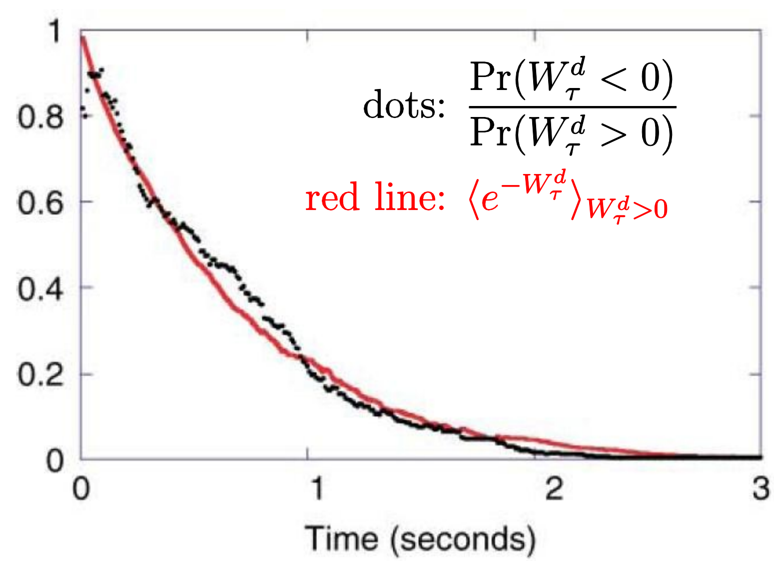
3. The Fluctuation Theorem [3/3]

The first experimental evidence. [Wang, Sevick, Mittag, Searles & Evans (2002)]



$$\frac{\rho^F(W_\tau^d)}{\rho^R(-W_\tau^d)} = e^{W_\tau^d} \Rightarrow \int_0^\infty dW_\tau^d e^{-W_\tau^d} \rho^F(W_\tau^d) = \int_0^\infty dW_\tau^d \rho^R(-W_\tau^d)$$

$$\Rightarrow \frac{\Pr(W_\tau^d < 0)}{\Pr(W_\tau^d > 0)} = \langle e^{-W_\tau^d} \rangle_{W_\tau^d > 0}$$



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 - prepare fundamental tools to study non-equilibrium physics

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4. Black Hole with Matter [1/3]

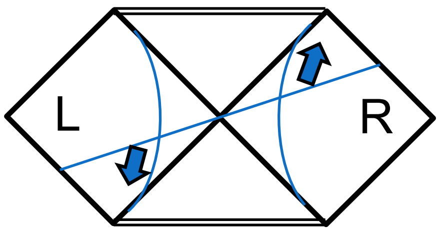
We will consider a spherically symmetric system. Back-reactions are neglected.
Scalar field in maximally extended Schwarzschild BH:

$$S = -\int_{r_H}^{r_H+\epsilon} dr d^3x \sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi_R\partial_\nu\phi_R + m^2\phi_R^2\right) + \int_{r_H}^{r_H+\epsilon} dr d^3x \sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi_L\partial_\nu\phi_L + m^2\phi_L^2\right).$$

We expand a solution in region R of $r = r_H \sim r_H + \epsilon$ as,

$$\phi_R(\omega, r) = a(\omega)F_\omega(r) + b(\omega)F_\omega^*(r)$$

which satisfy $e^{-i\omega t}F_\omega(r) \sim e^{-i\omega(t-r_*)}$, ingoing boundary condition.



We connect the solution to region L to satisfy

$F_\omega(r)$ is equivalent to Kruskal positive frequency modes,
$F_\omega^*(r)$ is equivalent to Kruskal negative frequency modes.

