Operator analysis of magnetized T^2/Z_N orbifolds

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Collaborating with

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Mysteries of the Standard Model

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- ❐ **Chiral fermion**

  What is the origin of the chiral fermion … ?

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Why so different the masses of the fermions are … ?

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- ❒ **Flavor structure**

  What determine the flavor structure … ?

⋮

Purpose

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We want to solve mysteries of the SM

- ❐ Chiral fermion
- ❐ Mass hierarchy
- ❐ Generations
- ❐ Flavor structure

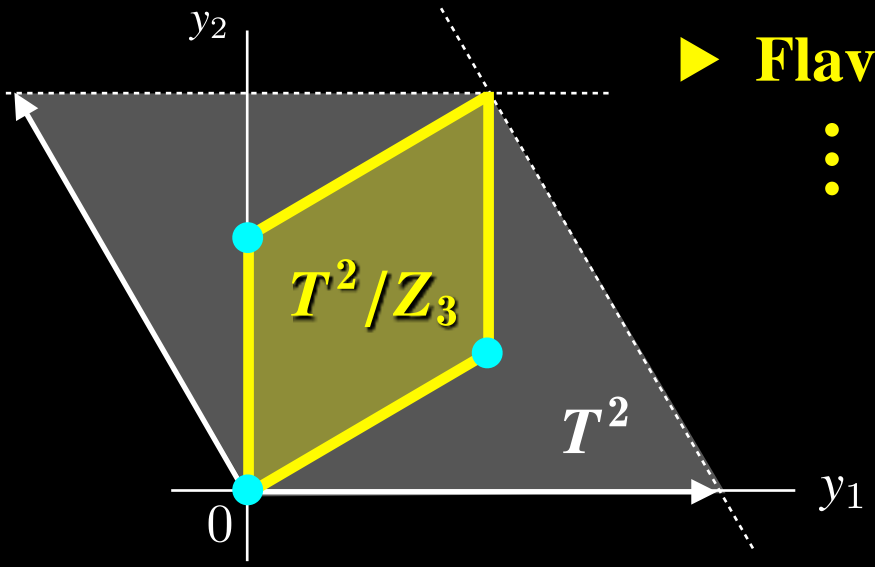
in the context of higher dimensional field theories.

Magnetic flux with Orbifold

Magnetic flux with Orbifold

❐ Orbifold

(e.g) T^2/Z_3



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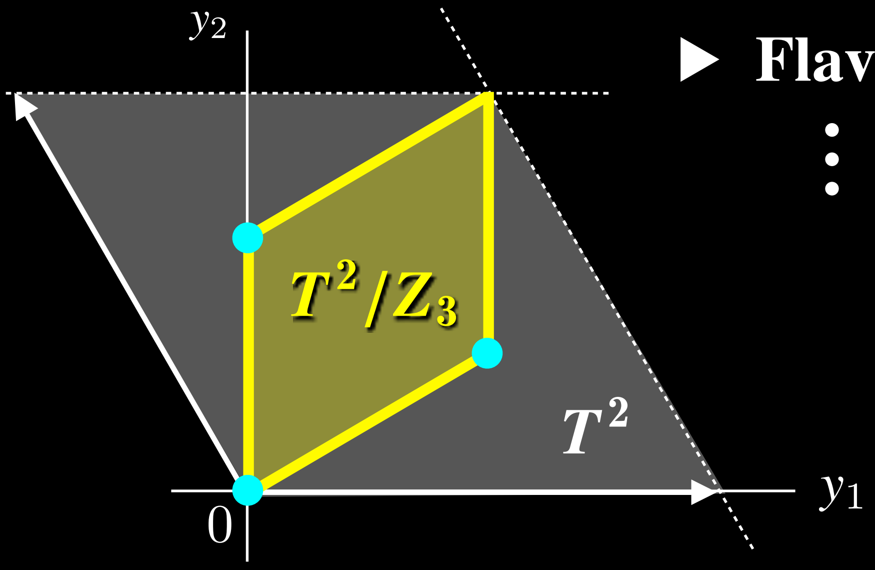

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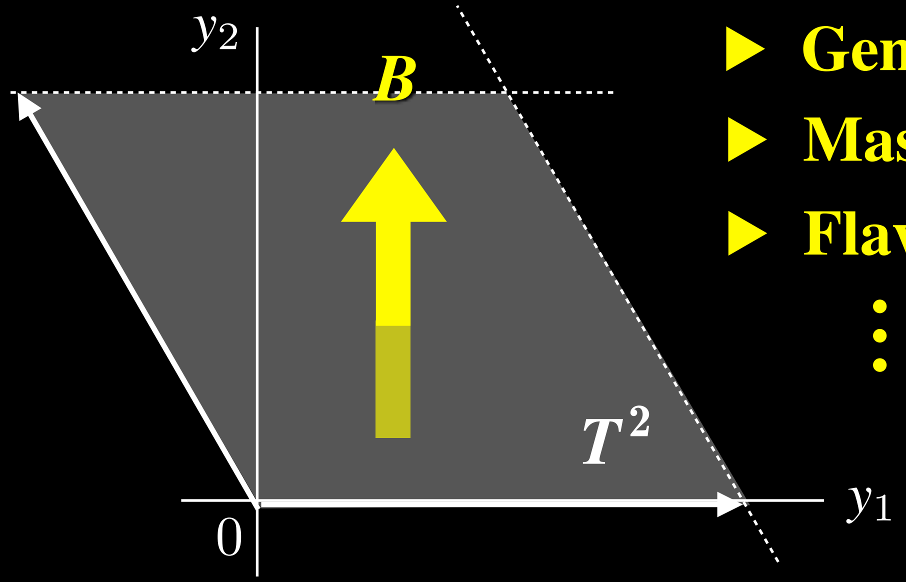

