Instanton effective action in Ω-background and D3/D(-1)-brane system in R-R background

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Theme of the work

Non-perturbation calculations in $\mathcal{N} = 2$ SYM via string theory

Examples of non-perturbative calculus in $\mathcal{N} = 2$ SYM

- Seiberg-Witten prepotential, 9407087
  → non-perturbative low-energy effective action
- Nekrasov, 0206161
  → instanton partition function

- Seiberg-Witten 9407087

The low-energy effective action of $\mathcal{N} = 2$ SYM is determined by some holomorphic function $\mathcal{F}$ called prepotential.

$$\mathcal{F}(\phi) = \mathcal{F}_{\text{pert}}(\phi) + \mathcal{F}_{\text{inst}}(\phi), \qquad \mathcal{F}_{\text{inst}}^{SU(2)}(\phi) = \sum_{k=1}^{\infty} \mathcal{F}_k \left(\frac{\Lambda}{\phi}\right)^{4k} \phi^2$$

- Nekrasov 0206161

The SW prepotential can be obtained via computation of the instanton partition function. (Nekrasov, 0206161)

$$Z_{\text{inst}}(\phi,\Lambda,\epsilon_1,\epsilon_2)=\sum_{k=1}^{\infty}\Lambda^{2Nk}\int_{\mathcal{M}_{N,k}}\mathrm{d}\mu\,\exp[-S_{\text{eff}}(\mu,\phi,\epsilon_1,\epsilon_2)]$$

$S_{\text{eff}}(\phi,\epsilon_1,\epsilon_2)$: instanton effective action deformed by ε's

$$\log Z_{\text{inst}}(\phi,\Lambda,\epsilon_1,\epsilon_2)=\frac{1}{\epsilon_1\epsilon_2}\left(\mathcal{F}^{\text{SW}}_{\text{inst}}(\phi)+(\epsilon_1+\epsilon_2)\mathcal{F}_1(\phi)+\mathcal{O}(\epsilon^2)\right)$$

Partition function of $\mathcal{N}$ = 2 SYM including the instanton effects

… Integral of the instanton moduli action on the instanton moduli space using localization technique

→ Introduce Ω-background (graviphoton background) → parameter ε's

Deformed supersymmetry = Equivariant BRST operator
→ Fixed points of the BRST operator on the instanton moduli space are isolated.

In string theory
Describe the Ω-background in terms of closed string backgrounds.

- Billo et. al.'s work 0606013

Self-dual Ω-background deformation of the instanton moduli action for $\mathcal{N}=2$ SYM

↕ Agree at the level of zero-modes

Self-dual R-R 3-form deformation of D(-1)-brane action in D(-1)/D3 system on orbifolding point (self-dual graviphoton effect)

What about general Ω-background ? (Not restricted to self-duality)

- Recent works on non-self-dual Ω-background

Topological closed string with non-self-dual graviphoton background
→ "Refinement" of the topological string
[Iqbal-Kozcaz-Vafa , Taki , …]

Non-self-dual Ω-background deformation of $\mathcal{N}=2$ SYM
→ Related with some integrable systems
[Nekrasov-Shatashvili , Mironov-Morozov , …]

- Our work

Main result

- General Ω-background deformation of the instanton effective action with R-sym. Wilson line

↕ Equivalent

Self-dual and anti-self-dual R-R 3-form deformation of D(-1) action (graviphiton + mass parameter)

- Consider general Ω-background deformation of the SYM

  → Found the ADHM construction of an instanton solution and the instanton eff. action with general Ω-background

- Checked the preserved supersymmetry (spacetime and instanton) of the deformed action introducing R-sym. Wilson line (Proposed by Nekrasov-Okounkov)

  = Equivariantly nilpotent supersymmetry

Contents

- Instanton effective action (field steory)
  - Meanings and derivation (deformed ADHM construction)
  - Equivariantly deformed SUSY
- Instanton effective action (string theory)
  - Instanton eff action and D(-1)-brane eff action (Review)
  - Insertion of R-R 3-forms
- Conclusions and Further topics

Instanton eff action (field theory)

- $\mathcal{N} = 2$ SYM Lagrangian with Ω-background (Euclidean spacetime)

We start from 6 dim. $\mathcal{N} = 1$ SYM Lagrangian

$$\mathcal{L}_{6\mathrm{D}} = \sqrt{-G}\,\mathrm{Tr}\left[\frac{1}{4}G^{MP}G^{NQ}F_{MN}F_{PQ} + \frac{i}{2}\bar{\Psi}\Gamma^{M}\mathcal{D}_{M}\Psi\right]$$

with metric

$$\mathrm{d}s^2_{6\mathrm{D}} = 2\mathrm{d}z\mathrm{d}\bar{z} + (\mathrm{d}x^m + \bar{\Omega}^m\mathrm{d}z + \Omega^m\mathrm{d}\bar{z})^2$$