Set the values : $\phi_{R,L}(\omega, r = r_H + \epsilon) \equiv \phi_{R,L}^\epsilon(\omega)$, $G^R(\omega) \equiv r_H^2 \partial_{r_*} \ln(F_\omega(r_H))$.

$$\Rightarrow S(r = r_H + \epsilon) = \int d^3x (\phi_R^\epsilon, \phi_L^\epsilon) \begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \begin{pmatrix} \phi_R^\epsilon \\ \phi_L^\epsilon \end{pmatrix}$$

Schwinger-Keldysh propagators. [Son, Teaney '09]

4. Black Hole with Matter [2/3]

It is possible to cast the propagators into the form

$$\begin{pmatrix} G_{11} & G_{12} \\ G_{21} & G_{22} \end{pmatrix} \to \begin{pmatrix} 0 & G^R \\ G^R & G_{\rm sym} \end{pmatrix}$$

when we use the bases

$$\begin{pmatrix} \phi_R^\epsilon \\ \phi_L^\epsilon \end{pmatrix} \to \begin{pmatrix} \phi_r^\epsilon \\ \phi_a^\epsilon \end{pmatrix} = \begin{pmatrix} \frac{1}{2}(\phi_R^\epsilon + \phi_L^\epsilon) \\ \phi_R^\epsilon - \phi_L^\epsilon \end{pmatrix} .$$

The method is called "retarded-advanced formalism". [for review Calzetta, Hu '08]

We introduce an auxiliary field ξ ,

$$e^{-\frac{1}{2}\int \frac{d\omega}{2\pi}\phi_a^\epsilon(-\omega)G_{\rm sym}(\omega)\phi_a^\epsilon(\omega)} = \int \mathcal{D}\xi e^{i\int \frac{d\omega}{2\pi}\phi_a^\epsilon(-\omega)\xi(\omega) - \frac{1}{2}\int \frac{d\omega}{2\pi}\xi(-\omega)G_{\rm sym}(\omega)^{-1}\xi(\omega)} .$$

We obtain the effective equation of motion for ϕ_r^ϵ ,

$$\Rightarrow \quad (\partial_t - \partial_{r_*})\phi_r(t,r)\big|_{r=r_H+\epsilon} = \xi(t) \quad , \ \langle \xi(t) \rangle = 0 \ , \ \langle \xi(t)\xi(t') \rangle \simeq 2\frac{T_H}{r_H^2}\delta(t-t') .$$

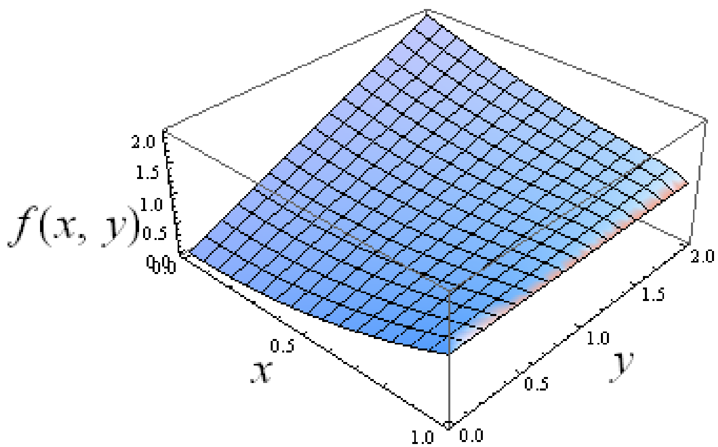
Langevin equation. Friction and noise $\Leftrightarrow$ absorption and Hawking radiation

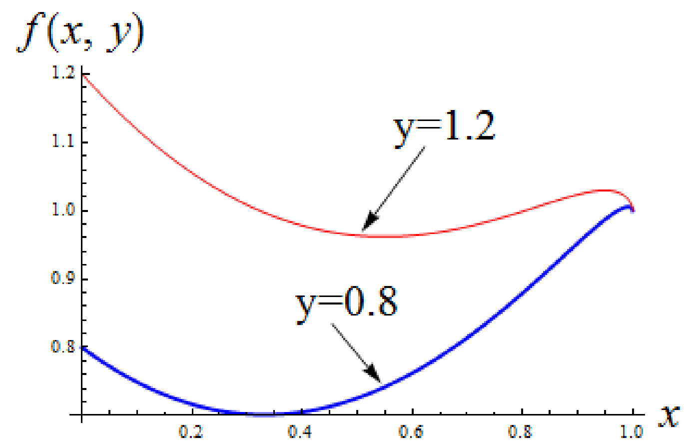
4. Black Hole with Matter [3/3]

