Extended Conformal Symmetry and Recursion Formulae for Nekrasov Partition Function

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Introduction and summary

- We derive infinite many recursion formulae of Nekrasov partition function for 4D N=2 U(N) linear quiver guage theory, which relates those have different instanton number.

- But their meaning is obscure in terms of gauge theory.

- AGT conjecture insist that

  U(N) Nekrasov function = W_N conformal block

  Instanton number = level of descendant state

Introduction and summary

- In 2D CFT, Ward identity is natural formula that relate different descendant contribution (= different instanton contribution) to conformal block function (= Nekrasov function)

$\Downarrow$

- We try to understand the recursion formulae as Ward identity and find that they can be interprited as the action of SH^c algebra (or degenerate double affine Hecke algebra)

This work is generalization of our previous paper (arXiv:1207.5658) for arbitrary Ω-background.

Contents

Recursion relation for Nekrasov partition function

Biulding block of Nekrasov partition function

$U(N)\times U(N)\times\cdots\times U(N)$ N=2 conformal linear quiver

vector, antifund. hyper

$$Z_{\text{inst}}=\sum_{\{\vec{Y}_1,\cdots,\vec{Y}_n\}}\left(\prod_{k=1}^{n} q_k^{|\vec{Y}_k|} z_{\text{vec}}(\vec{\hat{a}}_k,\vec{Y}_k)\right)\left(\prod_{\bar{p}=1}^{d_1} z_{\text{afd}}(\vec{\hat{a}}_1,\vec{Y}_1,\bar{\mu}_{\bar{p}})\right)$$

$$\times\left(\prod_{k=1}^{n-1} z_{\text{bfd}}(\vec{\hat{a}}_k,\vec{Y}_k;\vec{\hat{a}}_{k+1},\vec{Y}_{k+1};m_k)\right)\left(\prod_{p=1}^{d_n} z_{\text{fd}}(\vec{\hat{a}}_n,\vec{Y}_n,\mu_p)\right)$$

bifund.hyper, fund.hyper

$q_k=e^{2\pi i\tau_k}$

$\vec{Y}_k$ N-tuple Young diagram
\# of Instanton = \# of box

mass of matter filed

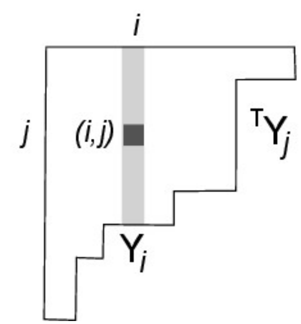
vev of adjoint scalar

$$z_{\text{bfd}}(\vec{\hat{a}},\vec{Y};\vec{\hat{b}},\vec{W};m)=\prod_{i,j}\prod_{s\in Y_i}(E(\hat{a}_i-\hat{b}_j,Y_i,W_j,s)-m)\times\prod_{t\in W_j}(\epsilon_+-E(\hat{b}_j-\hat{a}_i,W_j,Y_i,t)-m)$$

$$z_{\text{vec}}(\vec{\hat{a}},\vec{Y})=1/z_{\text{bfd}}(\vec{\hat{a}},\vec{Y};\vec{\hat{a}},\vec{Y};0)$$

$$z_{\text{fd}}(\vec{\hat{a}},\vec{Y},\mu)=\prod_{i}\prod_{s\in Y_i}(\phi(\hat{a}_i,s)-\mu+\epsilon_+)\quad z_{\text{afd}}(\vec{\hat{a}},\vec{Y},\bar{\mu})=z_{\text{fd}}(\vec{\hat{a}},\vec{Y},\epsilon_+-\bar{\mu})$$

Y



$$\epsilon_+=\epsilon_1+\epsilon_2\qquad E(\hat{a},Y,W,s)=\hat{a}-\epsilon_1(\lambda'_{W,j}-i)+\epsilon_2(\lambda_{Y,i}-j+1)$$