where

$$\Omega^m = \Omega^{mn}x_n, \quad \bar{\Omega}^m = \bar{\Omega}^{mn}x_n$$

$$\Omega^{mn} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & i\epsilon_1 & & \\ -i\epsilon_1 & & & \\ & & & -i\epsilon_2 \\ & & i\epsilon_2 & \end{pmatrix}, \bar{\Omega}^{mn} = \frac{1}{2\sqrt{2}}\begin{pmatrix} & i\bar{\epsilon}_1 & & \\ -i\bar{\epsilon}_1 & & & \\ & & & -i\bar{\epsilon}_2 \\ & & i\bar{\epsilon}_2 & \end{pmatrix}$$

constant anti-symmetric matrices

Insert R-symmetry Wilson line (gauging SU(2) R-symmetry)

$$\boldsymbol{A}^I{}_J = \bar{\mathcal{A}}^I{}_J \mathrm{d}z + \mathcal{A}^I{}_J \mathrm{d}\bar{z}$$

$\bar{\mathcal{A}}^I{}_J, \mathcal{A}^I{}_J$: constant parameter I, J : SU(2) R-symmetry indices

KK reduction of toric dim. → 4 dim. deformed SYM Lagrangian

$$\begin{aligned}\mathcal{L}_\Omega = \mathrm{Tr}\Big[&\frac{1}{4}F_{mn}F^{mn} - \frac{i\theta g^2}{32\pi^2}F_{mn}\tilde{F}^{mn} \\ &+ (D_m\varphi - gF_{mn}\Omega^n)(D_m\bar{\varphi} - gF_{mp}\bar{\Omega}^p) \\ &+ \Lambda^I\sigma^m D_m\bar{\Lambda} - \frac{i}{\sqrt{2}}g\Lambda^I[\bar{\varphi}, \Lambda_I] + \frac{i}{\sqrt{2}}g\bar{\Lambda}_I[\varphi, \bar{\Lambda}^I] \\ &+ \frac{g^2}{2}\left([\varphi, \bar{\varphi}] + i\Omega^m D_m\bar{\varphi} - i\bar{\Omega}^m D_m\varphi + ig\bar{\Omega}^m\Omega^n F_{mn}\right)^2 \\ &+ \frac{1}{\sqrt{2}}g\bar{\Omega}^m\Lambda^I D_m\Lambda_I - \frac{1}{2\sqrt{2}}g\bar{\Omega}_{mn}\Lambda^I\sigma^{mn}\Lambda_I \\ &- \frac{1}{\sqrt{2}}g\Omega^m\bar{\Lambda}_I D_m\bar{\Lambda}^I + \frac{1}{2\sqrt{2}}g\Omega_{mn}\bar{\Lambda}_I\bar{\sigma}^{mn}\bar{\Lambda}^I \\ &- \frac{1}{\sqrt{2}}g\bar{\mathcal{A}}^J{}_I\Lambda^I\Lambda_J - \frac{1}{\sqrt{2}}g\mathcal{A}^J{}_I\bar{\Lambda}^I\bar{\Lambda}_J\Big]\end{aligned}$$

Generally all the SUSY is broken …
But when we choose R-sym. Wilson line as

$$\mathcal{A}^I{}_J = -\frac{1}{2}\Omega_{mn}(\bar{\sigma}^{mn})^I{}_J, \quad \bar{\mathcal{A}}^I{}_J = -\frac{1}{2}\bar{\Omega}_{mn}(\bar{\sigma}^{mn})^I{}_J$$

→ $\mathcal{L}_\Omega$ preserves one supersymmetry $\bar{Q}_\Omega$

= scalar supercharge of topologically twisted theory

- Nilpotency of $\bar{Q}_\Omega$

$\bar{Q}_\Omega^2$ = gauge transf. by $2\sqrt{2}\varphi$ + U(1)^2 rotation by $2\sqrt{2}\Omega_{mn}$

equivariantly deformed SUSY

- $\bar{Q}_\Omega$ -exactness

The action can be written in exact form as

$$\int \mathrm{d}^4 x \mathcal{L}_\Omega = \frac{8\pi^2 k}{g^2} + \bar{Q}_\Omega \Xi$$

(k : instanton number)

Partition function of the SYM :

$$Z = \int \mathcal{D}A\mathcal{D}\Lambda\mathcal{D}\overline{\Lambda}\mathcal{D}\varphi\, \mathrm{e}^{-S}$$

$$= \sum_k \int \mathrm{d}X\mathrm{d}\psi\, \mathcal{D}\delta' A\mathcal{D}\delta'\Lambda\mathcal{D}\delta'\overline{\Lambda}\mathcal{D}\delta'\varphi\mathcal{D}\delta'\overline{\varphi}\, \mathrm{e}^{-\tilde{S}}$$

ψ : fermionic zero-modes of the instanton

● Integration of the non-zero-mode is described by the functional determinants, and cancels between the bosonic and the fermionic contribution as a consequent of SUSY, leaving some constant.

$$= N\sum_k \int \mathrm{d}X\mathrm{d}\psi\, \mathrm{e}^{-S_{\mathrm{eff}}[X;\psi]}$$

$S_{\mathrm{eff}}[X;\psi]$: instanton effective action

We want to know the instanton effective action.

