A Classification of Bosonic Supercurrents

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Abstract

The heterotic covariant lattice (bosonic supercurrent) formalism is revisited and a classification of right-mover $c = 9$ lattice SCFTs that potentially lead to chiral four-dimensional models is performed. All these SCFTs are related to certain (asymmetric) orbifold constructions, and 19 SCFTs lead to $\mathcal{N} = 1$ spacetime supersymmetry. Modular invariance implies that the corresponding left-mover lattice CFTs can be studied using the theory of lattice genera. Then, using the Minkowski-Siegel mass formula, lower bounds on the number of left-mover CFTs are calculated.

Keywords: Heterotic string, covariant lattice, rational (S)CFT, lattice genera.

Introduction

Heterotic string theory could eventually lead to a unified description of particle physics. However, the moduli space of four-dimensional heterotic strings is huge. At special symmetry enhanced points in the moduli space, a model can be described by rational (S)CFTs. The classification of RCFTs is an open problem, even if one considers only those relevant for chiral models. There are different well known constructions, e.g. Gepner models, asymmetric orbifolds, free fermionic and free bosonic constructions, and they have some overlaps.

This work considers a certain type of free bosonic construction: the covariant lattice theories. Making heavy use of computing power, a classification of right-mover lattice SCFTs that is relevant for chiral models is performed. The result is that there are 80 SCFTs with $\mathcal{N} = 0$ and 19 with $\mathcal{N} = 1$.

Further, modular invariance requires certain $c = 22$ left-mover lattice CFTs. Making use of the theory of lattice genera, a lower bound on the number of these CFTs can be given. Again with heavy use of computing power, a classification of some genera was performed.

The Covariant Lattice Formalism

Let us consider heterotic string theory in four space-time dimensions. Anomaly cancellation requires an internal CFT with central charges $(c_{\text{int}}, \tilde{c}_{\text{int}}) = (22, 9)$:

$$\underbrace{c(X^0) + \ldots + c(X^3)}_{4} + \underbrace{c_{\text{int}}}_{22} + \underbrace{c(bc)}_{-26} = 0$$

$$\underbrace{\tilde{c}(X^0) + \ldots + \tilde{c}(X^3)}_{4} + \underbrace{\tilde{c}(\psi^0) + \ldots + \tilde{c}(\psi^3)}_{2} + \underbrace{\tilde{c}_{\text{int}}}_{9} + \underbrace{\tilde{c}(bc)}_{-26} + \underbrace{\tilde{c}(\beta\gamma)}_{11} = 0$$

Assume a special case: internal CFT is made of free bosons with periodic boundary conditions. $\Longrightarrow$ lattice CFT with even lattice $\Gamma^{\text{int}}_{22,9}$ [1]

Bosonic string map [2]:

$$\psi^\mu - \beta\gamma \text{ SCFT} \longrightarrow \overline{D}_5^{\text{space-time}} \text{ root lattice}.$$

Modular invariance: even and self-dual lattice ("covariant lattice")

$$\Gamma_{22,14} \supset \Gamma^{\text{int}}_{22,9} \oplus \overline{D}_5^{\text{space-time}}.$$

Supersymmetry: $\tilde{c}_{\text{int}} = 9$ CFT must realize the $N = 1$ super-Virasoro algebra:

$$T(z)T(w) \sim \frac{\tilde{c}_{\text{int}}/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{(z-w)} \tag{1}$$

$$T(z)G(w) \sim \frac{\frac{3}{2}G(w)}{(z-w)^2} + \frac{\partial G(w)}{(z-w)} \tag{2}$$

$$G(z)G(w) \sim \frac{2\tilde{c}_{\text{int}}/3}{(z-w)^3} + \frac{2T(w)}{(z-w)}. \tag{3}$$

Equation (1): Energy-momentum tensor of 9 chiral bosons $X^i(z)$:

$$T(z) = -\frac{1}{2} : \partial X(z) \cdot \partial X(z) : .$$

Equation (2): $G(z)$ is a primary field of conformal weight 3/2:

$$G(z) = : \sum_{s^2=3} A(s) e^{is\cdot X(z)} \varepsilon(s, \hat{p}) + \sum_{r^2=1} iB(r) \cdot \partial X(z) e^{ir\cdot X(z)} \varepsilon(r, \hat{p}) : .$$

$(r \cdot B(r) = 0)$. The vectors r and s generate an odd integral lattice Ξ.