$$\phi(\hat{a},s)=\hat{a}+\epsilon_1(i-1)+\epsilon_2(j-1)$$

Building block of Nekrasov partition function

Rewrite Nekrasov partition function as

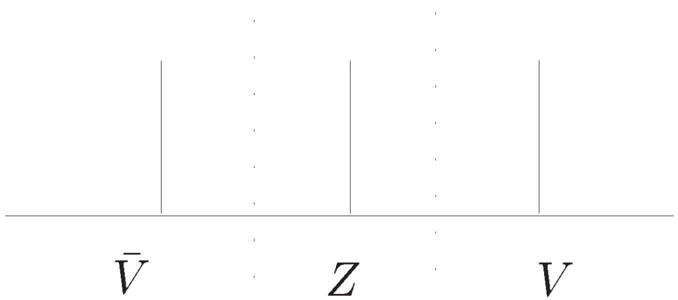
$$Z_{\text{inst}} = \sum_{\vec{Y}^{(1)},\cdots,\vec{Y}^{(n)}} q_i^{|\vec{Y}^{(i)}|} \bar{V}_{\vec{Y}^{(1)}} \cdot Z_{\vec{Y}^{(1)}\vec{Y}^{(2)}} \cdots Z_{\vec{Y}^{(n-1)}\vec{Y}^{(n)}} \cdot V_{\vec{Y}^{(n)}}$$

$$Z_{\vec{Y}^{(i)}\vec{Y}^{(i+1)}} = Z(\vec{a}^{(i)}, \vec{Y}^{(i)}; \vec{a}^{(i+1)}, \vec{Y}^{(i+1)}; \mu^{(i)})$$

Building block

$$\boxed{Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu) = \sqrt{z_{\text{vect}}(\vec{a}, \vec{Y}) z_{\text{vect}}(\vec{b}, \vec{W})}\, z_{\text{bifund}}(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu)}$$

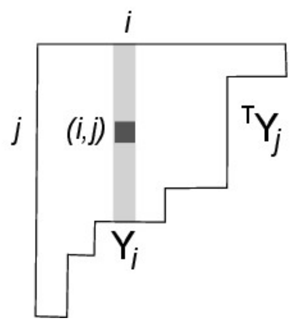
$$\bar{V}_{\vec{Y}^{(1)}} = Z(\vec{\lambda}, \vec{\emptyset}; \vec{a}^{(1)}, \vec{Y}^{(1)}; \mu^{(0)}) \qquad V_{\vec{Y}^{(n)}} = Z(\vec{a}^{(n)}, \vec{Y}^{(n)}; \vec{\lambda}', \vec{\emptyset}; \mu^{(n)})$$



Notation

$$\epsilon_1/\epsilon_2 = -\beta$$

Y



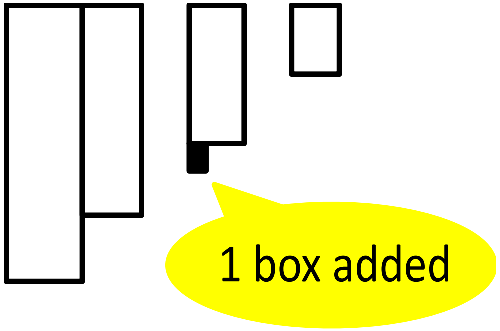
generalized hook length

$$c(s,Y) = \beta i - j \quad \text{for} \quad s=(i,j)\in Y$$

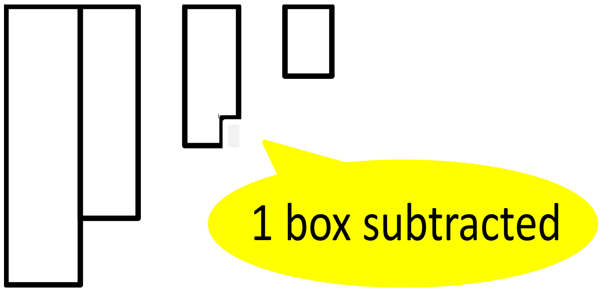
$Y^{(k,+)}$:One box added diagram at k-th rectangle of Y

$Y^{(k,-)}$:One box subtracted diagram at k-th rectangle of Y

$Y^{(k,+)}$



$Y^{(k,-)}$



Notation

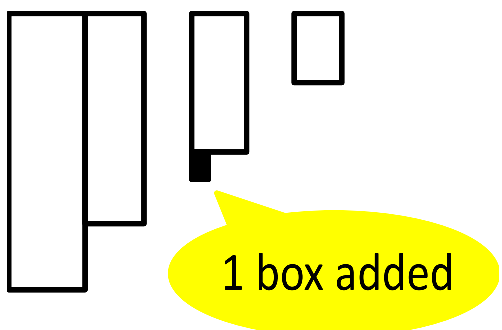
For U(N) instanton $\vec{Y}$ with adjoint scalar vev $\vec{a}$