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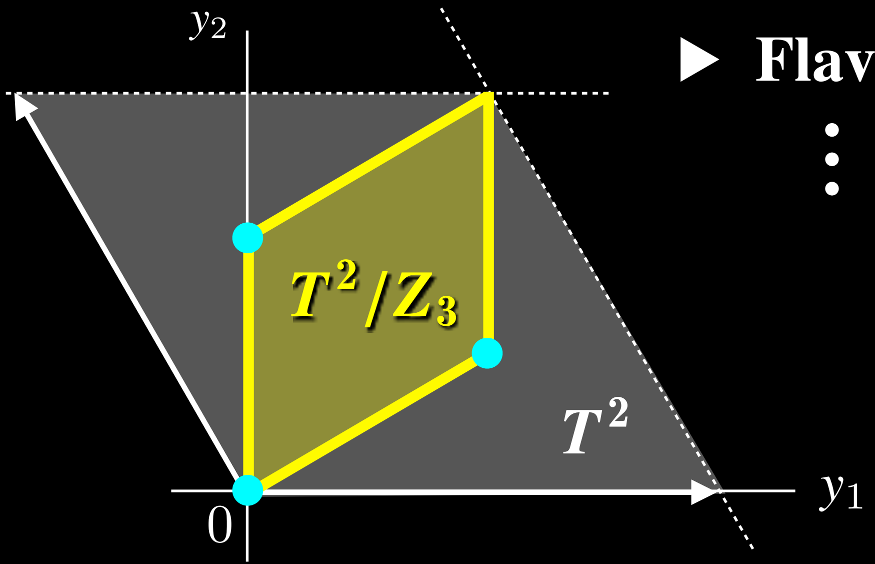

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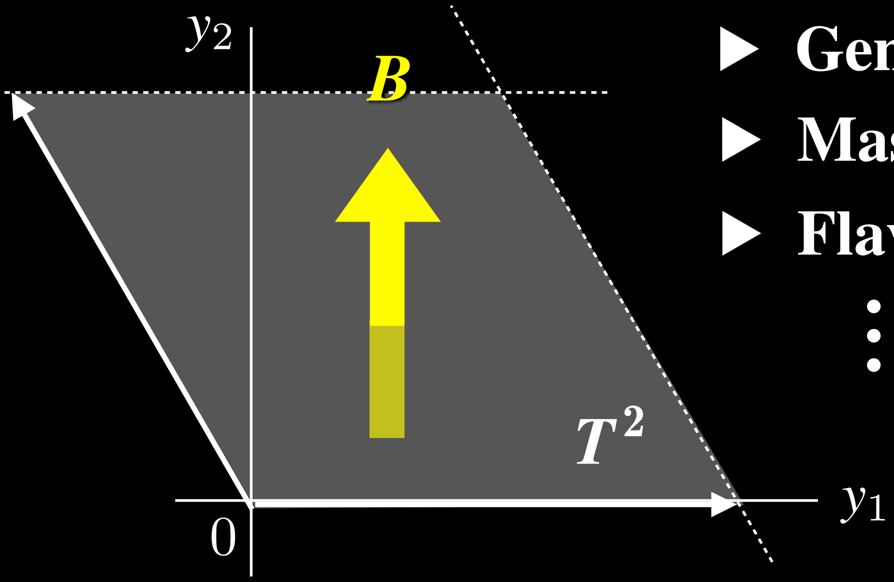