→ Need a complete set of instanton solutions with Ω-background.

But we cannot solve EOM exactly in presence of scalar VEVs…

→ Solve the EOM at the leading order in g

$$F^{-(0)}_{mn} = 0$$

$$\bar{\sigma}^m \nabla_m \Lambda^{(0)I} = 0$$

$$\nabla^2 \varphi^{(0)} - \sqrt{2} i \Lambda^{(0)I} \Lambda^{(0)}_I - \nabla^m (F^{(0)}_{mn} \Omega^n) = 0$$

$$\nabla^2 \bar{\varphi}^{(0)} - \nabla^m (F^{(0)}_{mn} \bar{\Omega}_n) = 0$$

$$\sigma^m \nabla_m \bar{\Lambda}^{(0)}_I - \sqrt{2} i [\bar{\varphi}^{(0)}, \Lambda^{(0)}_I] + \sqrt{2} \bar{\Omega}^m \nabla_m \Lambda^{(0)}_I - \frac{1}{\sqrt{2}} \bar{\Omega}^+_{mn} \sigma^{mn} \Lambda^{(0)}_I - \sqrt{2} \bar{\mathcal{A}}^J{}_I \Lambda^{(0)}_I = 0$$

where the SYM fields are expanded in the order of g as follows :

$$A_m = g^{-1} A^{(0)}_m + g^1 A^{(1)}_m + \cdots , \qquad \nabla_m * = \partial_m * + i [A^{(0)}_m, *]$$

$$\Lambda^I = g^{-\frac{1}{2}} \Lambda^{(0)I} + g^{\frac{3}{2}} \Lambda^{(1)I} + \cdots$$

$$\bar{\Lambda}_I = g^{+\frac{1}{2}} \bar{\Lambda}^{(0)}_I + g^{\frac{5}{2}} \bar{\Lambda}^{(1)}_I + \cdots$$

$$\varphi = g^0 \varphi^{(0)} + g^2 \varphi^{(1)} + \cdots$$

$$\bar{\varphi} = g^0 \bar{\varphi}^{(0)} + g^2 \bar{\varphi}^{(1)} + \cdots$$

- Deformed ADHM construction

  : Instanton solution of the EOM at the leading order in g
  in terms of instanton moduli (ADHM moduli)

Bosonic moduli	$(a'_m)_{ij}$	k × k Hermitian	"position moduli"
	$(w^{\dot{\alpha}})_{ui}$	N × k complex	"size moduli"
Fermionic moduli	$(\mathcal{M}'^{I}_{\alpha})_{ij}$	k × k Hermitian matrices	
	$(\mu^A)_{ui}$	N × k complex matrices	
Auxiliary variables	$\chi, \bar{\chi}, D^c$	k × k Hermitian	bosonic
	$\bar{\psi}^{\dot{\alpha}}_I$	k × k Hermitian	fermionic

Auxiliary variables are introduced to impose ADHM constraints between the moduli.

- Exlpicit forms of the solutions

Start with $\Delta_{\lambda j\dot{\alpha}} = a_{\lambda j\dot{\alpha}} + b^{\alpha}_{\lambda j} x_{\alpha\dot{\alpha}}$, $x_{\alpha\dot{\alpha}} \equiv x^m \sigma_{m\alpha\dot{\alpha}}$

Satisfying the "ADHM constraint" $\lambda = 1, \cdots N+2k$

$$\overline{\Delta}^{i\dot{\alpha}\lambda} \Delta_{\lambda j\dot{\beta}} = (f^{-1})^i_j \delta^{\dot{\alpha}}_{\dot{\beta}} \qquad i, j = 1, \cdots k$$

Define (N+2k) × N matrix U as $\overline{\Delta} U = 0$

⟶ Self-dual field strength for SU(N) YM with instanton # k

$$A^{(0)}_m = \frac{-i}{g} \overline{U} \partial_m U \quad \Rightarrow \quad F^{(0)}_{mn} = \frac{-i}{g} 4 b^{\alpha} (\sigma_{mn})_{\alpha}{}^{\beta} f \bar{b}_{\beta} U$$

Canonical form of Δ and the ADHM constraint

$$\Delta_{(u+i\alpha),j\dot{\alpha}} = \begin{pmatrix} w_{uj\dot{\alpha}} \\ (a'_{ij} + \delta_{ij} x^m \sigma_m)_{\alpha\dot{\alpha}} \end{pmatrix}$$

← "size moduli"
← "position moduli"

$$\vec{\tau}^{\dot{\alpha}}{}_{\dot{\beta}} (\overline{w}^{\dot{\beta}} w_{\dot{\alpha}} + \overline{a}'^{\dot{\beta}\alpha} a_{\alpha\dot{\alpha}}) = 0, \qquad a'_m = \overline{a}'_m$$

