Tinkertoys for Gaiotto Duality

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Introduction

A large class of $4d\,\mathcal{N}=2$ superconformal gauge theories arise as the partially-twisted compactification of a six-dimensional $(2,0)$ theory on a punctured Riemann surface C with certain real codimension-two defects of the six-dimensional theory at the punctures. The moduli space $\mathcal{M}_{g,n}$ of C can be identified with the space of exactly marginal deformations of the $4d$ SCFT. In a degeneration limit, C can be decomposed into three-punctured spheres ("fixtures") connected by cylinders, where the fixtures correspond to some kind of "matter" and the cylinders to weakly-coupled gauge groups. Different pair-of-pants decompositions of C correspond to different weakly-coupled gauge theory presentations of the same theory, related by S-duality. By classifying these basic building blocks, we can build up an arbitrary surface C as a "tinkertoy".

Seiberg-Witten solutions

This construction also realizes the Seiberg-Witten curve of the gauge theory as a branched cover of C. By reducing any $4d\,\mathcal{N}=2$ gauge theory on S^1, one obtains a $3d\,\mathcal{N}=4$ sigma model with hyperkähler target $\mathcal{M}$, which is a fibration over the 4d Coulomb branch $\mathcal{M}\to\mathcal{B}$ with generic fiber $\sim T^{2r}$. For these theories, $\mathcal{M}$ is the moduli space of solutions to Hitchin's equations on C . Denoting the Higgs field $\Phi(z)$, the $4d$ Coulomb branch is parametrized by

$$\phi_k(z)\sim Tr(\Phi(z)^k)\in H^0(C,K_C^{\otimes k})$$

and the Seiberg-Witten curve Σ is given by the spectral curve of the Hitchin system

$$\Sigma:\{\det(xdz-\Phi(z))=0\}\subset K_C$$

Codimension-two defects of the $6d\ (2,0)$ theory

Codimension-two defects are in 1-1 correspondence with homomorphisms

$$\rho:\mathfrak{su}(2)\to\mathfrak{g}$$

or, equivalently, with *nilpotent orbits* O_ρ in $\mathfrak{g}$. Nilpotent orbits in any simple lie algebra $\mathfrak{g}$ are classified by pairs $(\mathfrak{l},O^{\mathfrak{l}})$, where $\mathfrak{l}$ is a Levi subalgebra of $\mathfrak{g}$, and $O^{\mathfrak{l}}$ is a *distinguished* nilpotent orbit in $\mathfrak{l}$. For classical $\mathfrak{g}$, nilpotent orbits are equivalently classified by certain partitions, which we write as a Young diagram. For $\mathfrak{g}$ exceptional, nilpotent orbits are classified as above.

When $J=A_{N-1},D_N,E_6$, one can also introduce a sector of "twisted" defects, where, upon traversing a non-contractible cycle on C, the fields undergo a monodormy $o\in A$, the outer-automorphism group of J. When o is non-trivial, the defect is labeled by a nilpotent orbit in $\mathfrak{g}$, where $\mathfrak{g}^\vee$ is the subalgebra of $\mathfrak{j}$ invariant under o.

The effect on the Coulomb branch of the theory due to the presence of a defect labeled by $O_\rho\in\mathfrak{g}$ is determined by the properties of a nilpotent orbit $O_{\tilde\rho}\in\mathfrak{g}^\vee$. These nilpotent orbits are related by the *Spaltenstein map*:

$$d:\mathcal{N}_{\mathfrak{g}}/G\to\mathcal{N}_{\mathfrak{g}}^\vee/G^\vee$$

which is defined for any simple $\mathfrak{g}$. For a nilpotent orbit in $\mathfrak{su}(N)$, d is an isomorphism and is given by taking the transpose of the Young diagram labeling O_ρ. For other $\mathfrak{g}$, it is in general no longer an isomorphism, but satisfies $d^3=d$.

A puncture corresponds to a local boundary condition for the Higgs field $\Phi(z)$. For an untwisted defect, this is

$$\Phi(z)\sim\left[\frac{\Phi_{-1}}{z}+\Phi_0+\dots\right]dz$$

where $\Phi_{-1}\in d(O_\rho)$ and $\Phi_0\in\mathfrak{j}$.