Black holes in an asymptotically flat spacetime are thermodynamically unstable.
Rough estimates about stability of radiation in a box: [Gibbons, Perry '78]

$$\begin{cases} E = M + \frac{\pi^2}{30} V T^4 \\ S = 4\pi G M^2 + \frac{4}{3}\frac{\pi^2}{30} V T^3 \end{cases} \Rightarrow \text{Change of variables: } x = \frac{M}{E}\ ,\ y = \frac{1}{3\pi G}\left(\frac{\pi^2 V}{30 E^5}\right)^{\frac{1}{4}}$$

For given E, V (given y), maximize the entropy. $\frac{S}{4\pi G E^2} \equiv f(x,y) = x^2 + y(1-x)^{\frac{3}{4}}$



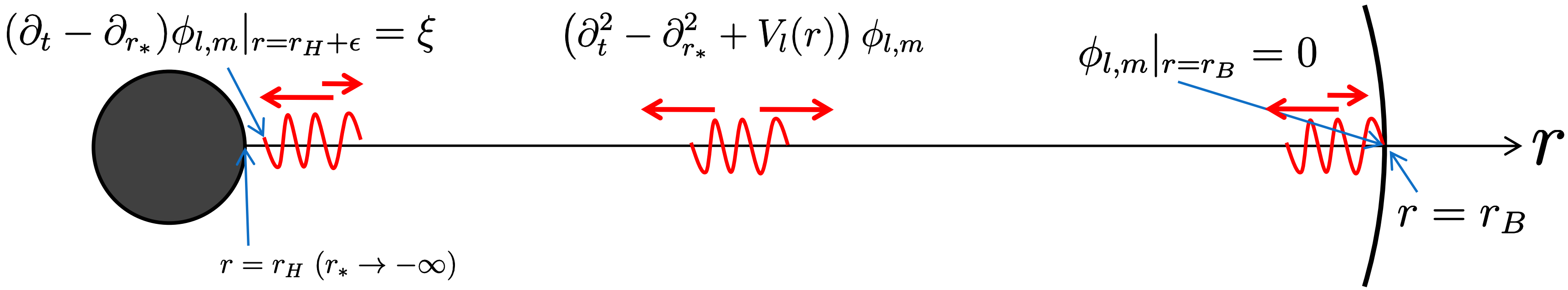


If $y > y_c \simeq 1.0$, $x = 0$ which corresponds to the radiation.

If $y < y_c$, $x \geq x_c \simeq 0.98$ which corresponds to the black hole in thermal equilibrium.

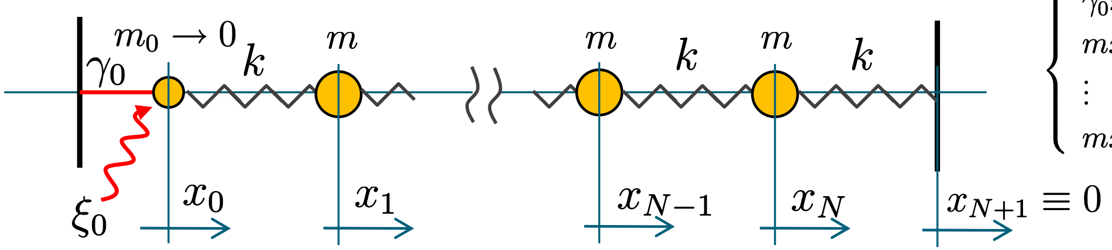
5. The Fluctuation Theorem for BH [1/5]

We consider a black hole in a box. This system is thermodynamically stable.



The Fokker-Planck equation for the scalar field
in black hole background is as follows: $\partial_t P(\phi_t|\phi_0) = \frac{\delta}{\delta\phi(t,r)}(-\dot{\phi}(t,r)P) + \cdots$.