For the fermion,

$$\Lambda^{(0)I}_{\ \ \ \alpha} = \overline{U}(\mathcal{M}^I f \bar{b}_\alpha - b_\alpha f \bar{\mathcal{M}}^I)U$$

$\mathcal{M}^I_{\lambda j}$: (N+2k) × k fermionic matrices with "fermionic ADHM constraints"

$$\bar{\mathcal{M}}^I \Delta + \overline{\Delta}\mathcal{M}^I = 0 \quad \rightarrow \quad \bar{\sigma}^m \nabla_m \Lambda^{(0)I} = 0$$

Canonical form of Δ and the fermionic ADHM constraint

$$\mathcal{M}^I_{(u+i\alpha)j} = \begin{pmatrix} \mu^I_{uj} \\ (\mathcal{M}'^I_\alpha)_{ij} \end{pmatrix} \rightarrow \begin{cases} \bar{\mu}^I w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot{\alpha}}] = 0 \\ \mathcal{M}'^I = \bar{\mathcal{M}}'^I \end{cases}$$

$w_{\dot{\alpha}}, a'_{\alpha\dot{\alpha}}, \mu^I, \mathcal{M}'^I_\alpha$: ADHM variables (instanton moduli)

Solution for the scalar fields

$$\varphi^{(0)} = -i\frac{\sqrt{2}}{4}\epsilon_{IJ}\overline{U}\mathcal{M}^I f \bar{\mathcal{M}}^J U + \overline{U}\begin{pmatrix}\phi^0 & 0 \\ 0 & \chi \mathbf{1}_2 - i\mathbf{1}_k \Omega^+_{mn}\sigma^{mn}\end{pmatrix} U$$

$$\overline{\varphi}^{(0)} = \overline{U}\begin{pmatrix}\overline{\phi}^0 & 0 \\ 0 & \overline{\chi}\mathbf{1}_2 - i\mathbf{1}_k \overline{\Omega}^+_{mn}\sigma^{mn}\end{pmatrix} U$$

Here $\phi^0, \overline{\phi}^0$ are VEVs of $\varphi, \overline{\varphi}$ respectively and k × k matrices $\chi, \overline{\chi}$ are defined as

$$\boldsymbol{L}\chi = i\frac{\sqrt{2}}{4}\epsilon_{IJ}\bar{\mathcal{M}}^I\mathcal{M}^J + \overline{w}^{\dot{\alpha}}\phi^0 w_{\dot{\alpha}} - i\Omega^{+mn}[a'_m, a'_n]$$

$$\boldsymbol{L}\overline{\chi} = \overline{w}^{\dot{\alpha}}\phi^0 w_{\dot{\alpha}} - i\Omega^{+mn}[a'_m, a'_n]$$

with $\boldsymbol{L}\chi_a \equiv \frac{1}{2}\{\overline{w}^{\dot{\alpha}} w_{\dot{\alpha}}, \chi_a\} + \left[a'_m, [a'^m, \chi_a]\right]$

Expanding the SYM action around the instanton solution, it can be written as :

$$S = \frac{8\pi^2|k|}{g^2} + ik\theta + S_{\text{eff}} + \mathcal{O}(g^2)$$

where

$$S_{\text{eff}} = \int \mathrm{d}^4x\, \mathrm{Tr}\Big[(\nabla_m\varphi^{(0)} - F^{(0)}_{mn}\Omega^n)(\nabla^m\overline{\varphi}^{(0)} - F^{(0)mp}\overline{\Omega}_p) \\ - \frac{i}{\sqrt{2}}\Lambda^{(0)I}\big[\overline{\varphi}^{(0)}, \Lambda^{(0)}_I\big] - \frac{1}{\sqrt{2}}\overline{\mathcal{A}}^J{}_I\Lambda^{(0)I}\Lambda^{(0)}_J \\ + \frac{1}{\sqrt{2}}\overline{\Omega}^m\lambda^{(0)I}\nabla_m\Lambda^{(0)}_I - \frac{1}{2\sqrt{2}}\overline{\Omega}_{mn}\Lambda^{(0)I}\sigma^{mn}\Lambda^{(0)}_J\Big]$$

is the instanton effective action written in terms of SYM fields.