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❒ Magnetized orbifold

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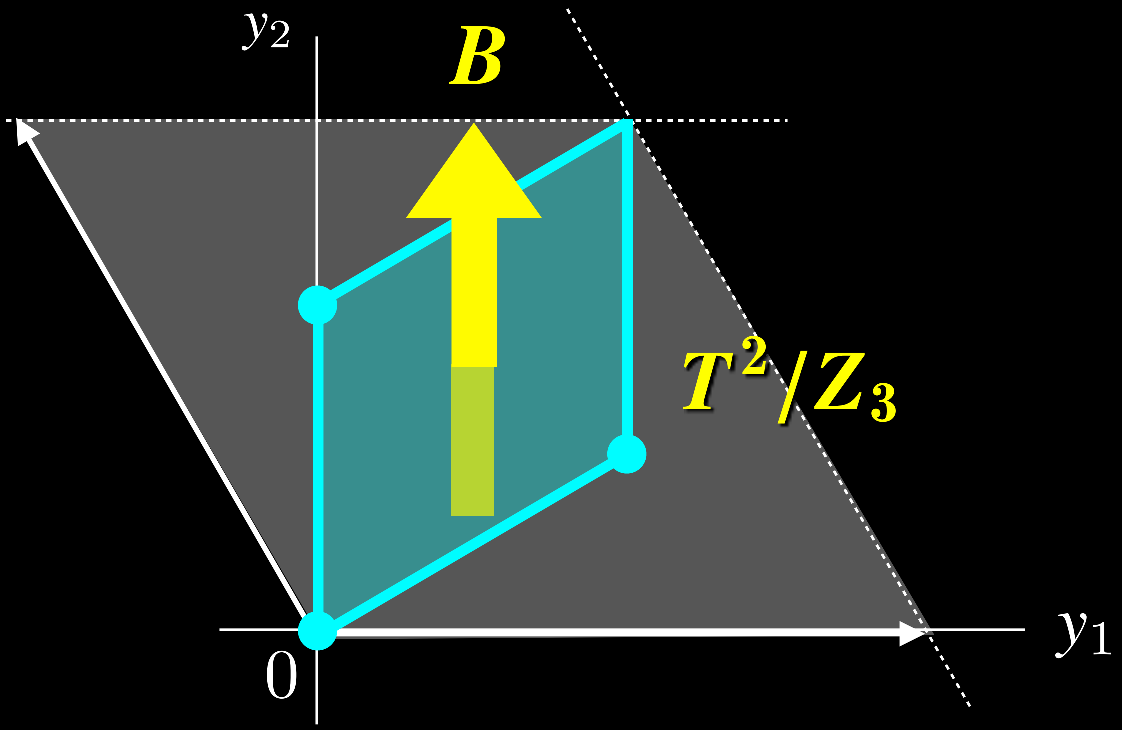

T-H.Abe, YF, T.Kobayashi, T.Miura, K.Nishiwaki, M.Sakamoto, JHEP01(2014)065

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Complex coordinate

Zero-mode solution on T^2 with a magnetic flux

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Eigenstate of T^2/Z_N

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index for degenerated sol. : $j \in \{0, 1, 2, \cdots, |M| - 1\}$

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ZN eigenvalue

$\eta \in \{1, \omega, \omega^2, \cdots, \omega^{N-1}\}$

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(e.g) T^2/Z_3

η \ M	2	4	6	8	$\cdots$
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$$\psi(z)_{T^2} \supset \vartheta\begin{bmatrix} a \\ b \end{bmatrix}(Mz, M\tau) = \sum_{l\in\mathbb{Z}} e^{i\pi(a+l)^2 M\tau}e^{2\pi i(a+l)(Mz+b)}$$

Our result

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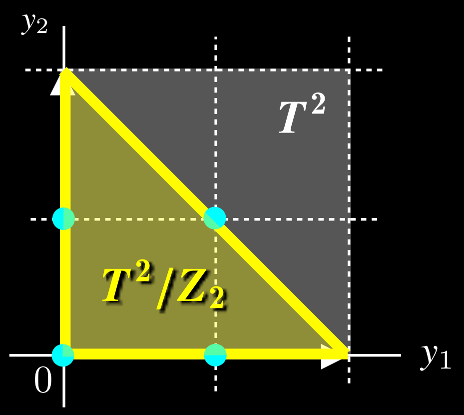
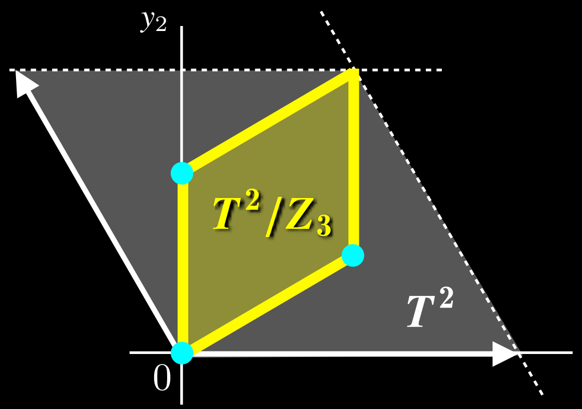
- Exact analytic results for

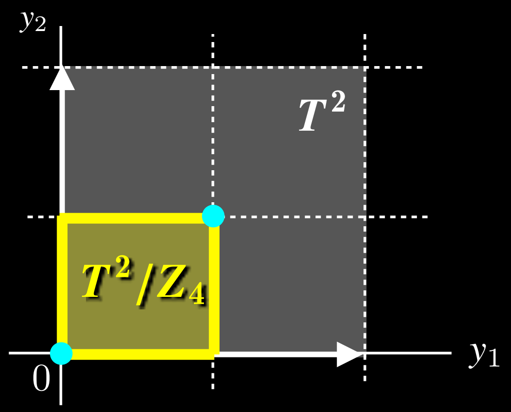
- The number of generations
- Expand coefficient

