Volume of Moduli Space of Vortices and Localization Formula

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Introduction

Topic:

* Calculation of the volume of the moduli space of BPS solitons

Moduli space?

* Kähler or hyper-Kähler quotient space

  \{\text{Solutions of F & D-term constraints}\} / \{\text{Gauge symmetry}\}

BPS soliton

* Abelian/non-Abelian vortex on a \textit{compact} Riemann surface with genus $h$ ($\Sigma_h$)
Introduction

BPS equations ($G = U(N_c)$ and $N_f$ flavors):

\[
\mu_r \equiv F - \frac{g^2}{2} (c - HH^\dagger) \omega = 0 \\
\mu_{\bar{z}} \equiv \mathcal{D}_{\bar{z}} H = 0 \\
\mu_z \equiv \mathcal{D}_z H^\dagger = 0
\]

where $g$: gauge coupling, $c$: FI parameter, $\omega$: Kähler 2-form on $\Sigma_h$, $H$: $N_c \times N_f$ matrix.

\[\mathcal{M}_k \equiv \{ \text{Solutions of } \mu_r = \mu_z = \mu_{\bar{z}} = 0 \text{ with } \frac{1}{2\pi} \int F = k \} \quad U(N_c)\]

Volume of $\mathcal{M}_k$?
Introduction

The volume of moduli space of the BPS solitons relates to

* Non-perturbative corrections in supersymmetric gauge theory (Nekrasov’s formula)

* Thermodynamical partition function of the BPS solitons (Manton et al.)

\[ Z = \frac{1}{\hbar^{2k}} \int d^k p d^k x \ e^{-\frac{T}{2} g^{ij} p_i p_j} = \left( \frac{2\pi^2 T}{\hbar^2} \right)^k \text{Vol}(\mathcal{M}_k) \]

The volume of moduli space of the BPS solitons (supersymmetric systems) is a key to understand the non-perturbative dynamics and dualities in gauge/string theory
Method

Straightforwardly,

BPS eqs. ⇒ BPS solutions ⇒ effective action ⇒ metric ⇒ volume

Instead, we define the field theoretical partition function to obtain the volume of the moduli space [Moore-Nekrasov-Shatashvili (1997)]:

\[
Z^{N_c,N_f}_k (\Sigma_h) = \int \mathcal{D} \Phi \mathcal{D}^2 A \mathcal{D}^2 \lambda \mathcal{D}^2 H \mathcal{D}^2 \psi \mathcal{D}^2 Y \mathcal{D}^2 \chi e^{-S_0 - S_1}
\]

where

\[
S_0 = \int_{\Sigma_h} \text{Tr} \left[ i\Phi \left\{ F - \frac{g^2}{2}(c - HH^\dagger)\omega \right\} + \frac{1}{2} \lambda \wedge \lambda + \frac{g^2}{2} \psi \psi^\dagger \omega \right]
\]

\[
S_1 = \int_{\Sigma_h} d^2 z \text{Tr} \left[ g^{z\bar{z}} (Y_z Y_{\bar{z}} + i\Phi \chi_{z} \chi_{\bar{z}}) \right] + \cdots
\]
Method

This is essentially a constrained system on the moduli space of the vortex

\[ Z_{k}^{N_c, N_f} = \int \mathcal{D}^2 A \mathcal{D}^2 H \, \delta(\mu_r) J_r \, \delta(\mu_z) J_z \, \delta(\mu_{\bar{z}}) J_{\bar{z}} \cdots \]

\[ = \text{Vol}(\mathcal{M}_{k}^{N_c, N_f}) \]

We can perform the path integral, which reduces to residue integrals over zero modes of Lagrange multiplier field \( \Phi \)

\[
Z_k^{N_c, N_f}(\Sigma_h) = \sum_{\sum_a k_a = k} (-1)^{\sigma} \int \prod_a \frac{d\phi_a}{2\pi} \frac{\prod_a (1 + \frac{N_f}{2\pi i \phi_a})^h \prod_{a<b} (i\phi_a - i\phi_b)^{2-2h}}{\prod_a (i\phi_a)^{N_f(1-h+k_a)}} \\
\times e^{2\pi i \sum_a \phi_a \left(\frac{g^2}{4\pi} A - k_a\right)}
\]