When o is non-trivial, $\mathfrak{j}$ splits into a direct sum of eigenspaces under the action of o:

$$\mathfrak{j}=\mathfrak{j}_1+\mathfrak{j}_{-1},\qquad \text{for } o \text{ of order 2}$$
$$\mathfrak{j}=\mathfrak{j}_1+\mathfrak{j}_\omega+\mathfrak{j}_{\omega^2},\quad \text{for } o \text{ of order 3}$$

The boundary condition for the Higgs field in each case is then

$$\Phi(z)\sim\left[\frac{\Phi_{-1}}{z}+\frac{\Phi_{-1/2}}{z^1/2}+\Phi_0+\dots\right]dz$$
$$\Phi(z)\sim\left[\frac{\Phi_{-1}}{z}+\frac{\Phi_{-2/3}}{z^2/3}+\frac{\Phi_{-1/3}}{z^1/3}+\Phi_0+\dots\right]dz$$

where $\Phi_{-1}\in d(O_\rho)$, $\Phi_{-1/2}\in\mathfrak{j}_{-1}$, $\Phi_{-1/3}\in\mathfrak{j}_\omega$, $\Phi_{-2/3}\in\mathfrak{j}_{\omega^2}$, and $\Phi_0\in\mathfrak{j}_1\equiv\mathfrak{g}^\vee$.

S-duality invariants

To check our identifications, we compute the following S-duality invariants

- Graded Coulomb branch dimensions
- Higgs branch dimension
- Global symmetry group (acts as hyperkähler isometries of the Higgs branch)
- Level k_{G_i} of each non-abelian factor $G_i\subset G_{\text{global}}$, defined by the current algebra

$$J^a_\mu(x)J^b_\nu(0)=\frac{3k_G}{4\pi^4}\delta^{ab}\frac{g_{\mu\nu}x^2-2x_\mu x_\nu}{(x^2)^4}+\frac{2}{\pi^2}f^{ab}_c\frac{x_\mu x_\nu x\cdot J^c}{(x^2)^3}$$

- Conformal anomaly coefficients (a,c), defined via

$$T_\mu^\mu=\frac{c}{16\pi^2}(\text{Weyl})^2-\frac{a}{16\pi^2}(\text{Euler})$$

Superconformal Index

The superconformal index contains all information about the protected spectrum of an SCFT which can be obtained from representation theory alone. It is evaluated by the following trace formula

$$\mathcal{I}(\mu_i)=Tr(-1)^Fe^{-\mu_iT_i}e^{-\beta\delta},\quad\delta=2\{Q,Q^\dagger\}$$

where

- The trace is over the states of the theory on S^3 in the radial quantization.
- $\{T_i\}$ is a set of generators for the Cartan of $SU(1,1|2)$, which is the subalgebra of the $4d\,\mathcal{N}=2$ superconformal algebra, $SU(2,2|2)$, commuting with Q.
- States with $\delta\neq0$ cancel pairwise, so the index counts states with $\delta=0$.

Since $SU(1,1|2)$ has rank 3, the SCI depends on 3 superconformal fugacities (p,q,t), as well as fugacities parametrizing the Cartans of the flavor symmetry of each puncture, which is given by the centralizer of $\rho_i(SU(2))$.

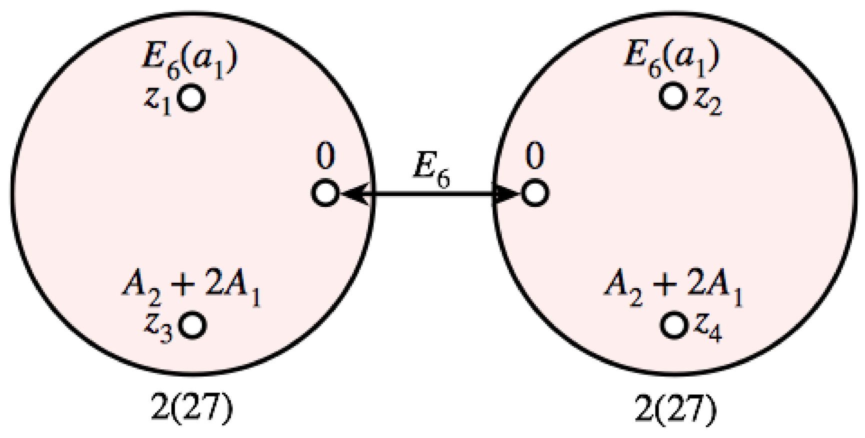
For these theories, the SCI has been shown by Gadde et al to take the form of a correlation function in a 2d TQFT on C:

$$\mathcal{I}_{g,n}(\mathbf{a}_1,\dots,\mathbf{a}_n)=\sum_\lambda(C_{\lambda\lambda\lambda})^{2g-2+n}\prod_{i=1}^n f_\lambda(\mathbf{a}_i)$$

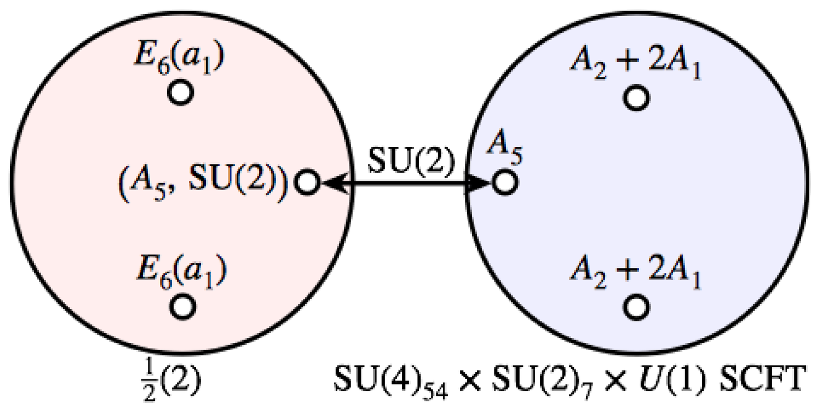
This expression also holds for *non-Lagrangian* SCFTs, allowing us to study the BPS spectrum of these theories. In particular, we can use the SCI to determine G_{global} for each fixture.