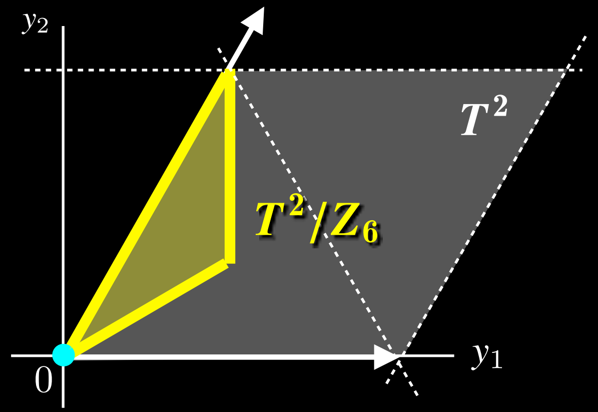
etc.

from 2d Quantum Mechanics analysis.









2d QM on magnetized orbifold

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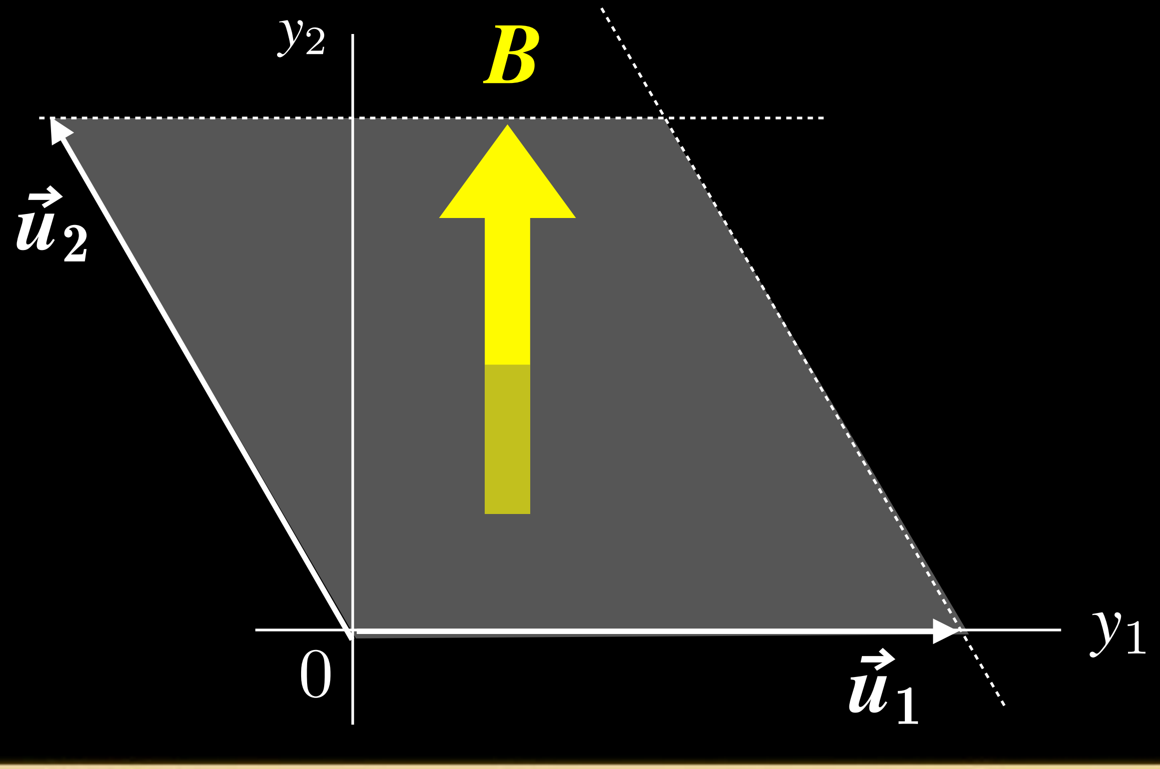
- **Wave-function form ($a_w = 0$ basis)**

