Hyperkahler Metrics from Monopole Walls

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Introduction

Recently some significant features of BPS monopoles with periodicity are revealed, which are called monopole chains and walls in the case of singly-periodic and doubly-periodic respectively. Also the asymptotic metrics of the moduli space are obtained in the former case along the line of thought of non-periodic case, which turns out to be the Hyperkahler metric. In this poster we try to obtain some asymptotic metrics of monopole walls.

References

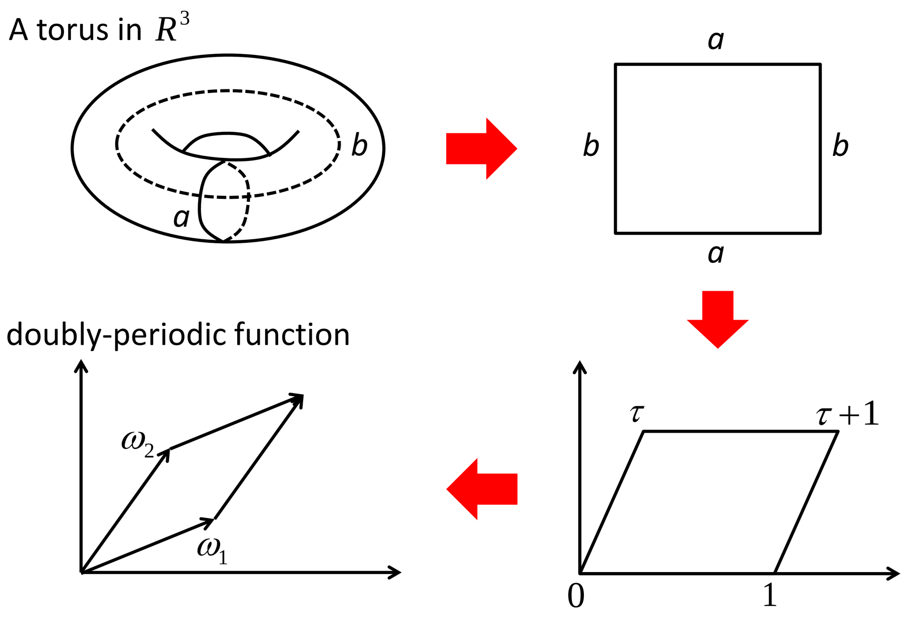
S. A. Cherkis and R. S. Ward, JHEP **1205, 090** (2012), arXiv:1202.1294 [hep-th]

S. A. Cherkis and A. Kapustin, Phys. Rev. **D65**, 084015 (2002), arXiv:hep-th/0109141

Complex torus coordinate

A torus in R^3

doubly-periodic function



For torus space $T^2 \times R(T^2 := R^2/Z^2)$, it is natural to use complex torus coordinate

$$\xi := x + \tau y\,,$$

where $\tau := \tau_1 + i\tau_2 (\tau_1, \tau_2 \in R)$ and we promise $\xi \approx \xi + m + \tau n (m, n \in Z)$. It is related to the doubly-periodic function with period ω_1, ω_2 by means of Mobius transform

$$\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix} \mapsto \begin{pmatrix}\omega_1'\\ \omega_2'\end{pmatrix} = \begin{pmatrix} d & c\\ b & a\end{pmatrix}\begin{pmatrix}\omega_1\\ \omega_2\end{pmatrix}$$

where $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{C})$. Then one can identify

$$\tau := \frac{\omega_2}{\omega_1}\,, \quad \mathrm{Im}\,\tau > 0\,.$$

At the complex torus coordinate, the metric can be written as

$$\mathrm{d}s^2 := \frac{\nu}{\tau_2}|\mathrm{d}\xi|^2 = \frac{\nu}{\tau_2}(\mathrm{d}x^2 + 2\tau_1 \mathrm{d}x\mathrm{d}y + |\tau|^2\mathrm{d}y^2) =: g_{ij}\mathrm{d}x^i\mathrm{d}x^j$$

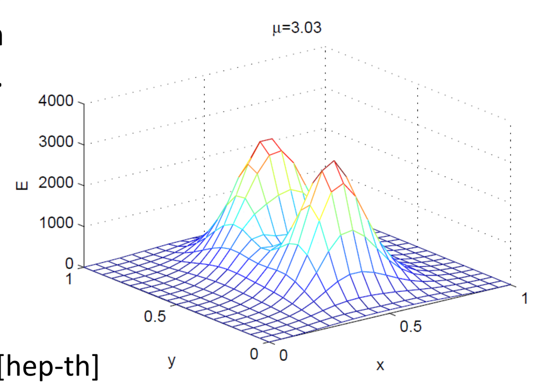
where $\nu := \sqrt{\det g},\ g := (g_{ij})$ and $\nu, \tau_1, \tau_2 \in C$ are parameters of complex torus.