After substituting the ADHM solution to Seff, (and a rather lengthy calculations), we obtain the following instanton effective action.

$$
\begin{aligned}
&S_{\text{eff}} \\
&= 2\pi^2 \mathrm{tr}_k \Big[-2\Big([\bar{\chi}, a'_m] + \bar{\Omega}_{mn} a'^n\Big)\Big([\chi, a'_m] + \Omega^{mp} a'_p\Big) \\
&+ \frac{i}{2\sqrt{2}} \mathcal{M}'^{\alpha I}\Big([\bar{\chi}, \mathcal{M}'_{\alpha I}] - \frac{i}{2}\bar{\Omega}^{+}_{mn}(\sigma^{mn})_\alpha{}^\beta \mathcal{M}'_{\beta I} - i\bar{\mathcal{A}}^J{}_I \mathcal{M}'_{\alpha J}\Big) \\
&- \frac{i}{\sqrt{2}} \bar{\mu}^I \Big(\mu_I \bar{\chi} - \bar{\phi}^0 \mu_I + i\bar{\mathcal{A}}^J{}_I \mu_J\Big) \\
&+ \Big(\bar{\chi}\bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\bar{\phi}^0 - \frac{i}{2}\bar{\Omega}^{-}_{mn}(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{w}^{\dot{\beta}}\Big)\Big(w_{\dot{\alpha}}\chi - \phi^0 w_{\dot{\alpha}} - \frac{i}{2}\Omega^{-}_{mn}(\bar{\sigma}^{mn})^{\dot{\gamma}}{}_{\dot{\alpha}} w_{\dot{\gamma}}\Big) \\
&+ \Big(\chi\bar{w}^{\dot{\alpha}} - \bar{w}^{\dot{\alpha}}\phi^0 - \frac{i}{2}\Omega^{-}_{mn}(\bar{\sigma}^{mn})^{\dot{\alpha}}{}_{\dot{\beta}}\bar{w}^{\dot{\beta}}\Big)\Big(w_{\dot{\alpha}}\bar{\chi} - \bar{\phi}^0 w_{\dot{\alpha}} - \frac{i}{2}\bar{\Omega}^{-}_{mn}(\bar{\sigma}^{mn})^{\dot{\gamma}}{}_{\dot{\alpha}} w_{\dot{\gamma}}\Big) \\
&- i\bar{\psi}^{\dot{\alpha}}_I(\bar{\mu}^I w_{\dot{\alpha}} + \bar{w}_{\dot{\alpha}}\mu^I + [\mathcal{M}'^{\alpha I}, a'_{\alpha\dot{\alpha}}]) + iD^c(\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}}(\bar{w}^{\dot{\beta}} w_{\dot{\alpha}} + \bar{a}'^{\dot{\beta}\alpha} a'_{\alpha\dot{\alpha}})\Big]
\end{aligned}
$$

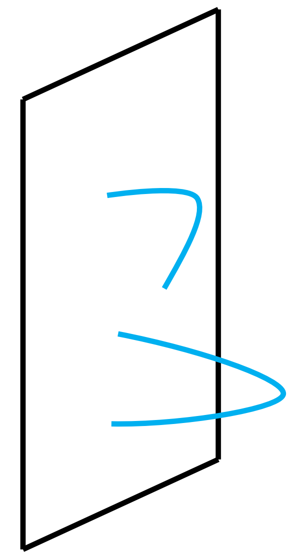
This action also preserves one supersymmetry.

= equivariantly deformed SUSY

Instanton calculus and the string theory

- Put N coincident D3-branes in Type IIB superstring theory.

  → U(N) super Yang-Mills theory as the low-energy eff. theory on the D3-branes



N D3-branes

- Modifying the transverse 6 dim.

Flat	→	$\mathcal{N}=4$ SYM
$\mathbb{R}^2\times(\mathbb{R}^4/\mathbb{Z}_2)$	→	$\mathcal{N}=2$ SYM

- Overviews of the $\mathcal{N} = 2$ stringy instanton calculus

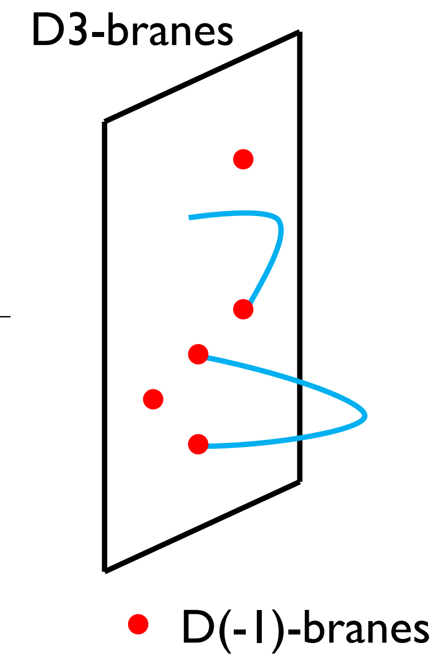
Embed k D(-1)-branes into the D3-branes and compute the disk amplitudes of strings connecting D(-1)/D(-1) and D(-1)/D3 branes.

→ Effective action of the D(-1)-branes

= (equivalent)

k-instanton effective action in the $\mathcal{N} = 2$ SYM

→ We consider that the D(-1)-branes play the role of the instanton from the view point of the D3-branes.