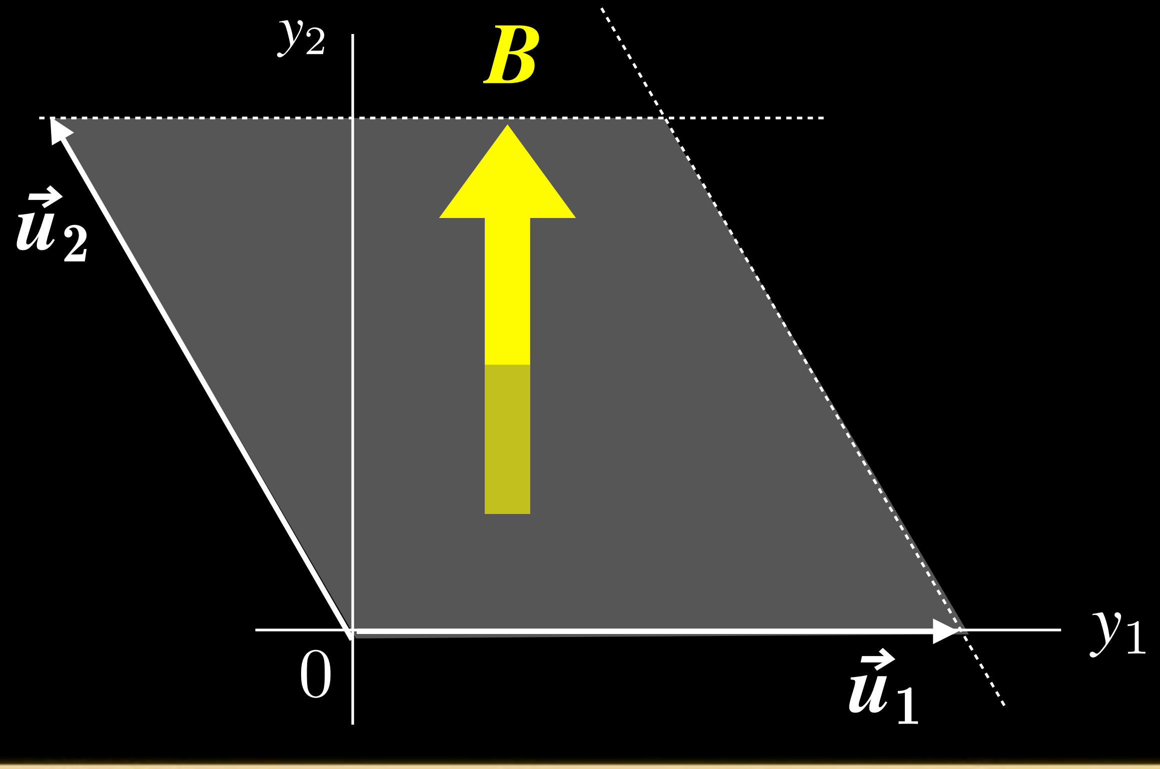
  Wilson line phase



2d QM on magnetized orbifold

❒ Wave-function form ($a_w = 0$ basis)

- **Hamiltonian** $H = (-i\vec{\nabla} - q\vec{A}(y))^2$
- **Vector potential** $\vec{A}(y) = -\frac{1}{2}\Omega\vec{y}$



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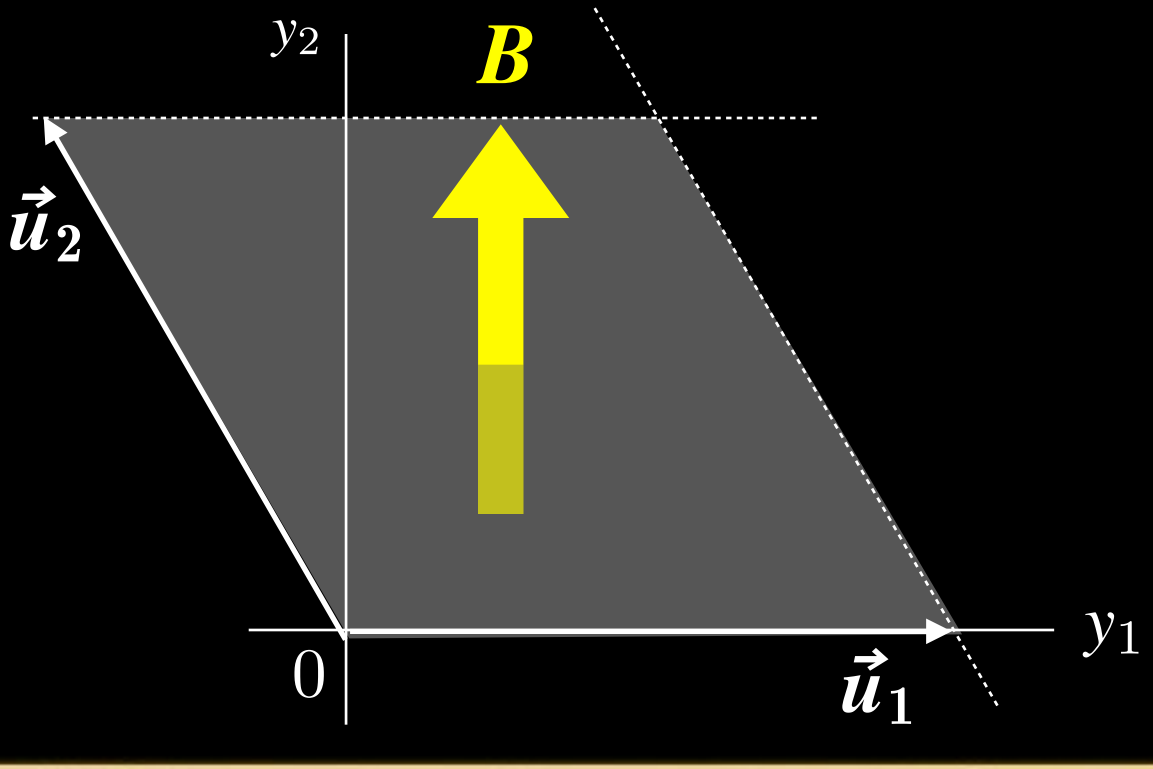
  $(a = 1, 2)$