S-duality of $E_6+4(27)$

E_6 gauge theory with 4 fundamentals is realized as the 4-punctured sphere:



The S-dual is an $SU(2)$ gauging of the $SU(4)_{54}\times SU(2)_7\times U(1)$ SCFT, with an additional half-hyper in the fundamental.

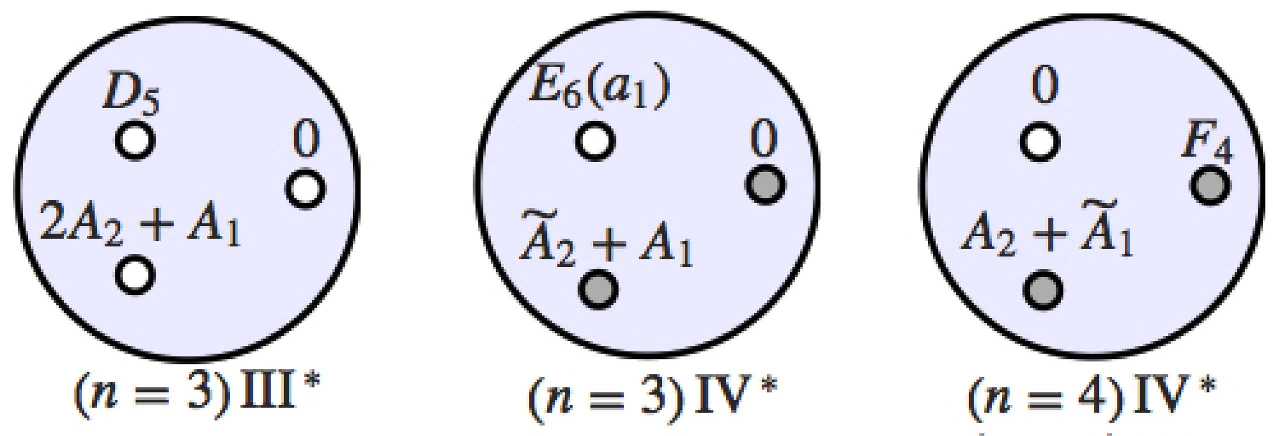


Connections with F-theory

The worldvolume theory on n D3-branes at a IV^*, III^*, or II^* singularity in F-theory is an $\mathcal{N}=2$ SCFT. For $n=1$, these are, respectively, the $(E_6)_6$, $(E_7)_8$, and $(E_8)_{12}$ SCFTs of Minahan-Nemeschansky. For higher n, the properties of these theories were computed by Aharony and Tachikawa:

F-Theory singularity	Flavour symmetry	Graded Coulomb branch dimensions	(n_h,n_v)
IV*	$(E_6)_{6n}\times SU(2)_{(n-1)(3n+1)}$	$n_{3l}=1,\quad l=1,2,\dots,n$	$(3n^2+14n-1,n(3n+2))$
III*	$(E_7)_{8n}\times SU(2)_{(n-1)(4n+1)}$	$n_{4l}=1,\quad l=1,2,\dots,n$	$(4n^2+21n-1,n(4n+3))$
II*	$(E_8)_{12n}\times SU(2)_{(n-1)(6n+1)}$	$n_{6l}=1,\quad l=1,2,\dots,n$	$(6n^2+35n-1,n(6n+5))$

Besides numerous examples of $n=1,2$, we find examples of $n=3,4$ by compactifying the $E_6\ (2,0)$ theory on the following three-punctured spheres:



Work in progress

1. Tinkertoy catalog for E_7 and E_8.
2. S_3-twisted D_4 theory
   - Outer-automorphism group, A, enhances to S_3 for D_4
   - S_3 is non-abelian, so we are no longer measuring twists by $H^1(C,A)$, but by $Hom(\pi_1(C),A)$.
3. Z_2-twisted A_{2N} theory
   - Compactifying the $A_{2N}\ (2,0)$ theory on S^1 with a $\mathbb{Z}_2$-twist gives $5d\,\mathcal{N}=2$ SYM with a non-trivial discrete theta angle.
   - This leads to various subtleties.
   - Recent work by mathematicians on "exotic nilpotent cones" might be relevant.