Supersymmetric Extension of Gradient Flow Equation

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Ⅰ. Introduction

Gradient Flow Equation for SU(N) Yang-Mills Theory

$$\dot{B}_\mu = \frac{\delta S_{\rm YM}}{\delta B_\mu} + \alpha_0 D_\mu \xi$$

Gradient flow equation

$$B_\mu|_{t=0} = A_\mu$$

Boundary condition

where

A_μ :Usual gauge field

α_0 :Gauge parameter

$$\xi \equiv \partial \cdot B$$

$$D_\mu = \partial_\mu + [B_\mu, \cdot]$$

The dot in L.H.S means the derivative of fictitious time called flow time.
The B is a new gauge field and related to the usual gauge field A through the boundary condition of this equation .

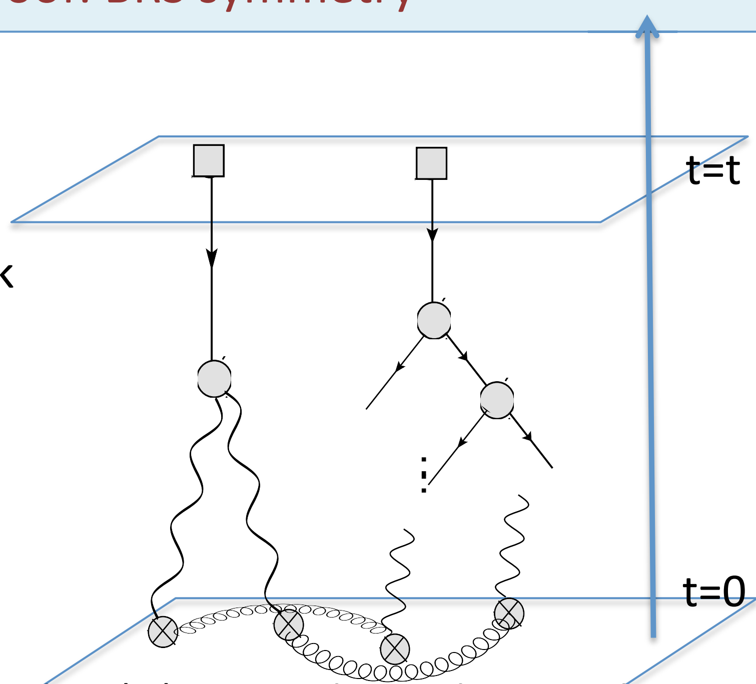
Luscher-Weisz Theorem

Claim

The expectation values in terms of the B field are finite without additional renormalization to all order!

Key point of proof: BRS symmetry

“B” fields lives in bulk



“A” fields of the usual SU(N) gauge theory lives at the boundary t=0

Applications of Gradient Flow

Various applications of the gradient flow equation

The gradient flow equation of matter fields

$$\dot{\bar{\chi}} = \bar{\chi}\overleftarrow{\Delta} + \alpha_0 \bar{\chi} \partial_\nu B_\nu,$$
$$\dot{\chi} = \Delta\chi - \alpha_0 \partial_\nu B_\nu \chi.$$

- Chiral condensate

Using the gradient flow equation of matter field, Luscher calculate the flow time dependent chiral condensate as a order parameter of chiral symmetry breaking.

$$\Sigma_t = -\frac{1}{2}\left\langle S_t^{uu} + S_t^{dd} \right\rangle$$

- Step scaling

The gradient flow gives new scheme of the renormalization group in the lattice theory.

$$g^2(L) = \text{constant} \times \{t^2 \langle E_t \rangle\}_{\sqrt{8t}=\frac{1}{3}L}.$$

- Improved action

The method of gradient flow is useful to construct improved action in the lattice QCD

$$\mathrm{tr}\{G_{\mu\nu}G_{\nu\mu}\}, \quad \bar{\chi}\chi, \quad \bar{\chi}\sigma_{\mu\nu}G_{\mu\nu}\chi, \quad (\bar{\chi}\Gamma\chi)(\bar{\chi}\Gamma\chi)$$

Problem of Gradient Flow Equation of Matter Field

What physical system the gradient flow method can be applied.

Luscher proposed the equation of the matter field.

$$\dot{\bar{\chi}} = \bar{\chi}\overleftarrow{\Delta} + \alpha_0 \bar{\chi}\partial_\nu B_\nu,$$
$$\dot{\chi} = \Delta\chi - \alpha_0 \partial_\nu B_\nu \chi.$$

- They do not satisfy the Luscher-Weisz theorem.
  There is no power divergence, but there are logarithmic divergences.

- The equation is no longer defined from the gradient of the action.

Can we construct the gradient flow equation of the matter field which satisfies the Luscher-Weisz theorem exactly?

In other words, what is the correct form of the gradient flow equation of the matter fields?

How to Derive Gradient Flow Equation of Matter field?

→ We focus on the Super Yang-Mills theory.