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Scherk-Schwarz phase

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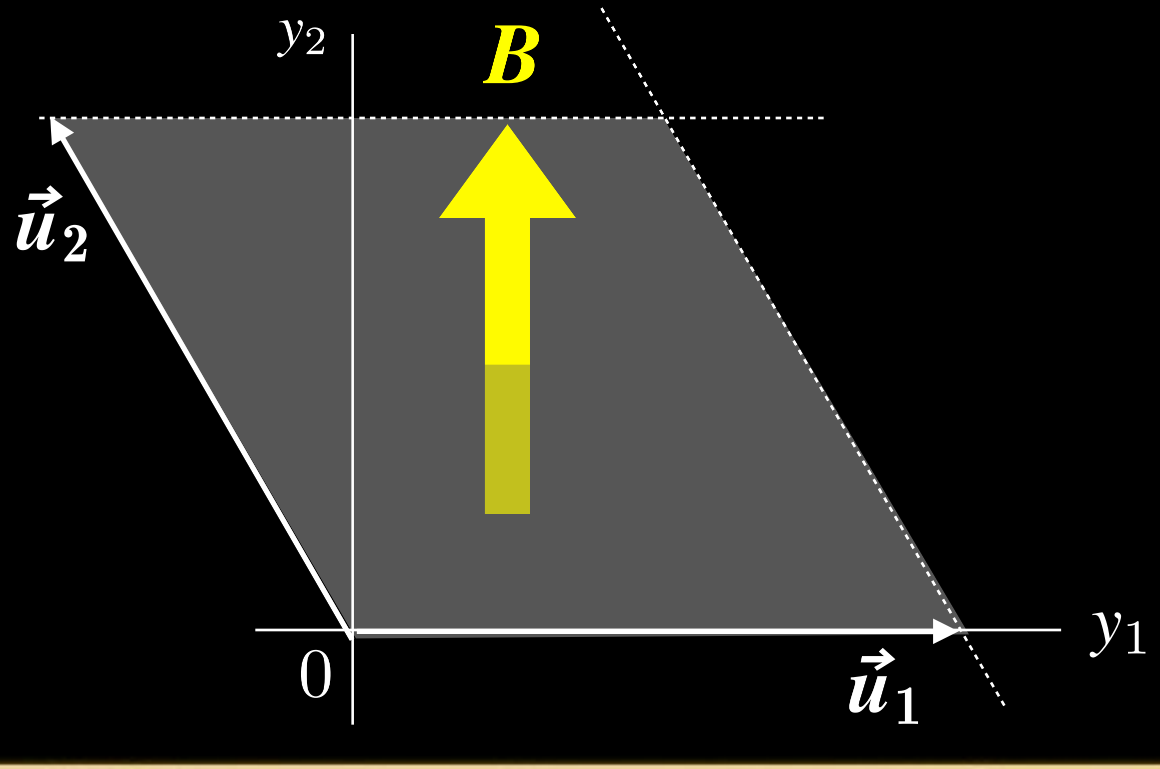
Basis of the torus