Constraint vectors: For any vector weight $\mathbf{v} = (\underline{\pm 1, 0, 0, 0, 0})$ of $\overline{D}_5^{\text{space-time}}$:

$$(0, r, \mathbf{v}) \text{ and } (0, s, \mathbf{v}) \in \Gamma_{22,14}$$

Chirality: Existence of $r \in \Xi$ of norm 1 $\Longrightarrow \overline{D}_5^{\text{space-time}} \subset \overline{D}_{5+n}^{\text{space-time}} \Longrightarrow$ non-chiral

Hermiticity: $G(z)^\dagger = (z^*)^{-3} G(1/z^*) \Longrightarrow A(s)^* = A(-s)$.

Then, the GG-OPE (3) gives a system of quadratic equations:

$$\sum_{s^2=3} |A(s)|^2 s^i s^j = 2\delta^{ij} \tag{4}$$

$$\sum_{\substack{s^2=t^2=3 \\ s+t=u}} A(s)A(t)\varepsilon(s,t) = 0 \quad \text{for all } u^2 = 4 \tag{5}$$

$$\sum_{\substack{s^2=t^2=3 \\ s+t=u}} A(s)A(t)\varepsilon(s,t)\,(s^i - t^i) = 0 \quad \text{for all } u^2 = 2 \tag{6}$$

From (4) $\Longrightarrow \dim \Xi = 9 \Longrightarrow$ the constraint vectors span a lattice $\overline{(\Gamma_{14})_{\text{R}}}$.

$\Longrightarrow$ The left-mover and right-mover CFTs are rational.

$$\Gamma_{22,14} \supset (\Gamma_{22})_{\text{L}} \oplus \overline{(\Gamma_{14})_{\text{R}}}.$$

Classification Problem

Problem of classifying chiral covariant lattice theories can be broken down as follows:

1. Enumerate all possible $(\Gamma_{14})_{\text{R}}$. Such a $(\Gamma_{14})_{\text{R}}$ is constructed from an odd integral lattice Ξ of rank 9 using the (even) constraint vectors.
2. Given a specific $(\Gamma_{14})_{\text{R}}$, enumerate the possible $(\Gamma_{22})_{\text{L}}$ allowing modular invariance.
3. Consider possibly inequivalent embeddings of $(\Gamma_{22})_{\text{L}} \oplus \overline{(\Gamma_{14})_{\text{R}}}$ in a covariant lattice.

Regarding 1.

A positive definite odd integral lattice Ξ is *admissible* if

1. *Chirality.* It is spanned by vectors s with $s^2 = 3$ and contains no vectors r with $r^2 = 1$.
2. *Supersymmetry.* There exists a solution $A(s)$ to equations (4-6)

It is sufficient to classify lattices that satisfy both of the following:

1. *Elementarity.* Ξ is admissible and does not contain an admissible sublattice $\Xi_{\text{sub}} \subset \Xi$ of the same dimension.
2. *Primitivity.* Ξ is not isomorphic to an orthogonal sum $\Xi_1 \oplus \ldots \oplus \Xi_k$, $k > 1$.

Any other admissible Ξ can be built from these building blocks.

Methods. All *chiral* lattices Ξ are enumerated by induction over $\dim \Xi$, up to $\dim \Xi = 9$. The algorithm makes use of ideas from [3]. Equation (4), as a linear equation in the $|A(s)|^2$ is solved with standard methods. Equations (5-6) are attacked by calculating Gröbner bases and fixing some symmetries.

Results. In $\dim \Xi \leq 9$ there are in total 26 primitive elementary lattices (Fig. 1). With some combinatorics one constructs in total 32 elementary lattices in $\dim \Xi = 9$. To these correspond 32 lattices $(\Gamma_{14})_{\text{R}}$ (Fig. 2).

Lattice inclusion digraph: For these $(\Gamma_{14})_{\text{R}}$, calculate all (even) overlattices

$$(\Gamma_{14})'_{\text{R}} \supset (\Gamma_{14})_{\text{R}}$$

$\Longrightarrow$ create directional graph. It contains 414 nodes and splits into 9 connected components. Out of these 414 nodes, 99 fulfill the *chirality* condition and 19 lead to $\mathcal{N} = 1$ spacetime supersymmetry (Fig. 3). Subgraph with supersymmetric nodes in Fig. 5.