$$A_k(Y_p) = c(s, Y_p) \quad s \in Y_p^{(k,+)} \setminus Y_p$$

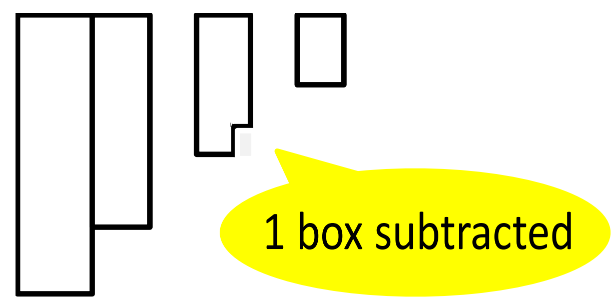
$$B_k(Y_p) = c(s, Y_p) \quad s \in Y_p \setminus Y_p^{(k,-)}$$

(generalized hook length assosiated with the added/subtracted box)

$Y^{(k,+)}$



$Y^{(k,-)}$



Recursion formulae

By explicit calculation, we find the following relation

$$\boxed{\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0}$$

where

$$\delta_{1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N} \Bigg(- \sum_k (a_p + B_k(Y_p))^n \Lambda_p^{(k,-)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,-),p};\vec{b},\vec{W};\mu)$$

$$+ \sum_k (b_p + \mu + A_k(W_p) + \xi)^n \Lambda_p^{(k,+)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,+),p};\mu) \Bigg)$$

$$U_{1,n} = \beta^{-1/2} q_{n+1}(\{a_p + B_k(Y_p)\},\{b_p + \mu + A_k(W_p)\}) \qquad \prod_{I=1}^{\mathcal{N}} \frac{\zeta - y_I}{\zeta - x_I} = 1 + \sum_{n=1}^{\infty} q_n(\{x_I\},\{y_I\})\zeta^{-n}$$

$$\Lambda_p^{(k,+)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell} \frac{a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + A_k(Y_p) - B_\ell(Y_q)} \prod{}'_{\ell} \frac{a_p - a_q + A_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + A_k(Y_p) - A_\ell(Y_q)} \right) \right)^{1/2}$$

$$\Lambda_p^{(k,-)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q+1} \frac{a_p - a_q + B_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + B_k(p) - A_\ell(q)} \prod{}'^{f_q}_{\ell=1} \frac{a_p - a_q + B_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + B_k(Y_p) - B_\ell(Y_q)} \right) \right)^{1/2}$$

Recursion formulae

By explicit calculation, we find the following relation

$$\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0$$

where

$$\delta_{-1,n} Z(\vec{a},\vec{Y};\vec{b},\vec{W};\mu) = \sum_{p=1}^{N} \left(\sum_{k=1} (a_p + A_k(Y_p))^n \Lambda_p^{(k,+)}(\vec{a},\vec{Y}) Z(\vec{a},\vec{Y}^{(k,+),p};\vec{b},\vec{W};\mu) \right.$$
$$\left. - \sum_{k=1} (b_p + \mu + B_k(W_p))^n \Lambda_p^{(k,-)}(\vec{b},\vec{W}) Z(\vec{a},\vec{Y};\vec{b},\vec{W}^{(k,-),p};\mu) \right)$$

$$U_{-1,n} = \beta^{-1/2} q_{n+1}(\{a_p + A_k(Y_p)\},\{b_p + \mu + B_k(W_p)\}) \qquad \prod_{I=1}^{\mathcal{N}} \frac{\zeta - y_I}{\zeta - x_I} = 1 + \sum_{n=1}^{\infty} q_n(\{x_I\},\{y_I\}) \zeta^{-n}$$

$$\Lambda_p^{(k,+)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q} \frac{a_p - a_q + A_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + A_k(Y_p) - B_\ell(Y_q)} \prod_{\ell=1}^{\prime f_q+1} \frac{a_p - a_q + A_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + A_k(Y_p) - A_\ell(Y_q)} \right) \right)^{1/2}$$

$$\Lambda_p^{(k,-)}(\vec{a},\vec{Y}) = \left(\prod_{q=1}^{N} \left(\prod_{\ell=1}^{f_q+1} \frac{a_p - a_q + B_k(Y_p) - A_\ell(Y_q) - \xi}{a_p - a_q + B_k(p) - A_\ell(q)} \prod_{\ell=1}^{\prime f_q} \frac{a_p - a_q + B_k(Y_p) - B_\ell(Y_q) + \xi}{a_p - a_q + B_k(Y_p) - B_\ell(Y_q)} \right) \right)^{1/2}$$