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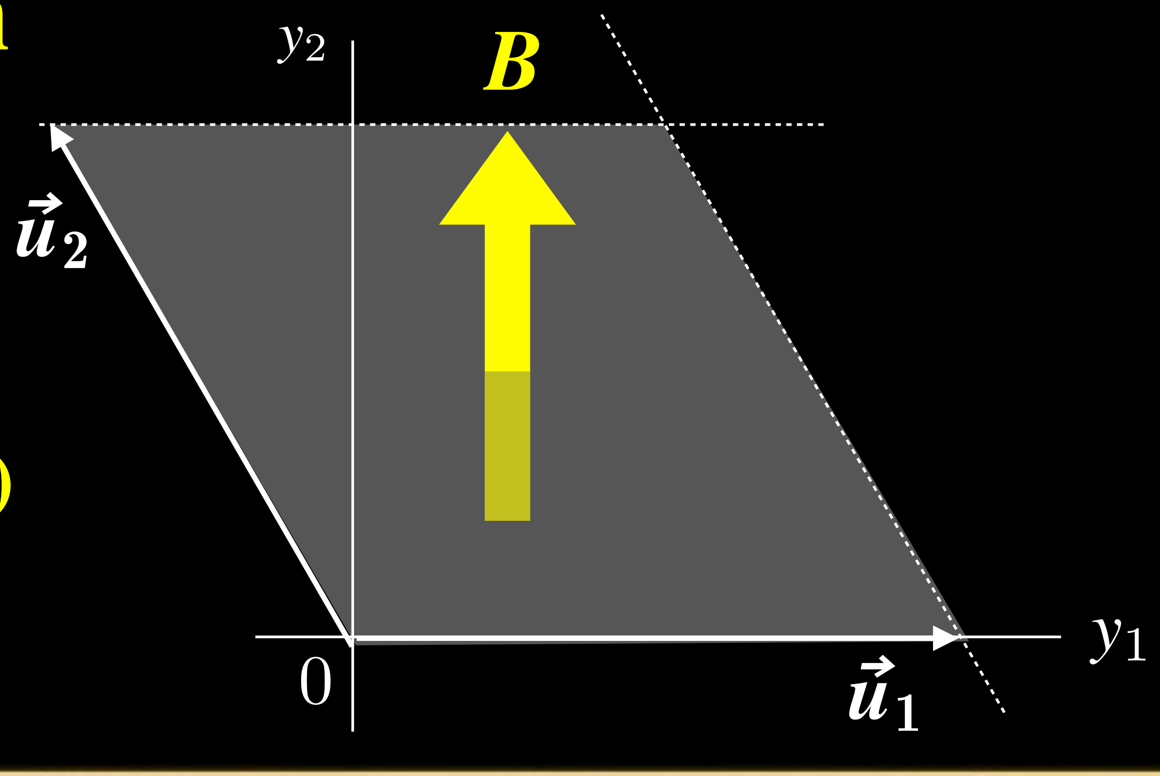
Basis of the torus

- ▸ **Magnetic flux quantization**

$$\frac{q(\vec{u}_1^{\mathrm{T}}B\vec{u}_2)}{2\pi} = M \in \mathbb{Z}$$

where $B = \frac{1}{2}(\Omega - \Omega^{\mathrm{T}})$



2d QM on magnetized orbifold

- **Operator formalism ($a_w = 0$ basis)**

$$\psi(y) = \langle y|\psi\rangle, \quad \vec{p} \equiv -i\vec{\nabla}, \quad [\hat{y}_i, \hat{p}_j] = i\delta_{i,j}$$

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Canonical transformation

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- **Hamiltonian** **(same as the harmonic oscillator)**

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$$e^{i\hat{\tilde{P}}}|\psi\rangle = |\psi\rangle$$

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Simultaneously diagonalizable.

2d QM on magnetized orbifold

❒ Eigenstates on T^2

$$\left|n, \frac{j}{M}\right\rangle_{T^2}$$

2d QM on magnetized orbifold

❒ Eigenstates on T^2

Index for the degeneracy
(Eigenvalue of the operator $\hat{\tilde{Y}}$)

$j \in \{0, 1, 2, \cdots, |M| - 1\}$

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KK-index (from Hamiltonian)

$n = 0, 1, 2, \cdots$

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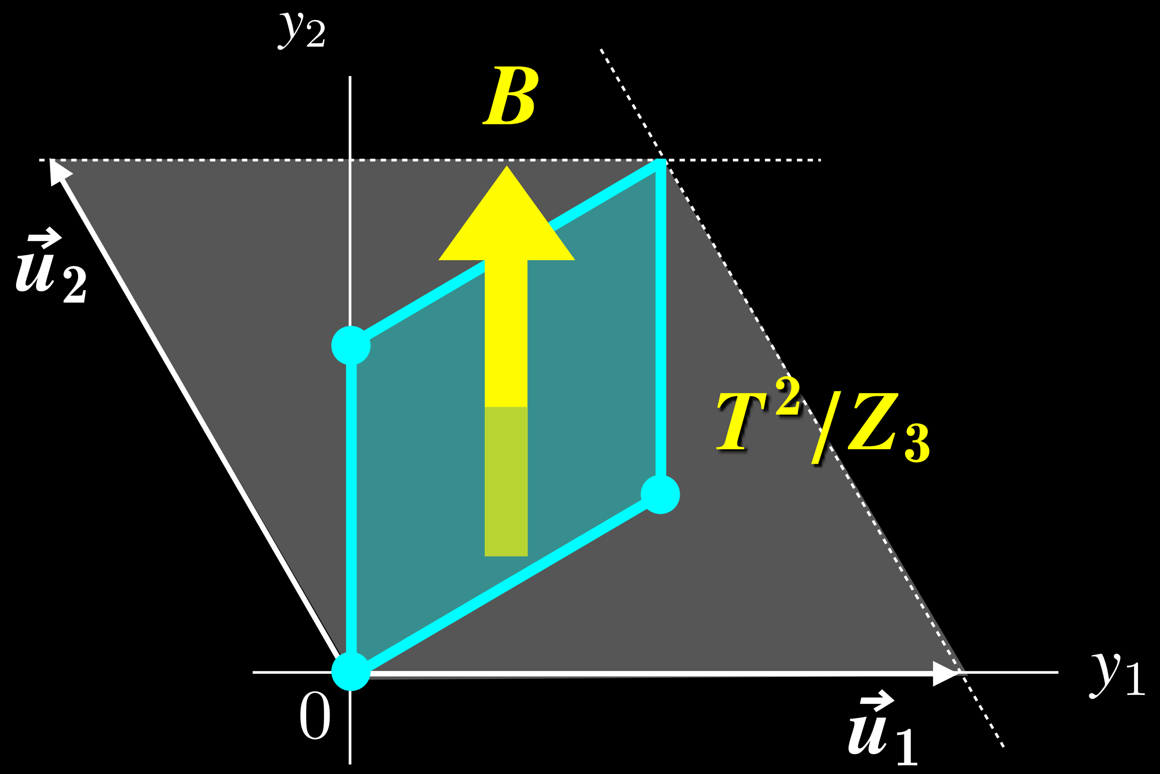
❐ Orbifold