Regarding 2.

The genus: Given two integral lattices Λ_1 and Λ_2 with Gram matrices G_1 and G_2. They are said to lie in the same genus if for all primes p there exist p-adic integral matrices U_p, such that

$$U_p G_1 U_p^T = G_2.$$

and if they have the same signature. A genus contains only finitely many lattices.

Discriminant forms: Alternative characterization due to Nikulin [4]: Even lattices Λ_1 and Λ_2 lie in the same genus if they have the same signature and their discriminant forms are isomorphic:

$$(\Lambda_1^*/\Lambda_1, (\cdot)^2 \bmod 2) \overset{\phi}{\cong} (\Lambda_2^*/\Lambda_2, (\cdot)^2 \bmod 2)$$

(the isomorphy already implies that the signature $p_1 - q_1 = p_2 - q_2 = 0 \mod 8$.)

Self-duality & glue: Given two even lattices L and R, when does there exist a self-dual lattice $S \supset L \oplus \overline{R}$, such that L and R are the *maximal* sublattices of S in their $\mathbb{R}$-span?

- Answer: L and R must have isomorphic *discriminant forms* (sufficient and necessary).
- *Glue*: $[l] \times \phi([l]) \subset S$ for $[l] \in L^*/L$ and $\phi : L^*/L \mapsto R^*/R$ isomorphism
- Given a *specific* R, all lattices L of a given signature that can be paired with R to a self-dual lattice form a *genus*.
- The bottom line: There are only finitely many lattices $(\Gamma_{22})_{\text{L}}$ that can be paired with a given $(\Gamma_{14})_{\text{R}}$ and they form a genus.

The Smith-Minkowski-Siegel mass formula (e.g. [5]): For a (positive definite) genus $\mathcal{G}$ the following identity holds:

$$\sum_{\Lambda' \in \mathcal{G}} \frac{1}{\text{Aut}(\Lambda')} = m(\Lambda), \ \Lambda \in \mathcal{G}.$$

Here, $m(\Lambda)$ is called the *mass* of $\mathcal{G}$ and it can be computed from a *single* $\Lambda \in \mathcal{G}$. The l.h.s. however, depends on all lattices in the genus. The mass formula can be used e.g. as follows:

1. Compute lower bounds on $|\mathcal{G}|$: $\text{Aut}(\Lambda) \geq 2$. $\Longrightarrow |\mathcal{G}| \geq 2m(\Lambda)$ for any $\Lambda \in \mathcal{G}$. However, this bound is very crude in most cases.
2. Check enumeration of lattices in $\mathcal{G}$ by computing automorphism groups.

Enumeration methods: Kneser's neighborhood method [6], lattice engineering [1].

Results: Some genera were enumerated. For other genera, lower bounds were calculated.

– Results & Comments –

Comments

1. *Asymmetric orbifolds.* Some of of the lattices $(\Gamma_{14})_R$ are related to asymmetric orbifolds as described in [7].

$$\Gamma_{22,14} = \text{Narain Lattice} \oplus \overline{E_8}^{\text{s.-t.}} \underset{\text{shift LM}}{\xrightarrow{\text{twist RM}}} \Gamma'_{22,14}$$

These are converted to shift-orbifolds by twist-shift correspondence. All lattices $(\Gamma_{14})_R$ in the inclusion graph can be seen to arise from shift-orbifolds.

2. *Model building.* Some models were constructed explicitly as asymmetric $\mathbf{Z}_3$ orbifolds in [8]. An enumeration now shows that there are 2030 models constructed from Narain lattices with $\overline{E_6}$ by the $\mathbf{Z}_3$ construction. In turn, the lower bound for models constructed from Narain lattices with $\overline{A_2^3}$ by $\mathbf{Z}_3$ is calculated as $\approx 6.9 \cdot 10^3$. However, the expected value is several orders of magnitude higher.

3. *Classification of left-mover lattices.* In most cases, the genera are to large to be enumerated completely. Instead, one could resort to randomized methods and just build a large number of models (and search for realistic ones).