Recursion formulae

$$\boxed{\delta_{\pm 1,n} Z_{\vec{Y},\vec{W}} - U_{\pm 1,n} Z_{\vec{Y},\vec{W}} = 0}$$

We want to understand it in terms of 2D CFT point of view, especially as Ward identity for “three-point function”

$\longrightarrow$ SH^c algebra

(or degenerate double affine Hecke algebra)

Two-dimensional interpretation

SH^c algebra

generator

$$D_{n,l}$$

spin l+1

level

fundamental commutation relation

$$[D_{0,l}, D_{1,k}] = D_{1,l+k-1} \quad l \geq 1$$

$$[D_{0,l}, D_{-1,k}] = -D_{-1,l+k-1}, \quad l \geq 1$$

$$[D_{-1,k}, D_{1,l}] = E_{k+l} \quad l, k \geq 1$$

$$1 + (1-\beta) \sum_{l\geq 0} E_l s^{l+1} = \exp\Big(\sum_{l\geq 0} (-1)^{l+1} c_l \phi_l(s)\Big) \exp\Big(\sum_{l\geq 0} D_{0,l+1} \varphi_l(s)\Big)$$

non linear

$$\phi_l(s) = s^l G_l(1 + (1-\beta)s) \quad \varphi_l(s) = \sum_{q=1,-\beta,\beta-1} s^l \big(G_l(1-qs) - G_l(1+qs)\big)$$

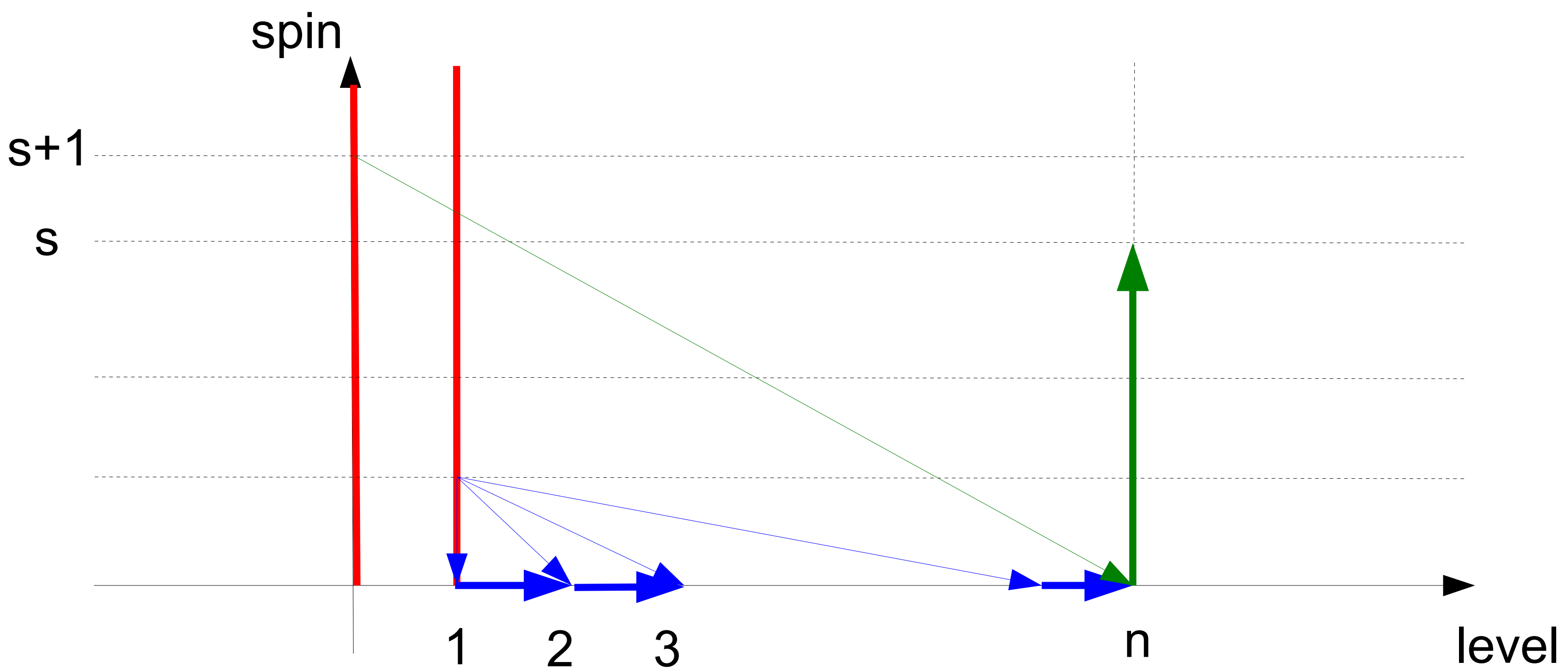
$$G_0(s) = -\log(s), \quad G_l(s) = (s^{-l} - 1)/l \quad l \geq 1$$