$$U\hat{\vec{y}}U^\dagger = R_\omega \hat{\vec{y}}$$
$$U\hat{\vec{p}}U^\dagger = R_\omega \hat{\vec{p}}$$

where

$$R_\omega = \begin{pmatrix} \cos\omega & -\sin\omega \\ \sin\omega & \cos\omega \end{pmatrix}$$

$\omega \equiv e^{i\frac{2\pi}{N}}$



2d QM on magnetized orbifold

- **Expand coefficient**

2d QM on magnetized orbifold

❒ Expand coefficient

▶ **Operator formalism:**

$$U_{Z_3}\left|n,\frac{j}{M}\right\rangle_{T^2}=\sum_k\tilde{C}_{jk}\left|n,\frac{k}{M}\right\rangle_{T^2}$$

2d QM on magnetized orbifold

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theta functions …

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(e.g.) T^2/Z_3 orbifold

2d QM on magnetized orbifold

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(e.g.) T^2/Z_3 orbifold

▶ Eigenstate of $\hat{\tilde{Y}}$ from Z_3-rotated state $U_{Z_3}\left|n, \frac{j}{M}\right\rangle_{T^2}$

2d QM on magnetized orbifold

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(e.g.) T^2/Z_3 orbifold

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$$|n, 0\rangle_{T^2} \propto \sum_{k=1}^{M-1} e^{2\pi i \eta_k} U_{Z_3} \left| n, \frac{k}{M} \right\rangle_{T^2}$$

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↓

$$U_{Z_3}\left|n,\frac{j}{M}\right\rangle_{T^2} = \sum_k \tilde{C}_{jk}\left|n,\frac{k}{M}\right\rangle_{T^2}$$

$$\tilde{C}_{jk} = \frac{1}{\sqrt{M}} e^{-i\frac{\pi}{12}+\frac{3\pi\alpha^2}{M}+i\pi\frac{k(k+6\alpha)}{M}+2\pi i\frac{j\cdot k}{M}}$$

: Exact form !!

2d QM on magnetized orbifold

❒ Physical states

2d QM on magnetized orbifold

❐ Physical states

$$\left| n, \frac{j}{M} \right\rangle_{T^2/Z_N, \eta} = \frac{1}{N} \sum_{l=0}^{N-1} \bar{\eta}^l \left(\hat{U}^l_{Z_N} \left| n, \frac{j}{M} \right\rangle_{T^2} \right)$$

2d QM on magnetized orbifold

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Eigenvalue of Z_N orbifold

$(\eta = \omega^k; \ \ \omega \equiv e^{i\frac{2\pi}{N}}, \ \ k = 0, 1, 2, \cdots, N-1)$

2d QM on magnetized orbifold

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$$=\sum_{k=0}^{M-1}M_{jk}\left|n,\frac{k}{M}\right\rangle_{T^2}$$

2d QM on magnetized orbifold

- **Physical states**

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- **# of physical states = Rank [M_{jk}]**

2d QM on magnetized orbifold

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(e.g)
T^2/Z_3

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← **Consistent with the numerical analysis !!**

2d QM on magnetized orbifold

- **New formula…?**

2d QM on magnetized orbifold

- ❒ **New formula…?**

  (e.g.) T^2/Z_3 case $\left(\tau = e^{i\frac{2\pi}{3}}\right)$

2d QM on magnetized orbifold

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Operator formalism said

$$= \frac{1}{\sqrt{M}} e^{-i\frac{\pi}{12} + \frac{3\pi\alpha^2}{M} + i\pi\frac{k(k+6\alpha)}{M} + 2\pi i\frac{j\cdot k}{M}}$$

$(\alpha \equiv \alpha_1 = \alpha_2)$

Conclusion and Discussion

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- **We obtained exact analytic results for**

  - **The number of generations**
  - **Expand coefficient**

  etc.

from 2d Quantum Mechanics analysis.