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Tables

dim Ξ	CP	CSP	CSEP	Naming of CSEP Lattices
1	1	1	1	$\Xi(1)$
2	2	0	0	-
3	7	1	1	$\Xi(3)$
4	28	0	0	-
5	136	1	1	$\Xi(5)$
6	911	6	3	$\Xi(6_A), \Xi(6_B), \Xi(6_C)$
7	8665	2	2	$\Xi(7_A), \Xi(7_B)$
8	131316	7	4	$\Xi(8_A), \dots, \Xi(8_D)$
9	3345309	36	14	$\Xi(9_A), \dots, \Xi(9_N)$

Fig. 1: Lattices Ξ as classified by an inductive algorithm.
C: *chirality*, S: *supersymmetry*, E: *elementarity*, P: *primitivity*.

Component	Genus $\mathcal{G}_{\mathrm{R}}$		Lattice $(\Gamma_{14})_{\mathrm{R}}$	
	$\lvert\mathcal{G}_{\mathrm{L}}\rvert$ Bound	Divisors	Ξ	$\Delta_\perp$
$\mathcal{C}_1$	$3.1 \cdot 10^{17}$	$3^7 6^2$	1^9	-
	$1.3 \cdot 10^{9}$	$2^4 4^1 12^1$	$1^1 8^1_A$	A_1^2
	$4.2 \cdot 10^{14}$	$3^1 6^4$	$1^1 8^1_D$	-
	$3.4 \cdot 10^{12}$	$3^1 18^2$	9^1_D	A_1
	$1.0 \cdot 10^{14}$	$2^1 6^1 12^2$	9^1_J	-
	$2.9 \cdot 10^{11}$	$3^2 6^1 18^1$	9^1_K	-
$\mathcal{C}_2$	$5.8 \cdot 10^{17}$	$6^2 12^2$	$1^2 7^1_B$	-
	$3.8 \cdot 10^{17}$	$2^6 12^2$	$1^1 3^1 5^1$	-
	$2.1 \cdot 10^{15}$	$4^2 12^2$	$1^1 8^1_B$	-
	$1.8 \cdot 10^{15}$	$2^2 4^6$	3^3	-
	$1.2 \cdot 10^{12}$	$2^4 4^4$	$3^1 6^1_A$	-
	$2.1 \cdot 10^{11}$	$2^4 8^2$	9^1_C	A_1^2
	$5.2 \cdot 10^{9}$	$2^4 8^2$	9^1_E	-
	$4.4 \cdot 10^{9}$	$2^2 4^4$	9^1_F	-
	$4.2 \cdot 10^{11}$	$4^3 16^1$	9^1_G	-
	$3.6 \cdot 10^{14}$	$2^2 20^2$	9^1_L	-
	$1.7 \cdot 10^{15}$	$2^2 4^2 8^2$	9^1_M	-
$\mathcal{C}_3$	$4.1 \cdot 10^{13}$	$7^1 14^2$	9^1_N	-
$\mathcal{C}_4$	$6.1 \cdot 10^{18}$	$2^1 4^1 24^2$	$1^1 8^1_C$	-
	$8.4 \cdot 10^{17}$	$2^1 4^3 8^2$	$3^1 6^1_B$	-
	$1.9 \cdot 10^{7}$	$2^4 4^1 8^1$	9^1_A 9^1_B	A_1^2 A_1^2
	$5.1 \cdot 10^{13}$	$2^3 4^1 8^2$	9^1_H	-
	$3.0 \cdot 10^{14}$	$4^1 8^3$	9^1_I	-
$\mathcal{C}_5$	$1.2 \cdot 10^{18}$	$2^1 6^5$	$1^4 5^1$	-
	$8.2 \cdot 10^{21}$	$2^2 4^1 12^3$	$1^3 3^2$	-
	$6.0 \cdot 10^{17}$	$2^3 6^1 12^2$	$1^3 6^1_A$	-
$\mathcal{C}_6$	$6.5 \cdot 10^{22}$	$15^1 30^2$	$1^3 6^1_C$	-
$\mathcal{C}_7$	$2.6 \cdot 10^{20}$	$2^1 10^1 20^2$	$3^1 6^1_C$	-
$\mathcal{C}_8$	$5.2 \cdot 10^{22}$	$2^1 12^1 24^2$	$1^3 6^1_B$	-
$\mathcal{C}_9$	$2.7 \cdot 10^{23}$	$3^2 6^2 12^2$	$1^6 3^1$	-
	$5.6 \cdot 10^{11}$	$2^6 6^2$	$1^2 7^1_A$	-

Fig. 2: Lattices $(\Gamma_{14})_{\mathrm{R}}$ generated from the 32 *elementary* lattices Ξ. A lower bound on the number of models is given.