Turn on the self-dual RR 3-form flux and compute the disk amplitudes of D(-1)-flux scattering

→ Deformed D(-1)-brane effective action

=

Instanton eff. action with self-dual Ω-background

to the 2nd order of the deformation

(Billo-Frau-Fucito-Lerda, 0606013)

What corresponds to general Ω-background in string theory?

- Undeformed D(-1)-brane effective action (no RR flux)

Embed k D(-1)-branes into the D3-branes and compute the disk amplitudes of strings connecting D(-1)/D(-1) and D(-1)/D3 branes.

List of the zero-modes

D(-1)/D(-1) strings (U(k) adjoint rep.)

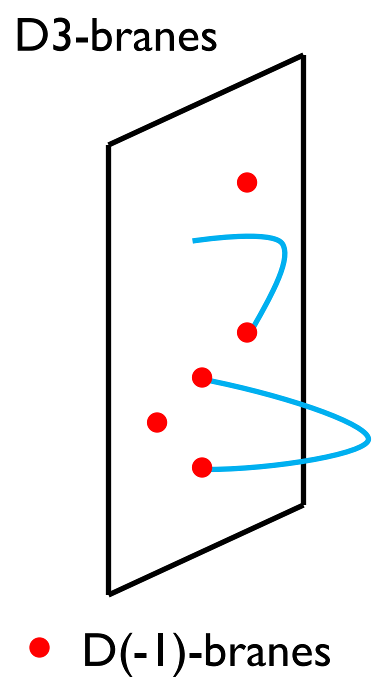
$$a'_m, \mathcal{M}'^I_\alpha, \chi, \overline{\chi}, \overline{\psi}_{\dot{\alpha}I}, \vec{D}$$

D(-1)/D3 strings (U(k) × U(N) bi-fundamental rep.)

$$w_{\dot{\alpha}}, \overline{w}_{\dot{\alpha}}, \mu^I, \overline{\mu}^I$$

Auxiliary fields to disentangle 4-pt. interaction

$$Y_m, Y^\dagger_m, X_{\dot{\alpha}}, \bar{X}_{\dot{\alpha}}, X^\dagger_{\dot{\alpha}}, \bar{X}^\dagger_{\dot{\alpha}}$$



Collecting non-vanishing amplitudes under the field theory limit

$$\alpha' \to 0, g_{\mathrm{D}(-1)} \to \infty \quad \text{with fixed } g_{\mathrm{s}}$$

we obtain the D(-1)-brane effective action.

$$
\begin{aligned}
S_{\mathsf{str}} = 2\pi^2\, \mathrm{tr}_k \Big[& 2Y^m Y_m^\dagger - X_{\dot\alpha}\bar{X}^{\dagger\dot\alpha} - X^\dagger_{\dot\alpha}\bar{X}^{\dot\alpha} + 2Y_m[\overline{\chi}, a'_m] + 2Y^{\dagger m}[\chi, a'_m] \\
& - X_{\dot\alpha}(\overline{\chi}\overline{w}^{\dot\alpha} - \overline{w}^{\dot\alpha}\bar{\phi}^0) - X^\dagger_{\dot\alpha}(\chi\overline{w}^{\dot\alpha} - \overline{w}^{\dot\alpha}\phi^0) \\
& - (w_{\dot\alpha}\overline{\chi} - \bar{\phi}^0 w_{\dot\alpha})\overline{X}^{\dot\alpha} - (w_{\dot\alpha}\chi - \phi^0 w_{\dot\alpha})X^{\dagger\dot\alpha}{}_a \\
& + \frac{\sqrt{2}}{2} i\epsilon_{IJ}\bar{\mu}^I(-\mu^J\bar{\chi} + \bar{\phi}^0\mu^J) + \frac{\sqrt{2}}{4} i\epsilon_{IJ}\mathcal{M}'^{\alpha I}[\overline{\chi}, \mathcal{M}'^J_\alpha] \\
& - i\overline{\psi}^{\dot\alpha}_I(\overline{\mu}^I w_{\dot\alpha} + \overline{w}_{\dot\alpha}\mu^I + [\mathcal{M}'^{\alpha A}, a'_{\alpha\dot\alpha}]) \\
& + iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}(\overline{w}^{\dot\beta} w_{\dot\alpha} + \overline{a}'^{\dot\beta\alpha} a'_{\alpha\dot\alpha}) \Big]
\end{aligned}
$$

Eliminating the auxiliary fields X and Y, we get precisely the same action as the undeformed part of instanton effective action.

$$
S_{\mathsf{str}} = \tilde{S}_{\mathsf{eff}}(\Omega, \overline{\Omega}, \mathcal{A}, \overline{\mathcal{A}} = 0)
$$

Deformed instanton eff. action (string theory)

Turn on R-R 3-form background
→ Scattering of string between R-R and D(-1) deforms the D(-1) eff. action.