Monopole walls

Monopole walls or doubly-periodic monopoles are solutions of the Bogomolny equation $*D_A\phi = F$ at the space $T^2 \times R(T^2 := R^2/Z^2)$ with coordinate $x^j := (x, y, z)$ (j = 1,2,3) in which x and y are periodic (with period 1). For example,

$$\phi = 2\pi\mathrm{i}z\sigma_3\,, \quad A_j = \pi\mathrm{i}(-y, x, 0)\sigma_3$$

is called the constant-energy solution which is doubly-periodic up to gauge. The solution have constant energy in the space but the pertubated one have non-constant energy and four-moduli.



S. A. Cherkis and R. S. Ward, JHEP **1205, 090** (2012), arXiv:1202.1294 [hep-th]

Linearized Bogomolny equation

Firstly, we assume the space $T^2 \times R(T^2 := R^2/Z^2)$ with coordinate $x^j := (x, y, z)$ (j = 1,2,3) in which x and y are periodic with period 1. We also take the torus coordinate $\xi := x + \tau y$ for the space T^2 where $\tau := \tau_1 + i\tau_2$. In this case the metric can be written as

$$\begin{aligned}\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} &:= \frac{\nu}{\tau_2}|\mathrm{d}\xi|^2 + \mathrm{d}z^2 =: g_{IJ}\mathrm{d}x^I\mathrm{d}x^J\\ &= \frac{\nu}{\tau_2}(\mathrm{d}x^2 + 2\tau_1\mathrm{d}x\mathrm{d}y + |\tau|^2\mathrm{d}y^2) + \mathrm{d}z^2 =: g_{ij}\mathrm{d}x^i\mathrm{d}x^j\,.\end{aligned}$$

In this space we consider the usual Bogomolny equation

$$*D_A\phi = F$$

where endomorphism ϕ is the Higgs field and 1-form A is the gauge field. The covariand derivative and the field strength are determined as follows.

$$D_A\phi := \mathrm{d}\phi + [A, \phi]\,, \quad F := \mathrm{d}A + A \wedge A\,.$$

We suppose the asymptotic field of SU(2) k-monopole walls with charge $Q_\pm$ as superposition of $n := |Q_-| + |Q_+|$ linearized doubly-periodic 't Hooft-Polyakov monopoles sitting at points $a_j := (\xi_j, z_j)$ (j=1,…,k),

$$\phi(\boldsymbol{x}) = v + \sum_{j=1}^{k}\phi^j(\boldsymbol{x} - \boldsymbol{a}_j)\,,$$

$$A_\xi(\boldsymbol{x}) = b + \sum_{j=1}^{k}A^j_\xi(\boldsymbol{x} - \boldsymbol{a}_j)\,, \quad A_z(\boldsymbol{x}) = 0$$

where v and b are the vacuum expectation value and the background field respectively. To avoid diverging, we suppose the linear superposition of the Higgs field arranged in a finite $(2M+1)\times(2N+1)$ rectangle

$$\phi^j(\boldsymbol{x}) = \sum_{m=-M}^{M}\sum_{n=-N}^{N}\frac{-g}{\sqrt{|\xi - m - n\tau|^2 + z^2}} = 2\pi g|z| - gC_{M,N}$$

where g is the magnetic charge and $C_{M,N}$ is a positive constant diverging linearly at the limit of M and N.