Component	Genus $\mathcal{G}_{\mathrm{R}}$		Lattice $(\Gamma_{14})_{\mathrm{R}}$	
	$\lvert\mathcal{G}^{\mathrm{L}}\rvert$	Divisors	$\Delta_\perp$	Δ_{S}
$\mathcal{C}_1$	31	3^1	E_6 -	E_8 D_{13}
$\mathcal{C}_2$	68	2^2	D_6 A_1^2 -	E_8 D_{12} D_{14}
$\mathcal{C}_3$	153	7^1	A_6	E_8
$\mathcal{C}_4$	326	$2^1 4^1$	$A_1 D_5$ $A_1 A_3$ A_1	E_8 D_{10} D_{13}
$\mathcal{C}_5$	382	$2^1 6^1$	$A_2 D_4$ A_1^3 A_2	E_8 D_{10} D_{12}
$\mathcal{C}_6$	1163	15^1	$A_2 A_4$	E_8
$\mathcal{C}_7$	4043	$2^1 10^1$	$A_1^2 A_4$ A_4	E_8 D_{10}
$\mathcal{C}_8$	9346	$2^1 12^1$	$A_1 A_2 A_3$ $A_1 A_2$	E_8 D_{11}
$\mathcal{C}_9$	19832	6^2	$A_1^2 A_2^2$ A_1^2 A_2^2 -	E_8 D_{10} D_{10} D_{12}

Fig. 3: Tops of the graph components. The corresponding left-mover lattices in $\mathcal{G}_{\mathrm{L}}$ were also classified.

Component	Genus $\mathcal{G}_{\mathrm{R}}$		Lattice $(\Gamma_{14})_{\mathrm{R}}$		
	Divisors	$\lvert\mathcal{G}^{\mathrm{L}}\rvert$	Nr.	$\Delta_\perp$	$\mathbf{Z}_N$ Twist Orbifold
$\mathcal{C}_1$	3^3	2030		A_2^4	$E_6/\mathbf{Z}_3$
	$3^1 6^2$	$> 1.5 \cdot 10^3$		A_1^4	$E_6/\mathbf{Z}_6^{\mathrm{I}}, E_6/\mathbf{Z}_6^{\mathrm{II}}$
	3^5	$> 6.9 \cdot 10^3$		A_2	$A_2^3/\mathbf{Z}_3$
	$3^3 9^1$	$> 2.7 \cdot 10^5$		-	
	$3^1 12^2$	$> 1.5 \cdot 10^9$		-	$E_6/\mathbf{Z}_{12}^{\mathrm{I}}, A_1 A_5/\mathbf{Z}_6^{\mathrm{II}}$
	3^7	$> 4.1 \cdot 10^8$		-	
$\mathcal{C}_2$	2^6	–		A_3	
	$2^2 4^2$	> 3		A_1^4	
	$2^2 4^2$	> 6		A_1^6	$D_6/\mathbf{Z}_4$
	$2^4 4^2$	$> 1.3 \cdot 10^5$		A_1^2	$A_1^2 D_4/\mathbf{Z}_4$
	$2^4 4^2$	$> 1.3 \cdot 10^4$		-	
	$2^2 8^2$	$> 4.8 \cdot 10^6$		A_1	$A_3^2/\mathbf{Z}_4, D_6/\mathbf{Z}_8^{\mathrm{I}}$
	$2^2 8^2$	$> 8.0 \cdot 10^5$		-	
	4^4	$> 8.0 \cdot 10^5$		-	
	$2^2 4^4$	$> 1.7 \cdot 10^9$		-	
$\mathcal{C}_3$	7^3	$> 4.0 \cdot 10^7$		-	$A_6/\mathbf{Z}_7$
$\mathcal{C}_4$	$2^2 4^1 8^1$	$> 4.4 \cdot 10^3$		A_1^2	
	$2^1 4^1 8^2$	$> 2.3 \cdot 10^9$		-	$A_1 D_5/\mathbf{Z}_8^{\mathrm{II}}$
$\mathcal{C}_5$	$2^1 6^3$	$> 5.2 \cdot 10^7$		-	$A_2 D_4/\mathbf{Z}_6^{\mathrm{II}}$

Fig. 4: Right-mover lattices with $\mathcal{N} = 1$.