Theory	Gradient Flow Equation	Gradient of what?
Yang-Mills theory	Gauge field A	Yang-Mills action
	Matter field χ	None
Super Yang-Mills theory	Vector superfield V including gauge and gaugino fields	Super Yang-Mills action

If we derive the gradient flow equation of the vector superfield by the analogy of the Yang-Mills gradient flow, we can obtain the equation of matter field very naturally, because the super Yang-Mills theory contains gaugino as a 'matter' field.

Ⅱ. Generalization of Gradient Flow Equation

Generalization of Gradient Flow Equation

We propose a generalization of the gradient flow equation for the quantum field theory with nontrivial metric in functional space

General Form of Gradient Flow Equation

$$\frac{\partial \phi^a}{\partial t} = g^{ab}(\phi) \frac{\delta S(\phi)}{\delta \phi^b}$$

g^{ab} is the metric, which constructs the invariant norm on the functional space

$$||\delta\phi||^2 = \int d^4x g_{ab}(\phi(x)) \delta\phi^a(x) \delta\phi^b(x), \qquad a = 1, 2, \cdots, D.$$

In the case of Yang-Mills theory,

$$g^{ab}(A_\mu) = \delta^{ab}$$

Ⅲ. Appling Equation to SU(N) Super Yang-Mills Theory.

Metric of SU(N) Super Yang-Mills Theory

Super gauge transformation

$$e^V \to e^{-i\Lambda^\dagger} e^V e^{i\Lambda}$$

$i\Lambda$: Chiral field

$-i\Lambda^\dagger$: Anti-chiral field

In the case of super Yang-Mills theory, the metric is not flat.

$$\begin{aligned} ||\delta V||^2 &= \int d^8 z \mathrm{Tr}[e^{-V}\delta e^V e^{-V}\delta e^V] \\ &= \int d^8 z \delta V^a(z)\delta V^b(z) \mathrm{Tr}\left[\left(\frac{1-e^{-L_V}}{L_V}T^a\right)\left(\frac{1-e^{-L_V}}{L_V}T^b\right)\right] \end{aligned}$$

$$g^{ab}(V) = \mathrm{Tr}\left[\left(\frac{1-e^{-L_V}}{L_V}T^a\right)\left(\frac{1-e^{-L_V}}{L_V}T^b\right)\right]$$

where $L_V \cdot = [V, \cdot]$

Gradient Flow Equation of Super Yang-Mills Theory

$$\frac{\partial B^a(x)}{\partial t} = \delta^{ab}\frac{\delta S_{\rm YM}}{\delta B^b(x)} + \alpha_0 \delta B^a(x)$$

$$\frac{\partial v^a(z)}{\partial t} = g^{ab}(v(z))\frac{\delta S_{\rm SYM}}{\delta v^b(z)} + \alpha_0 \delta v^a(z)$$

$$\dot{B}_\mu = D_\nu G_{\nu\mu} + \alpha_0 D_\mu \partial_\nu B_\nu$$

$$B_\mu|_{t=0} = A_\mu$$

$$S_{YM} = \int d^D x Tr[F_{\mu\nu}(x)F_{\mu\nu}(x)]$$

The general form of gradient flow equation of super Yang-Mills theory.

$$v|_{t=0} = V$$

$$\begin{aligned} S_{SYM} &= \int d^4x {\rm Tr}[W^\alpha W_\alpha|_{\theta\theta} + \bar{W}_{\dot\alpha}\bar{W}^{\dot\alpha}|_{\bar\theta\bar\theta}] \\ &= -\int d^8 z {\rm Tr}[e^{-V}(D^\alpha e^V)W_\alpha] + h.c. \end{aligned}$$

$$W_\alpha = -\bar{D}^2(e^{-V}D_\alpha e^V)$$

Replacement

- Yang-Mills action $S_{\rm YM}$ $\rightarrow$ Super Yang-Mills action $S_{\rm SYM}$.
- New gauge field B^a $\rightarrow$ New superfield v^a.
- Gauge transformation δB^a $\rightarrow$ Super gauge transformation δv^a.
- Metric δ^{ab} $\rightarrow$ $g^{ab}(v)$

Explicit Form of Super Yang-Mills Gradient Flow Equation

General form of gradient flow equation of super Yang-Mills theory.

$$\frac{\partial v^a(z)}{\partial t} = g^{ab}(v(z)) \frac{\delta S_{\text{SYM}}}{\delta v^b(z)} + \alpha_0 \delta v^a(z)$$

We substitute the explicit form of $g^{ab}, \frac{\delta S_{\text{SYM}}}{\delta v}, \delta v$ into the gradient flow equation

Explicit form of super Yang-Mills gradient flow equation

$$\frac{\partial v}{\partial t} = \frac{L_v}{1 - e^{-L_v}} (F + \alpha_0 \Phi_v) + h.c.$$

where $F = D^\alpha w_\alpha + \{e^{-v} D^\alpha e^v, w_\alpha\}.$

Or We can also rewrite this equation in terms of e^v as a nonlinear field.

$$\frac{\partial e^v}{\partial t} = e^v (F + \alpha_0 \Phi_v) + h.c.$$

Ⅳ. Flow Equation Closed under Wess Zumino Gauge

Can We Reduced to Wess Zumino Gauge?