SH^c algebra

higher level generators are defined by

$$[D_{1,1}, D_{l,0}] = l D_{l+1,0} \quad [D_{-l,0}, D_{-1,1}] = l D_{-l-1,0}$$

$$[D_{0,l+1}, D_{r,0}] = D_{r,l} \quad [D_{-r,0}, D_{0,l+1}] = D_{-r,l}$$



Representation of SH^c algebra

◆representation space

$$\mathcal{H}_{\vec{b}} = \{|\vec{b}, \vec{W} > |\vec{W} : \text{N-tuple Young diagram}\}$$

◆action of the generators

$$D_{-1,l}|\vec{b}, \vec{W} >= (-1)^l \sum_{q=1}^{N} \sum_{k} (b_q + B_k(W_q))^l \Lambda(\vec{W}, q; k)|\vec{b}, \vec{W}_p^{(k,-)} >$$

$$D_{1,l}|\vec{b}, \vec{W} >= (-1)^l \sum_{q=1}^{N} \sum_{k} (b_q + A_k(W_q))^l \bar{\Lambda}(\vec{W}, q; k)|\vec{b}, \vec{W}_q^{(k,+)} >$$

$$D_{0,l+1}|\vec{b}, \vec{W} >= (-1)^l \sum_{q=1}^{N} \sum_{s \in W_q} (b_q + c(s, W_q))^l |\vec{b}, \vec{W} >$$

U(1) × Virasoro subalgebra of $\mathbf{SH^c}$ algebra

◆ U(1) current

$$J_n = D_{-n,0}, \quad J_{-n} = \beta^{-n} D_{n,0}, \quad J_0 = E_1/\beta$$

◆ Virasoro algebra

$$L_n = D_{-n,1}/n + (1-n)c_0(1-\beta)J_n/2$$

$$L_{-n} = D_{n,1}/n + (1-n)c_0(1-\beta)J_{-n}/2 \quad L_0 = [L_1, L_{-1}]/2$$

In our representation

Virasoro central charge $c = N - Q^2(N^3 - N) \quad Q = \sqrt{\beta} - 1/\sqrt{\beta}$

- ◆ independent of momenta
- ◆ same as the central charge of $U(1) \times W_N$

U(1) × Virasoro subalgebra of $\mathbf{SH^c}$ algebra

◆ eigen value of zero mode

$$J_0|\vec{a},\vec{Y}\rangle = \frac{1}{\beta}\left(-\sum_i(a_i-\xi)+\frac{\xi N(N-1)}{2}\right)|\vec{a},\vec{Y}\rangle = \frac{1}{\sqrt{\beta}}(\vec{p}\cdot\vec{e})\,|\vec{a},\vec{Y}\rangle \qquad \vec{e}=(1,1,\cdots,1)$$

$$L_0|\vec{a},\vec{Y}\rangle = \left(|\vec{Y}|+\frac{1}{2\beta}\left(\sum_i(a_i-\xi)^2+(1-N)\xi\sum_i(a_i-\xi)+\frac{\xi^2}{6}N(N-1)(N-2)\right)\right)|\vec{a},\vec{Y}\rangle$$

$$= \left(|\vec{Y}|+\Delta(\vec{p})\right)|\vec{a},\vec{Y}\rangle$$

where $\quad p_i := -\frac{a_i}{\sqrt{\beta}} - Qi\,, \quad i=1,\cdots,N$

$$\Delta(\vec{p}) := \frac{\vec{p}\cdot(\vec{p}-2Q\vec{\rho})}{2} \qquad \vec{\rho} : A_{N-1} \text{ Weyl vector}$$

Conformal dimension of the state with Toda momentum $\vec{p}$

"Vertex operator"

- To understand the recursion formulae as Ward identity, we must know a commutater between "vertex operator" and the generators.