Lattice Inclusion Graphs for $\mathcal{N} \geq 1$

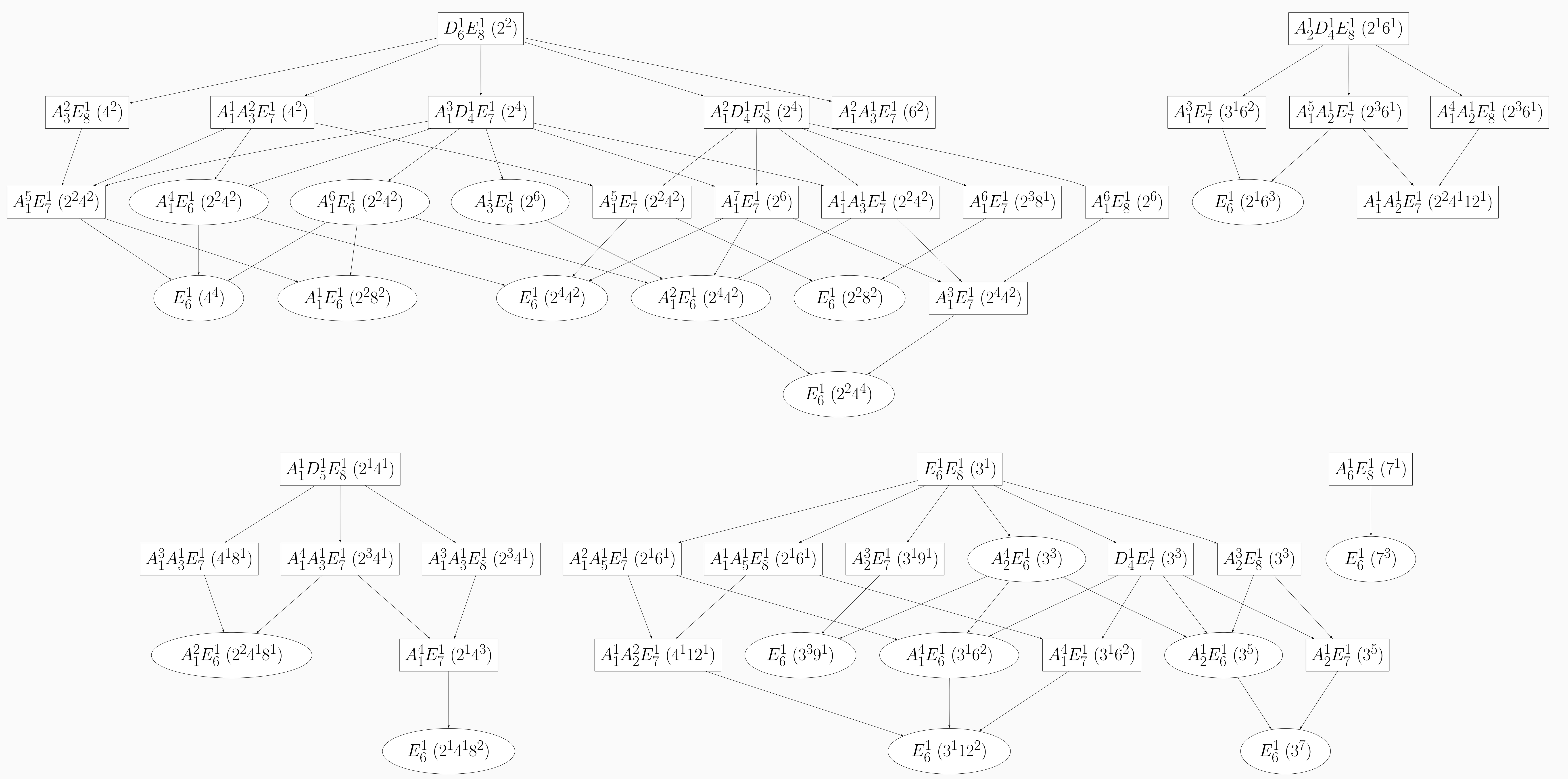

Fig. 5: Graph showing the right-mover lattices $(\Gamma_{14})_{\mathrm{R}}$. Ellipses correspond to $\mathcal{N} = 1$, boxes to $\mathcal{N} \geq 2$