Substituting $\phi^j(x)$ into $\phi(x)$ we obtain

$$\phi(\boldsymbol{x}) = v_{\mathrm{ren}} + 2\pi g\sum_{j=1}^{k}|z - z_j|$$

where $v_{\mathrm{ren}} := v - kgC_{M,N}$ which can be controlled to converge at the limit of M and N with v diverge. From the Bogomolny equation the asymptotic form of the gauge field can also be obtained

$$A^j_\xi(\boldsymbol{x}) = -\frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z)\,\bar{\xi}\,, \quad \bar{A}^j_{\bar{\xi}}(\boldsymbol{x}) = \frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z)\,\xi\,, \quad A^j_z(\boldsymbol{x}) = 0$$

where the field is doubly-periodic up to gauge. For convenience, we introduce

$$u(z) := 2\pi|z| - C_{M,N}\,, \quad w(\boldsymbol{x}) := -\frac{\mathrm{i}\pi\nu}{2\tau_2}\mathrm{sign}(z)\,\bar{\xi}\,.$$

Then the solution can be summarized as

$$\phi^j(\boldsymbol{x}) = gu(z)\,, \quad A^j_\xi(\boldsymbol{x}) = gw(\boldsymbol{x})\,, \quad \bar{A}^j_{\bar{\xi}}(\boldsymbol{x}) = g\bar{w}(\boldsymbol{x})\,, \quad A^j_z(\boldsymbol{x}) = 0\,.$$

2-monopole wall metric

The interaction of moving monopoles can be considered not only with the relative coordinates but also with the relative phases, whose rate of change generates electric charge and makes monopoles into dyons. Then the conceivable charges couple to the fields and the Lagrangian of k^{th} dyon can be written as

$$L_k = -(g^2 + q_k^2)^{1/2}\phi(1 - \boldsymbol{V}_k^2)^{1/2} + q_k\boldsymbol{V}_k\cdot\boldsymbol{A} - q_kA_0 + g\boldsymbol{V}_k\cdot\tilde{\boldsymbol{A}} - g\tilde{A}_0$$

where $(g^2 + q_k^2)^{1/2}$, q_k and $V_k := (\dot{\xi}_k, \dot{z}_k)$ are the scalar charge, electric charge and velocity of k^{th} dyon, respectively. Moreover $\tilde{A}$ and $\tilde{A}_0$ are dual potentials which satisfy $\tilde{F} = *F$. From the solution of the Bogomolny equation we obtain

$$\begin{aligned}&\phi^j(\boldsymbol{x}) = (g^2 + q_j^2)^{1/2}u(z)\,,\\ &A^j_\xi(\boldsymbol{x}) = gw(\boldsymbol{x})\,, \quad \bar{A}^j_{\bar{\xi}}(\boldsymbol{x}) = g\bar{w}(\boldsymbol{x})\,, \quad A^j_z(\boldsymbol{x}) = 0\,, \quad A^j_0(\boldsymbol{x}) = -q_ju(z)\\ &\tilde{A}^j_\xi(\boldsymbol{x}) = -q_jw(\boldsymbol{x})\,, \quad \tilde{A}^j_{\bar{\xi}}(\boldsymbol{x}) = -q_j\bar{w}(\boldsymbol{x})\,, \quad \tilde{A}^j_z(\boldsymbol{x}) = 0\,, \quad \tilde{A}^j_0(\boldsymbol{x}) = -gu(z)\,.\end{aligned}$$

These fields can be rewritten for the one of moving dyons in terms of ordinary Lorentz boost in R^3.

$$\phi^j(\boldsymbol{x}) \simeq (g^2 + q_j^2)^{1/2}u(z)(1 - \boldsymbol{V}_j^2)^{1/2} \simeq gu(z)\left(1 + \frac{q_j^2}{2g^2} - \frac{\boldsymbol{V}_j^2}{2}\right)$$

$$\begin{aligned}A^j_\xi(\boldsymbol{x}) &\simeq -q_ju(z)V_{j\xi} + gw(\boldsymbol{x})\,,\\ A^j_z(\boldsymbol{x}) &\simeq -q_ju(z)V_{jz}\,,\\ A^j_0(\boldsymbol{x}) &\simeq -q_ju(z) + \frac{2\tau_2}{\nu}g(wV_{j\bar{\xi}} + \bar{w}V_{j\xi}) - \frac{1}{2}q_ju(z)\boldsymbol{V}_j^2\,,\\ \tilde{A}^j_\xi(\boldsymbol{x}) &\simeq -gu(z)V_{j\xi} - q_jw(\boldsymbol{x})\,,\\ \tilde{A}^j_z(\boldsymbol{x}) &\simeq -gu(z)V_{jz}\,,\\ \tilde{A}^j_0(\boldsymbol{x}) &\simeq -gu(z) - \frac{2\tau_2}{\nu}q_j(wV_{j\bar{\xi}} + \bar{w}V_{j\xi}) - \frac{1}{2}gu(z)\boldsymbol{V}_j^2\end{aligned}$$

keeping terms of order V_j^2, q_j^2, V_jq_j.