Instead, we consider a discrete model for simplicity.



$$\begin{cases} \gamma_0 \dot{x}_0 = -k(x_0 - x_1) + \xi_0 \ , \ \langle \xi_0(t)\xi_0(t')\rangle = 2\gamma_0 T_0 \delta(t-t') \\ m\ddot{x}_1 = -k(x_1 - x_2) + k(x_0 - x_1) \\ \vdots \\ m\ddot{x}_N = -k(x_N - x_{N+1}) + k(x_{N-1} - x_N) \ , \ x_{N+1} \equiv 0 \end{cases}$$

5. The Fluctuation Theorem for BH [2/5]

We can construct a path integral representation for the discrete model.

$$P(x_f, t=\tau | x_i, t=0) = \int_{x_i}^{x_f} \mathcal{D}\Gamma_\tau e^{-\frac{1}{4\gamma_0 T_0}\int^{\Gamma_\tau} dt \left(\gamma_0 \dot{x}_0 + \partial_{x_0} U\right)^2} \prod_{i,t} \delta\left(m\ddot{x}_i + \partial_{x_i} U\right)|_{x_{N+1}=0}$$

$$U(x_i) \equiv \frac{1}{2}k\sum_{i=1}^{N}\left((x_i - x_{i-1})^2 + V(r_i)x_i^2\right)$$

We choose the initial distribution as a stationary distribution.
In this case, it is a equilibrium distribution i.e. $P^{eq} = e^{-\frac{1}{T_0}\sum_{i=1}^N \left(\frac{1}{2}mv_i^2 + \frac{1}{2}k(x_i - x_{i-1})^2 + \frac{1}{2}kV(r_i)x_i^2\right)}/Z.$

$$\Rightarrow \quad \frac{P^F[\Gamma_\tau | x_i] P^{eq}(x_i)}{P^R[\Gamma_\tau^\dagger | x_f] P^{eq}(x_f)} = e^{-\frac{1}{T_0}\int^{\Gamma_\tau} dt \dot{x}_0 \partial_{x_0} U + \frac{1}{T_0}(\Delta H_M - \Delta F_M)}.$$

Here, we introduce a externally controlled potential $U(x_i; \lambda_t^F)$.

Since $\int^{\Gamma_\tau} dt \dot{x}_0 \partial_{x_0} U \to -\sum_{l,m}\int^{\Gamma_\tau} dt \dot{\phi}_{l,m}^{\epsilon} \partial_{r_*}\phi_{l,m}^{\epsilon} = -\int^{\Gamma_\tau} dt A_{BH} T_t^r(r_\epsilon)$, $\frac{1}{T_H}\int^{\Gamma_\tau} dt A_{BH} T_t^r(r_\epsilon) = \Delta S_{BH}$,

we can reinterpret the entropy production as

$$S[\Gamma_\tau] = -\frac{1}{T_0}\int^{\Gamma_\tau} dt \dot{x}_0 \partial_{x_0} U + \frac{1}{T_0}(\Delta H_M - \Delta F_M) = \Delta S_{BH} + \Delta S_M$$

This quantity fluctuates due to Hawking radiation and externally controlled potential.

5. The Fluctuation Theorem for BH [3/5]

We establish the fluctuation theorem for black holes with matters. [Iso, Okazawa, Zhang '10]

$$\Rightarrow \frac{\rho^F(\Delta S_{BH}+\Delta S_M)}{\rho^R(-(\Delta S_{BH}+\Delta S_M))} = e^{\Delta S_{BH}+\Delta S_M}$$

We also establish the Jarzynski type equality.

$$\Rightarrow \langle e^{-(\Delta S_{BH}+\Delta S_M)} \rangle = 1$$

From the above, an inequality $\langle e^x \rangle \geq e^{\langle x \rangle}$ gives the generalized second law.