Decomposition of 10D R-R field strength

$\mathcal{F}^{\hat{A}\hat{B}}$ $\hat{A}, \hat{B}$: 10D spinor index (16 component)

↓

$$\mathcal{F}^{\alpha\beta AB} \quad \rightarrow \quad \mathcal{F}_a, \boxed{\mathcal{F}_{abc}}, \boxed{\mathcal{F}_{mna}}, \mathcal{F}_{mnabc}$$

$$\mathcal{F}^{\alpha}{}_{\dot\beta}{}^{A}{}_{B} \quad \rightarrow \quad \mathcal{F}_m, \mathcal{F}_{mab}$$

$$\mathcal{F}_{\dot\alpha}{}^{\beta}{}_{A}{}^{B} \quad \rightarrow \quad \mathcal{F}_m, \mathcal{F}_{mab}$$

$$\mathcal{F}_{\dot\alpha\dot\beta AB} \quad \rightarrow \quad \mathcal{F}_a, \boxed{\mathcal{F}_{abc}}, \boxed{\mathcal{F}_{mna}}, \mathcal{F}_{mnabc}$$

m,n : 4D spacetime indices (D3-brane)
a,b,c : 6D spacetime indices (transverse)
A,B : 6D spinor indices (4 component)

Details of decomposition : (anti-)symmetrization of spinor indices

$$\mathcal{F}^{\alpha\beta AB} \rightarrow \epsilon^{\alpha\beta}(\Sigma^a)^{AB}\mathcal{F}_a, \quad \epsilon^{\alpha\beta}(\Sigma^a\overline{\Sigma}^b\Sigma^c)^{AB}\mathcal{F}_{abc},$$
$$(\sigma^{mn})^{\alpha\beta}(\Sigma^a)^{AB}\mathcal{F}^+_{mna}, \quad (\sigma^{mn})^{\alpha\beta}(\Sigma^a\overline{\Sigma}^b\Sigma^c)^{AB}\mathcal{F}_{mnabc}$$

$$\mathcal{F}_{\dot{\alpha}\dot{\beta}AB} \rightarrow \epsilon_{\dot{\alpha}\dot{\beta}}(\overline{\Sigma}^a)_{AB}\mathcal{F}_a, \quad \epsilon_{\dot{\alpha}\dot{\beta}}(\overline{\Sigma}^a\Sigma^b\overline{\Sigma}^c)_{AB}\bar{\mathcal{F}}_{abc},$$
$$(\overline{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}(\overline{\Sigma}^a)_{AB}\mathcal{F}^-_{mna}, \quad (\overline{\sigma}^{mn})_{\dot{\alpha}\dot{\beta}}(\overline{\Sigma}^a\Sigma^b\overline{\Sigma}^c)_{AB}\mathcal{F}_{mnabc}$$

$\Sigma, \overline{\Sigma}$: 6D sigma matrices

After orbifold projection, surviving R-R 3-forms among the above are

$$\mathcal{F}^+_{mna} \rightarrow \mathcal{F}^+_{mn}, \bar{\mathcal{F}}^+_{mn}$$
$$\mathcal{F}^-_{mna} \rightarrow \mathcal{F}^-_{mn}, \bar{\mathcal{F}}^-_{mn}$$

} (S,A)-type background

$$\mathcal{F}^{(AB)} \rightarrow \mathcal{F}^{(IJ)}, \mathcal{F}'^{(I'J')}$$
$$\bar{\mathcal{F}}_{(AB)} \rightarrow \bar{\mathcal{F}}_{(IJ)}, \bar{\mathcal{F}}'_{(I'J')}$$

} (A,S)-type background

where $I, J = 1, 2, \quad I', J' = 3, 4$

Under the zero-slope limit, $(2\pi\alpha')^{1/2}\mathcal{F}$ is kept finite.

$$(2\pi\alpha')^{1/2}\mathcal{F}_{(\mathrm{S,A})} \sim C_{mn}$$
$$(2\pi\alpha')^{1/2}\mathcal{F}_{(\mathrm{A,S})} \sim m_{(IJ)}$$

Collecting the non-vanishing string amplitudes including R-R 3-forms, the new interaction terms are

$$\begin{aligned}\delta\tilde{S} = 2\pi^2\,\mathrm{tr}_k\Big[&2C^{+mn}Y_m^\dagger a'_n + \bar{C}^{+mn}Y_m a'_n - \frac{\sqrt{2}}{2}i\epsilon_{IJ}\mathcal{M}'^I_\alpha\mathcal{M}'^J_\beta\bar{C}^{+(\alpha\beta)} \\ &+ 2Y_m a'_n\bar{C}^{-mn} + 2Y_m^\dagger a'_n C^{-mn} \\ &+ \frac{1}{2}X_{\dot{\alpha}}\overline{w}_{\dot{\beta}}(\overline{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}\bar{C}^{-mn} + \frac{1}{2}X_{\dot{\alpha}}^\dagger\overline{w}_{\dot{\beta}}(\overline{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}C^{-mn} \\ &+ \frac{1}{2}w_{\dot{\alpha}}\bar{X}_{\dot{\beta}}(\overline{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}\bar{C}^{-mn} - \frac{1}{2}w_{\dot{\alpha}}\bar{X}_{\dot{\beta}}^\dagger(\overline{\sigma}_{mn})^{\dot{\alpha}\dot{\beta}}C^{-mn} \\ &- \mathcal{M}'^{\alpha I}\mathcal{M}'^J_\alpha m_{(IJ)} - 2\overline{\mu}^I\mu^J m_{(IJ)}\Big]\end{aligned}$$