Substituting them into the Lagrangian for the case of k=2 we obtain

$$\begin{aligned}L_2 = &-m_2 + \frac{1}{2}m_2\boldsymbol{V}_2^2 + q_2(bV_2^\xi + \bar{b}V_2^{\bar{\xi}})\\ &-\frac{1}{2}u(z_2 - z_1)(q_2 - q_1)^2 + \frac{g^2}{2}u(z_2 - z_1)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2\\ &+(q_2 - q_1)\{gw_{21}(V_2^\xi - V_1^\xi) + g\bar{w}_{21}(V_2^{\bar{\xi}} - V_1^{\bar{\xi}})\} + \frac{g^2}{2}u(z_2 - z_1)\boldsymbol{V}_1^2\end{aligned}$$

where $m_k := v(g^2 + q_k^2)^{1/2}$ is the rest mass of k^{th} dyon. Furthermore, expanding m_2 and symmetrizing terms, we obtain the total Lagrangian

$$\begin{aligned}L_{21} = &-\frac{v}{2g}(q_2^2 + q_1^2) + \frac{vg}{2}(\boldsymbol{V}_2^2 + \boldsymbol{V}_1^2) + q_2(bV_2^\xi + \bar{b}V_2^{\bar{\xi}}) + q_1(bV_1^\xi + \bar{b}V_1^{\bar{\xi}})\\ &-\frac{1}{2}u(z_2 - z_1)(q_2 - q_1)^2 + \frac{g^2}{2}u(z_2 - z_1)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2\\ &+(q_2 - q_1)\{gw_{21}(V_2^\xi - V_1^\xi) + g\bar{w}_{21}(V_2^{\bar{\xi}} - V_1^{\bar{\xi}})\}\,.\end{aligned}$$

The total Lagrangian L_{21} can be divided as $L_{21} = L_{\mathrm{CM}} + L_{\mathrm{rel}}$,

$$\begin{aligned}L_{\mathrm{CM}} = &-\frac{v}{4g}(q_2 + q_1)^2 + \frac{vg}{4}(\boldsymbol{V}_2 + \boldsymbol{V}_1)^2\\ &+\frac{b}{2}(q_2 + q_1)(V_2^\xi + V_1^\xi) + \frac{\bar{b}}{2}(q_2 + q_1)(V_2^{\bar{\xi}} + V_1^{\bar{\xi}})\,,\\ L_{\mathrm{rel}} = &-\left(\frac{v_{\mathrm{ren}}}{4g} + \pi|z_2 - z_1|\right)(q_2 - q_1)^2 + g^2\left(\frac{v_{\mathrm{ren}}}{4g} + \pi|z_2 - z_1|\right)(\boldsymbol{V}_2 - \boldsymbol{V}_1)^2\\ &+\left\{\frac{b}{2} - \frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z_2 - z_1)\,(\bar{\xi}_2 - \bar{\xi}_1)\right\}(q_2 - q_1)(V_2^\xi - V_1^\xi)\\ &+\left\{\frac{\bar{b}}{2} + \frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z_2 - z_1)\,(\xi_2 - \xi_1)\right\}(q_2 - q_1)(V_2^{\bar{\xi}} - V_1^{\bar{\xi}})\end{aligned}$$

in which the center of mass Lagrangian L_{CM} would diverge but the relative Lagrangian L_{rel} would converge at the limit of M and N.

We concentrate on the relative Lagrangian L_{rel} which include the electric charge but it can be replaced by the phase in terms of Legendre transform. For convenience, we put $\xi := \xi_2 - \xi_1$, $z := z_2 - z_1$, $V := V_2 - V_1$, $q := q_2 - q_1$ and the relative phase $\chi := \chi_2 - \chi_1$. After the Legendre transform, the metric read from the Lagrangian can be summarized in the form of the Gibbons-Hawking ansatz

$$\frac{1}{g^2}\mathrm{d}s^2 = U\mathrm{d}\boldsymbol{x}\cdot\mathrm{d}\boldsymbol{x} + \frac{1}{U}(\mathrm{d}\chi + W_J\mathrm{d}x^J)^2$$