Integrals are localized at poles (Localization formula)
Results

\( N_c=N_f=1 \) (Abrikosov–Nielsen–Olesen vortex)

\[ \mathcal{Z}^{1,1}_k(\Sigma_h) = (2\pi)^{k-h} \sum_{j=0}^{h} \frac{h!}{j!(k-j)!(h-j)!} \left( \frac{g^2 c}{4\pi} A - k \right)^{k-j} \]

We find

\[ A \geq \frac{4\pi}{g^2 c} k \quad \text{Bradlow limit} \]

For \( h=0 \) (sphere)

\[ \mathcal{Z}_k(S^2) = \frac{(2\pi)^k}{k!} \left( \frac{g^2 c}{4\pi} A - k \right)^k \xrightarrow{A \to \infty} \text{Vol}((S^2)^k/S_k) \sim \frac{A^k}{k!} \]

Eq of state: \( P \left( A - \frac{4\pi}{g^2 c} k \right) = kT \)
Results

\( N_c=2, \ N_f \) (non-Abelian semi-local vortex) on the sphere

\[
\mathcal{Z}^{2,N_f}_{0} (S^2) = \frac{2!}{(N_f - 1)! (N_f - 2)!} (2\pi \tilde{A})^{2(N_f - 2)}
\]

\[
\mathcal{Z}^{2,N_f}_{1} (S^2) = \frac{(2\pi)^{3N_f - 4}}{(2N_f - 1)(N_f - 1)! (2N_f - 3)!} \tilde{A}^{N_f - 3} (\tilde{A} - 1)^{2N_f - 3}
\times \left( (N_f - 2)\tilde{A}^2 + 2(N_f + 1)\tilde{A} + (N_f - 2) \right)
\]

\[
\mathcal{Z}^{2,N_f}_{2} (S^2) = 2(2\pi)^{4N_f - 4} \left[ \frac{-2}{(N_f - 1)! (3N_f - 1)!} \tilde{A}^{N_f - 3} (\tilde{A} - 2)^{3N_f - 3}
\times \left( (2N_f^2 - 2N_f + 1)\tilde{A}^2 + 2(2N_f + 1)(N_f - 1)\tilde{A} + 2(N_f - 1)(N_f - 2) \right)
\right.
\left. + \frac{1}{(2N_f - 1)! (2N_f - 2)!} (\tilde{A} - 1)^{4N_f - 4} \right]
\]

where \( \tilde{A} \equiv \frac{g^2 c}{4\pi} A \)
Comments

On the sphere, $k=0$ (perturbative part) gives the volume of the vacuum moduli space

$$
\mathcal{Z}_0^{N_c,N_f}(S^2) = N_c! \times \text{Vol}(G_{N_c,N_f}) \tilde{A}^{N_c(N_f-N_c)}
$$

Vol. of Grassmannian

The case of $N_c=N_f$ (non-Abelian local vortex) is rather special

$$
\mathcal{Z}_0^{2,2}(S^2) = 2 \\
\mathcal{Z}_1^{2,2}(S^2) = 2 \times (2\pi)^2(\tilde{A} - 1) \\
\mathcal{Z}_2^{2,2}(S^2) = 2 \times \frac{(2\pi)^4}{2!} \left( \tilde{A}^2 - \frac{20}{6} \tilde{A} + \frac{17}{6} \right)
$$

In the limit of $\mathcal{A} \to \infty$, the divergence comes from the position moduli only
Conclusion

- We evaluate the volume of the moduli space of BPS vortices on the Riemann surface by using the localization formula.

- We can see not only the volume itself but also the geometrical structure of the moduli space.

- Using the deconstruction, we can also apply the localization method to the solitons in the Chern-Simons-Higgs system, etc.

- The volume of the moduli space (aka localization formula) is a key to understand the dualities between gauge theories, string theories, matrix models, and more!