Combining them with the undeformed D(-1) action and integrating out the auxiliary fields X and Y, the final result is …

$$
\begin{aligned}
&S_{\text{str}}(C,\bar{C},m)\\
&= 2\pi^2 \mathrm{tr}_k\Big[-2\Big([\bar{\chi},a'_m]+(\bar{C}^+_{mn}+\bar{C}^-_{mn})a'^n\Big)\Big([\chi,a'_m]+(C^{+mp}+C^{-mp})a'_p\Big)\\
&+\frac{i}{2\sqrt{2}}\mathcal{M}'^{\alpha I}\Big([\bar{\chi},\mathcal{M}'_{\alpha I}]+\frac{1}{2}\bar{C}^+_{mn}(\sigma^{mn})_\alpha{}^\beta\mathcal{M}'_{\beta I}+2\sqrt{2}im_{(IJ)}\mathcal{M}'^J_\alpha\Big)\\
&+\frac{i}{\sqrt{2}}\bar{\mu}^I\Big(-\mu_I\bar{\chi}+\bar{\phi}^0\mu_I+2\sqrt{2}im_{(IJ)}\mu^J\Big)\\
&+\Big(\bar{\chi}\bar{w}^{\dot\alpha}-\bar{w}^{\dot\alpha}\bar{\phi}^0+\frac{1}{2}\bar{C}^-_{mn}(\bar{\sigma}^{mn})^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta}\Big)\Big(w_{\dot\alpha}\chi-\phi^0w_{\dot\alpha}+\frac{1}{2}C^-_{mn}(\bar{\sigma}^{mn})^{\dot\gamma}{}_{\dot\alpha}w_{\dot\gamma}\Big)\\
&+\Big(\chi\bar{w}^{\dot\alpha}-\bar{w}^{\dot\alpha}\phi^0+\frac{1}{2}C^-_{mn}(\bar{\sigma}^{mn})^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta}\Big)\Big(w_{\dot\alpha}\bar{\chi}-\bar{\phi}^0w_{\dot\alpha}+\frac{1}{2}\bar{C}^-_{mn}(\bar{\sigma}^{mn})^{\dot\gamma}{}_{\dot\alpha}w_{\dot\gamma}\Big)\\
&-i\bar{\psi}^{\dot\alpha}_I(\bar{\mu}^Iw_{\dot\alpha}+\bar{w}_{\dot\alpha}\mu^I+[\mathcal{M}'^{\alpha I},a'_{\alpha\dot\alpha}])+iD^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}(\bar{w}^{\dot\beta}w_{\dot\alpha}+\bar{a}'^{\dot\beta\alpha}a'_{\alpha\dot\alpha})\Big]
\end{aligned}
$$

where $C_{mn}=C^+_{mn}+C^-_{mn},\ \bar{C}_{mn}=\bar{C}^+_{mn}+\bar{C}^-_{mn}$

This action agrees with the deformed instanton effective action when we identify

$$C_{mn}=-i\Omega_{mn},\qquad \bar{C}_{mn}=-i\bar{\Omega}_{mn},\qquad m_{(IJ)}=-\frac{1}{2\sqrt{2}}\epsilon_{IK}\bar{\mathcal{A}}^K{}_J$$

and preserves one supersymmetry

= Nekrasov’s equivariant BRST operator

Conclusions

- We derived the $\mathcal{N} = 2$ SYM deformed by Ω-background.
  → Found one preserved SUSY = Equivariantly deformed SUSY

- We constructed instanton solutions with general Ω-background.
  → ADHM construction deformed by Ω-background
  → Instanton effective action deformed by Ω-background

- We computed string scattering amplitudes between D(-1)/D3 system and with R-R 3-form backgrounds.
  → Deformed D(-1)-brane effective action

- We showed that the instanton effective action deformed by Ω-background precisely coincides with
  the deformed D(-1)-brane effective action deformed by two types of R-R 3-form background.

Further topics

- The meaning of R-R 3-forms in terms of 4D field theory
  … self-dual Ω → graviphoton field strength
  ASD parameters → ??? (vector multiplet?)

- Similar calculations for $\mathcal{N}$ =2* and $\mathcal{N}$ =4 SYM
  → Extended Ω-background for $\mathcal{N}$ =4 SYM is needed.

- Apply to higher dimensional instanton calculus
  …D6/D0, D7/D3/D(-1)-brane systems, etc.