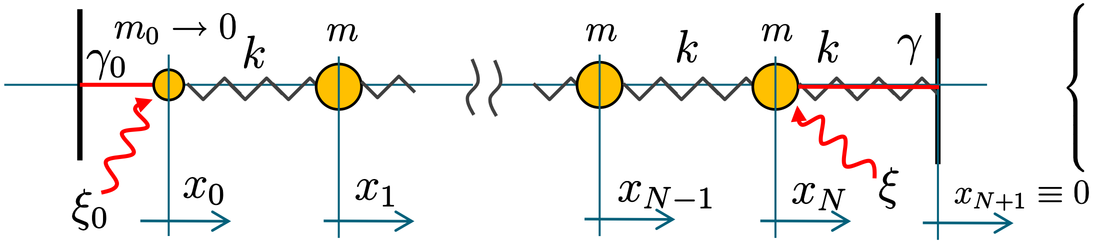
$$\Rightarrow \langle (\Delta S_{BH} + \Delta S_M) \rangle \geq 0$$

To satisfy the Jarzynski equality, entropy decreasing trajectory must exist.
A violation of the generalized second law.

Next: we examine steady state case.

5. The Fluctuation Theorem for BH [4/5]

For the purpose to make steady states, we introduce artificial thermal bath at the wall.



$$\begin{cases} \gamma_0 \dot{x}_0 = -k(x_0 - x_1) + \xi_0 \ , \ \langle \xi_0(t)\xi_0(t')\rangle = 2\gamma_0 T_0 \delta(t-t') \\ m\ddot{x}_1 = -k(x_1 - x_2) + k(x_0 - x_1) \\ \vdots \\ m\ddot{x}_N = -\gamma \dot{x}_N - k(x_N - x_{N+1}) + k(x_{N-1} - x_N) + \xi \\ \qquad , \ \langle \xi(t)\xi(t')\rangle = 2\gamma T \delta(t-t') \end{cases}$$

The path integral representation becomes

$$P(x_f,\tau|x_i,0) = \int_{x_i}^{x_f} \mathcal{D}\Gamma_\tau e^{-\frac{1}{4\gamma_0 T_0}\int^{\Gamma_\tau} dt\left(\gamma_0\dot{x}_0 + \partial_{x_0}U\right)^2} e^{-\frac{1}{4\gamma T}\int^{\Gamma_\tau} dt\left(m\ddot{x}_N + \gamma\dot{x}_N + \partial_{x_N}U\right)^2} \prod_t \prod_{i=1}^{N-1} \delta\left(m\ddot{x}_i + \partial_{x_i}U\right)|_{x_{N+1}=0}.$$

The key property: $\dfrac{P[\Gamma_\tau|x_i]}{P[\Gamma_\tau^\dagger|x_f]} = e^{-\frac{1}{T_0}\int^{\Gamma_\tau} dt\dot{x}_0\partial_{x_0}U - \frac{1}{T}\int^{\Gamma_\tau} dt\dot{x}_N(m\ddot{x}_N + \partial_{x_N}U)}.$

The system reaches a steady state in late time. Entropy production becomes the form

$$S[\Gamma_\tau] \to (\beta_0 - \beta)\int^{\Gamma_\tau} dt A_{BH} T_t^r(r_\epsilon).$$

5. The Fluctuation Theorem for BH [5/5]

When we denote $\mathcal{A} = \beta_0 - \beta\ ,\ k = \int dt A_{BH} T^r_t(r_\epsilon)$, one can prove the fluctuation theorem

$$\frac{\rho(k,\mathcal{A})}{\rho(-k,\mathcal{A})} = e^{k\mathcal{A}}.$$

The theorem can be recast in the relation of the generating function.

$$Z(\alpha,\mathcal{A}) \equiv \ln\left(\int dk e^{ik\alpha}\rho(k,\mathcal{A})\right) = \ln\left(\int dk e^{ik(\alpha - i\mathcal{A})}\rho(-k,\mathcal{A})\right) = Z(i\mathcal{A}-\alpha,\mathcal{A})$$

n-th cumulant : $Z(\alpha,\mathcal{A}) = \sum_{n=1}^{\infty} \frac{(i\alpha)^n}{n!} K_n(\mathcal{A})$. Expansion: $K_1(\mathcal{A}) = L^{(1)}\mathcal{A} + L^{(2)}\mathcal{A}^2 + \cdots$

We get the relations from the GF. ➡ $L^{(1)} = \frac{1}{2}K_2(0)\ ,\ L^{(2)} = \frac{1}{2}\partial_{\mathcal{A}} K_2(0)\ ,\ \cdots$

the Green-Kubo relation and non-linear response coefficients.