- Vertex oprerator is product of W_N part V^W and U(1) part $\tilde{V}^H$

$$V = \tilde{V}^H V^W$$

- According to AGT conjeture, V^W is W_N primary field

  with the Toda momentum $\frac{\kappa}{N}(1, 1, \cdots, 1-N)$

  Conformal dimension $\Delta_W = \frac{\kappa(\kappa - Q(N-1))}{2} - \frac{\kappa^2}{2N}$

U(1) part vertex operator

U(1) vertex operator must satisfy the following two conditions

(1) reproduces correct U(1) factor

$$\langle \tilde{V}^H_{\kappa_1}(z_1)\cdots\tilde{V}^H_{\kappa_n}(z_n)\rangle = \prod_{i<j}(z_i - z_j)^{\frac{-\kappa_i(NQ-\kappa_j)}{N}}$$

(2) becomes a primaly field at $\beta = 1$

Free U(1) part vertex operator should be (Fateev-Litvinov 1109.4042)

$$\tilde{V}^H_{\kappa}(z) = e^{\frac{1}{\sqrt{N}}(NQ-\kappa)\phi_-}e^{\frac{-1}{\sqrt{N}}\kappa\phi_+}$$ Not primary field

$$\phi_+ = \alpha_0 \log z - \sum_{n=1}^{\infty}\frac{\alpha_n}{n}z^{-n} \qquad \phi_- = q + \sum_{n=1}^{\infty}\frac{\alpha_{-n}}{n}z^n$$

$$[\alpha_n, \alpha_m] = n\delta_{n+m,0} \qquad [\alpha_m, q] = \delta_{m,0}$$

The free boson representation implies that commutators with U(1) and Virasoro should be

- U(1) current

$$[J_n, V_\kappa(z)] = \frac{1}{\sqrt{\beta}}(NQ-\kappa)z^n V_\kappa(z) \quad (m \geq 0), \quad [J_{-n}, V_\kappa(z)] = \frac{-1}{\sqrt{\beta}}\kappa z^{-n} V_\kappa(z) \quad (n > 0)$$

- Virasoro algebra

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{(NQ-\kappa)^2}{2N}(n+1)z^n V_\kappa(z)$$

$$+\sqrt{N}Q\sum_{m=0}^{n} z^{n-m}V_\kappa(z)\alpha_m + (n+1)z^n \Delta_W V_\kappa(z), \;\; n \geq 0$$

$$[L_n, V_\kappa(z)] = z^{n+1}\partial_z V_\kappa(z) + \frac{\kappa^2}{2N}(n+1)z^n V_\kappa(z)$$

$$-\sqrt{N}Q\sum_{m=1}^{|n|} z^{n+m}\alpha_{-m}V_\kappa(z) + (n+1)z^n \Delta_W V_\kappa(z), \;\; n < 0$$

Recursion formulae as Ward identity

Assumption

(1) There exist three point function $< \vec{a}, \vec{Y} | V_\kappa | \vec{b} + \mu \vec{e}, \vec{W} >$

(2) It equals to building block of Nekrasov function

$$< \vec{a}, \vec{Y} | \tilde{V}_\kappa | \vec{b} + \mu \vec{e}, \vec{W} > = Z(\vec{a}, \vec{Y}; \vec{b}, \vec{W}; \mu)$$

Write Ward identity for U(1)×Virasoro generators

$$< \vec{a}, \vec{Y} | J_{\pm 1} V_\kappa | \vec{b} + \mu \vec{e}, \vec{W} > - < \vec{a}, \vec{Y} | V_\kappa J_{\pm 1} | \vec{b} + \mu \vec{e}, \vec{W} >$$

$$= < \vec{a}, \vec{Y} | [J_{\pm 1}, V_\kappa] | \vec{b} + \mu \vec{e}, \vec{W} >$$

Do they coincide with the recursion relation or not ?

Result

- We have confirmed $J_{\pm 1}$ $L_{\pm 1}$ constraint for three point function same as the recursion relation for l=0,1

Note

- The anomalous form of the U(1) vertex operator is essential to reproduce the recursion formulae

- The special form of the Toda momentum of W_N vertex operater is neccesary.

  (vertex operator corresponding to “simple puncture”)

Conclusion

We found the recursion formulae for Nekrasov partition function which connect different instanton contributions (instanton numbers Differ by one.)

They might be understood as Ward identity of $\mathbf{SH}^c$ algebra, which is symmetry of U(N) instanton moduli space.

Future direction

Examine the recursion for l > 1 generators

More general quiver cases

Interpretation in string or M-theory