The gradient flow equation has infinite number of commutators in general gauge

Very difficult to solve it

➡ **How about WZ gauge?**

In order to obtain the flow equation with finite number of terms, we choose Wess-Zumino gauge.

Problem

The time evolution can carry the system away from the WZ gauge.

Solution

Find a special choice of α_0 term which keeps WZ gauge under time evolution!

Determination of α_0 Term

$+\alpha_0 \delta v$ Does the α_0 term, which satisfies the following requirements, exist?

- It is positive.
- The mass dimension is two.
- It is described by the super gauge transformation.
- The flow of the vector field keeps the WZ gauge at any flow time.

The answer is "YES"! We found the special form, which satisfy the requirements.

$$\alpha_0 = 1$$

$$\delta v = \Phi_v + \Phi_v^\dagger + \frac{1}{2}[v, \Phi_v - \Phi_v^\dagger] + \frac{1}{12}[v, [v, \Phi_v + \Phi_v^\dagger]].$$

$$\Phi_v = \bar{D}^2(D^2 v + [D^2 v, v]).$$

Gradient Flow Equation of Super Yang-Mills Theory for Each Component of Vector Multiplet

$$
\begin{aligned}
\dot{c} &= 0, \\
\dot{\chi} &= 0, \\
\dot{\bar{\chi}} &= 0, \\
\dot{m} &= 0, \\
\dot{m}^* &= 0,
\end{aligned}
$$

$$
\begin{aligned}
\dot{v}_m &= -16\mathscr{D}^k v_{mk} + 16\mathscr{D}_m \partial_k v^k - 8\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}_m \lambda)^{\dot{\alpha}}\}, \\
\dot{\bar{\lambda}} &= -16\bar{\sigma}^k \sigma^m \mathscr{D}_k \mathscr{D}_m \bar{\lambda} + 8[\bar{\lambda}, d + i\partial_m v^m], \\
\dot{\lambda} &= -16\sigma^k \bar{\sigma}^m \mathscr{D}_k \mathscr{D}_m \lambda - 8[\lambda, d - i\partial_m v^m], \\
\dot{d} &= 16\Box d + 16i[v_m, \partial^m d] \\
&\quad +2i\mathrm{Tr}[\bar{\sigma}^m \sigma^l \bar{\sigma}^n \sigma^k - \bar{\sigma}^m \sigma^k \bar{\sigma}^n \sigma^l]\mathscr{D}_n \mathscr{D}_l v_{mk} \\
&\quad +8i\{\bar{\lambda}_{\dot{\alpha}}, (\bar{\sigma}^m \mathscr{D}_m \lambda)^{\dot{\alpha}}\} - 8i\{\lambda^\alpha, (\sigma^m \mathscr{D}_m \bar{\lambda})_\alpha\} \\
&\quad -4[v_m, [v^m, d]].
\end{aligned}
$$

Compare Our Results with Luscher's One

The flow of the matter field

Luscher's one

$$\dot{\bar{\chi}} = \bar{\chi}\overleftarrow{\Delta} + \alpha_0 \bar{\chi}\partial_\nu B_\nu,$$
$$\dot{\chi} = \Delta\chi - \alpha_0 \partial_\nu B_\nu \chi.$$

Our results

$$\dot{\bar{\lambda}} = -16\bar{\sigma}^k \sigma^m \mathscr{D}_k \mathscr{D}_m \bar{\lambda} + 8[\bar{\lambda}, d + i\partial_m v^m],$$
$$\dot{\lambda} = -16\sigma^k \bar{\sigma}^m \mathscr{D}_k \mathscr{D}_m \lambda - 8[\lambda, d - i\partial_m v^m].$$

We find that if we regard Δ as $\not{D}^2$, Lucher's one are almost similar to our results except for $[\lambda, d]$ terms and the point that α0 terms are described in terms of commutation relations.

Comments on Symmetry of Gradient Flow Equation

Supersymmetry of the gradient flow equation

Manifestly satisfied

Flow Time Dependence of Super Gauge Transformation

When we demand the super gauge symmetry of the gradient flow equation at any flow time, Λ has to satisfy the following condition.

$$i\frac{\partial \Lambda}{\partial t} = \alpha_0(\delta\Phi_V + i[\Lambda, \Phi_V])$$

This symmetry is a key to prove the Luscher-Weisz theorem in the case of super Yang-Mills theory.

V. Summary and Discussion

Summary and Discussion 1

There are four important results.

- We proposed the general form of the gradient flow equation for the QFT with nontrivial functional space.

- Applying this equation to super Yang-Mills theory, we obtained the gradient flow equation of the Super Yang-Mills gradient flow, which has manifest supersymmetry and super gauge symmetry.

- We found the special α_0 term to keep the flow under the WZ gauge consistently.

- We derived the gradient flow equation of matter field very naturally.

Summary and Discussion 2

Future's work

- Examining whether the gradient flow of super Yang-Mills theory satisfies Lusche-Weisz theorem or not.

- Studying whether the gradient flow of matter field which is obtained in this study has better property more than Luscher's one or not.

- Appling the general gradient flow equation to other theories such as the nonlinear sigma model.

Thank you for your kind attention.