In our case, for $\bar{J} \equiv \lim_{\tau\to\infty} \frac{1}{\tau}\int_0^\tau dt A_{BH} T^r_t(r_\epsilon) = L^{(1)}(\beta_H - \beta) + L^{(2)}(\beta_H - \beta)^2 + \cdots$

we obtain

➡ $L^{(1)} = \frac{1}{2}\int_0^\infty dt A_{BH}^2 \langle T^r_t(0,r_\epsilon) T^r_t(t,r_\epsilon)\rangle|_{T=T_H}\ ,\ L^{(2)} = \frac{1}{2}\int_0^\infty dt A_{BH}^2 \partial_{\mathcal{A}} \langle T^r_t(0,r_\epsilon) T^r_t(t,r_\epsilon)\rangle|_{T=T_H}\ ,\cdots$

8. Summary and Discussion [1/3]

- The fluctuation theorem for black holes with matters

$$\frac{\rho^F(\Delta S_{BH}+\Delta S_M)}{\rho^R(-(\Delta S_{BH}+\Delta S_M))} = e^{\Delta S_{BH}+\Delta S_M}$$

- Violation of the second law

- The Jarzynski type equality $\langle e^{-(\Delta S_{BH}+\Delta S_M)}\rangle = 1$

- The generalized second law $\langle(\Delta S_{BH}+\Delta S_M)\rangle \geq 0$

- The steady state fluctuation theorem $\dfrac{\rho(k,\mathcal{A})}{\rho(-k,\mathcal{A})} = e^{k\mathcal{A}}$ $\quad ,\mathcal{A} = \beta_H - \beta\ ,\ k = \int dt A_{BH} T^r_t(r_\epsilon)$

- Green-Kubo relation, non-linear coefficients

$$L^{(1)} = \frac{1}{2}\int_0^\infty dt A_{BH}^2 \langle T^r_t(0,r_\epsilon)T^r_t(t,r_\epsilon)\rangle|_{T=T_H}\ ,\ L^{(2)} = \frac{1}{2}\int_0^\infty dt A_{BH}^2 \partial_{\mathcal{A}}\langle T^r_t(0,r_\epsilon)T^r_t(t,r_\epsilon)\rangle|_{T=T_H}\ ,\cdots$$

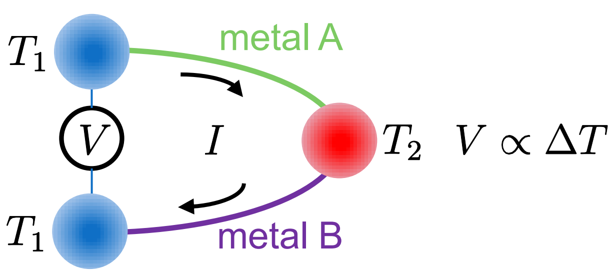
8. Summary and Discussion [2/3]

Future directions

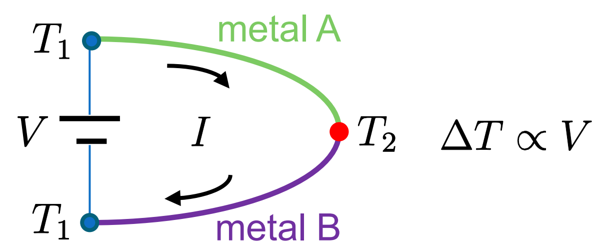
- We want to import more from non-equilibrium physics
  – Are there any study about the system which has negative specific heat?
  – e.g. evaporating liquid droplet