where

$$U = \frac{v_{\mathrm{ren}}}{2g} + 2\pi|z|\,, \quad W_\xi = \frac{b}{2g} - \frac{\mathrm{i}\pi\nu}{2\tau_2}\mathrm{sign}(z)\,\bar{\xi}\,, \quad W_{\bar{\xi}} = \bar{W}_\xi\,, \quad W_z = 0$$

which is one of the Hyperkahler metric. This metric contains the original torus metric of $T^2 \times R$ not only in the first term but also in the second term in the sense of $\left(\begin{smallmatrix} a & b\\ c & d\end{smallmatrix}\right) \in SL(2,\mathbb{Z})$ transform.

$$\tau \mapsto \frac{a\tau + b}{c\tau + d}\,, \quad \tau_2 \mapsto \frac{\tau_2}{|c\tau + d|^2}\,, \quad \xi \mapsto \frac{\xi}{c\tau + d}\,.$$

Multi-monopole wall metric

Multi-monopole wall case can also be derived in the same way as k=2. The relative Lagrangian L_{rel} can easily be generalized as follows.

$$\begin{aligned}L_{\mathrm{rel}} = &-\frac{1}{2}\sum_{1\le i<j\le k}\left(\frac{v_{\mathrm{ren}}}{gk} + 2\pi|z_j - z_i|\right)(q_j - q_i)^2\\ &+\frac{g^2}{2}\sum_{1\le i<j\le k}\left(\frac{v_{\mathrm{ren}}}{gk} + 2\pi|z_j - z_i|\right)(\boldsymbol{V}_j - \boldsymbol{V}_i)^2\\ &+\sum_{1\le i<j\le k}\left\{\frac{b}{k} - \frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z_j - z_i)\,(\bar{\xi}_j - \bar{\xi}_i)\right\}(q_j - q_i)(V_j^\xi - V_i^\xi)\\ &+\sum_{1\le i<j\le k}\left\{\frac{\bar{b}}{k} + \frac{\mathrm{i}\pi\nu g}{2\tau_2}\mathrm{sign}(z_j - z_i)\,(\xi_j - \xi_i)\right\}(q_j - q_i)(V_j^{\bar{\xi}} - V_i^{\bar{\xi}})\,.\end{aligned}$$

Then the metric can be read from the relative Lagrangian

$$\frac{1}{g^2}\mathrm{d}s^2 = U_{ij}\mathrm{d}\boldsymbol{x}_i\cdot\mathrm{d}\boldsymbol{x}_j + U_{ij}^{-1}(\mathrm{d}\chi_i + \boldsymbol{W}_{ik}\cdot\mathrm{d}\boldsymbol{x}_k)(\mathrm{d}\chi_j + \boldsymbol{W}_{j\ell}\cdot\mathrm{d}\boldsymbol{x}_\ell)$$

where

$$\begin{aligned}U_{jj} &= \frac{v_{\mathrm{ren}}}{gk}(k - 1) + 2\pi\sum_{i\neq j}|z_j - z_i|\,,\\ U_{ij} &= -\frac{v_{\mathrm{ren}}}{gk} - 2\pi|z_j - z_i|\,, \quad (i \neq j)\\ (W_\xi)_{jj} &= \frac{b}{gk}(k - 1) - \frac{\mathrm{i}\pi\nu}{2\tau_2}\sum_{i\neq j}\mathrm{sign}(z_j - z_i)(\bar{\xi}_j - \bar{\xi}_i)\,,\\ (W_\xi)_{ij} &= -\frac{b}{gk} + \frac{\mathrm{i}\pi\nu}{2\tau_2}\mathrm{sign}(z_j - z_i)(\bar{\xi}_j - \bar{\xi}_i)\,, \quad (i \neq j)\\ (W_{\bar{\xi}})_{ij} &= (\bar{W}_\xi)_{ij}\,,\\ (W_z)_{ij} &= 0\,.\end{aligned}$$

Conclusion

- We derive the Hyperkahler metric of monopole walls as the metric of geodesic motion of linearized doubly-periodic monopoles.
- The metric contains the torus metric of the original space $T^2 \times R$.
- Outlook: The formulation can be applied for the case of monopole walls with Dirac singularities which have some constraint of the boundary conditions of monopole walls.