- We expect the Onsager reciprocal relation
  – for example “thermoelectric effect”: the Seebeck effect ⟺ the Peltier effect

thermocouple



solid state refrigerator

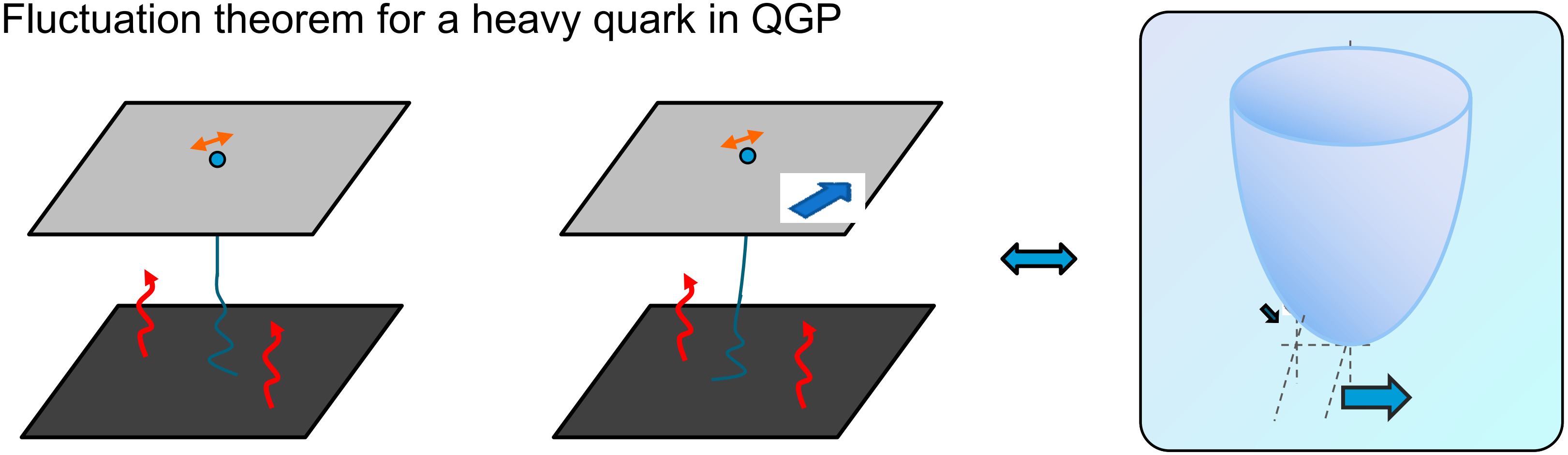


  – Reissner-Nordstrom BH with charged matter may provide a reciprocal relation
  (But, reciprocal relation will be violated by non-equilibrium fluctuations)

8. Summary and Discussion [3/3]

- Including the back reaction
  - Energy conservation: $\Delta A = 8\pi G \int da d\lambda \; \lambda T_{\mu\nu} k^\mu k^\nu$ $\qquad k = \frac{d}{d\lambda}$: null geodesic generator
  - Modification of the Fokker-Planck equation
  - Analyzing asymptotically flat black holes (without box i.e. as an unstable system)

- Application of the method to gauge/gravity duality
  - Stochastic string model [de Boer, Hubeny, Rangamani, Shigemori '08, Son, Teaney '09]
  - Fluctuation theorem for a heavy quark in QGP



- Can we refine Jacobson's idea by using the fluctuation theorem?

Backup

Definitions.

$$S = -\int d^4x\sqrt{-g}\frac{1}{2}\left(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + m^2\phi^2\right)$$
$$= -\sum_{l,m}\int dt dr_*\phi_{l,m}\left[\partial_t^2 - \partial_{r_*}^2 + V_l(r)\right]\phi_{l,m}$$

$$ds^2 = -f(r)dt^2 + \frac{dr^2}{f(r)} + r^2 d\Omega^2 \ , \ f(r) = 1 - \frac{2GM}{r}$$

$$r_* = \int \frac{dr}{f(r)}$$
$$V_l(r) = f(r)\left(\frac{l(l+1)}{r^2} + \frac{\partial_r f(r)}{r} + m^2\right)$$
$$\phi(t, r, \Omega) = \sum_{l,m}\frac{\phi_{l,m}(t,r)}{r}Y_{l,m}(\Omega)$$