Higgs branch localization of 3d N=2 theories

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mainly based on
M.Fujistuka(Sokendai), M. Honda(HRI) and Y.Y
arXiv:1312.3627 [hep-th]
and partially based on Y. Y arXiv:1403.0891[hep-th]

Factorization of 3d partition function(PF)

The partition function of $\mathcal{N}=2$ $G=U(1)$ $2N_f$-flavor chiral multiplets (R-charge:$\Delta=0$) on S_b^3 :

$$Z(S_b^3) = \int d\sigma_0 e^{2\pi i\zeta\sigma} \prod_{j=1}^{2N_f} s_b\left(\sigma_0 + m_j + i\frac{Q}{2}\right)$$

σ_0: saddle point value of the scalar in vector multiplet

m_j: real masses ζ : FI-parameter

$s_b\left(\sigma_0 + m_j + i\frac{Q}{2}\right)$: 1-loop determinants of chiral multiplets

Factorization of 3d partition function(PF)

Pasquetti (arXiv:1111.6905) showed that the PF on S_b^3 factorized to vortex and anti-vortex partition functions

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$$= \sum_{l=1}^{2N_f} e^{\pi i \zeta m_l} \prod_{j\neq l} s_b\left(m_j - m_l + i\frac{Q}{2}\right) Z_V^{3\mathrm{d},l} \bar{Z}_V^{3\mathrm{d},l}$$

$Z_V^{3\mathrm{d},l}$: 3d vortex PF on $S_\beta^1 \times \mathbb{R}_\varepsilon^2$ $(\beta\varepsilon = 2\pi i/b^2, \cdots)$
$\bar{Z}_V^{3\mathrm{d},l}$: 3d anti-vortex PF on $S_\beta^1 \times \mathbb{R}_\varepsilon^2$ $(\beta\varepsilon = 2\pi i b^2, \cdots)$

Superconformal indices ($S^1 \times S^2$) also factorize to vortex and anti-vortex partition functions (Hwang-Kim-Park 1211.6023)

generalization to $G = U(N)$ on S_b^3 (Taki 1303.5915)

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$(\varepsilon,\ \beta)$ are determined ?

Higgs branch localization

can answer these questions.

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]}$$

one-parameter family of Q-exact term

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi] - tQV[\Phi]}$$

one-parameter family of Q-exact term

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]-tQV[\Phi]}$$

Kapsutin-Willet-Yaakov

Hama-Hosomichi-Lee

$$t \to \infty$$

$$QV[\Phi] = S_{\rm YM} + S_{\psi}$$

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Evaluating the multi-contour integrals

(Pasquetti, Hwang-Kim-Park, Taki)

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Evaluating the multi-contour integrals

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Content of the talk

- Introduction
- 3d N=2 theories on ellipsoid
- Higgs branch localization
- vortex partition function
- Factorization of 4d N=1 superconformal index
- Summary

3d $\mathcal{N}=2$ theories on ellipsoid

We consider $G=U(N)$ SYM theory +
N_f- fundamental chiral multiplets ($N_f \geq N$)
with generic real masses $M=\mathrm{diag}(m_1,\cdots m_{N_f})$ on S_b^3

3d ellipsiod S_b^3 is defined by

$$x_0^2+x_1^2+x_2^2+x_3^2=1$$

Torus fibration coordinate: $(\vartheta,\varphi_1,\varphi_2)$

$$(x_0,x_1,x_2,x_3)=(\cos\vartheta\cos\varphi_2,\cos\vartheta\sin\varphi_2,\sin\vartheta\cos\varphi_1,\sin\vartheta\sin\varphi_1)$$

metric

$$ds^2=R^2\left(f(\vartheta)^2 d\vartheta^2+b^2\sin^2\vartheta d\varphi_1^2+b^{-2}\cos^2\vartheta d\varphi_2^2\right)$$

Hopf fibration coordinate $\quad \vartheta=\frac{1}{2}\theta,\quad \varphi_1=\frac{1}{2}(\psi-\phi),\quad \varphi_2=\frac{1}{2}(\psi+\phi)$

Q-exact terms

Vector multiplet: $(A_\mu, \sigma, D, \lambda, \bar{\lambda})$

$$\mathcal{L}_{\rm YM} = Q\mathrm{Tr}\frac{(Q\lambda)^\dagger\lambda + (Q\bar{\lambda})^\dagger\bar{\lambda}}{4}$$

$$= \frac{1}{4}F_{\mu\nu}F^{\mu\nu} + \frac{1}{2}D_\mu\sigma D^\mu\sigma + \frac{1}{2}\left(D + \frac{\sigma}{Rf(\vartheta)}\right)^2 + \cdots$$

chiral multiplet: (ϕ, ψ, F)

$$\mathcal{L}_\psi = Q\frac{(Q\psi)^\dagger\psi + (Q\bar{\psi})^\dagger\bar{\psi}}{2}$$

$$= D^\mu\bar{\phi}D_\mu\phi + \frac{\Delta(2-\Delta)}{(Rf(\vartheta))^2}\bar{\phi}\phi + \bar{F}F + i\bar{\phi}D\phi$$

$$+\bar{\phi}(\sigma+M)^2\phi + \frac{i(2\Delta-1)}{Rf(\vartheta)}\bar{\phi}(\sigma+M)\phi + \cdots$$

$\mathcal{L}_{\mathrm{YM}}$ and $\mathcal{L}_{\psi}$ are taken as QV in the ordinary localization

saddle points equation $(\mathcal{L}_{\mathrm{YM}} = 0, \mathcal{L}_{\psi} = 0)$

$$A_\mu = 0,\ \sigma = \sigma_0 = \text{consant},\ D + \frac{\sigma}{Rf(\vartheta)} = 0$$

$$\phi = 0, F = 0$$

$\cdot Q$-closed terms

$$\mathcal{L}_{\mathrm{CS}} = \epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda + 2D\sigma$$

$$\mathcal{L}_{\mathrm{FI}} = -\frac{\zeta}{2\pi R}\left(D - \frac{\sigma}{R f(\vartheta)}\right)$$

The saddle point values of Q-closed terms contribute to the localization computation.

In the ordinary localization, the partition function is expressed by multi-contour integrals.

$$Z = \int d^{\mathrm{rank}(G)}\sigma_0 e^{-S(\sigma_0)} Z_{\text{1-loop}}(\sigma_0)$$

Summation over the saddle points

$\cdot Q$-closed terms

$$\mathcal{L}_{\rm CS} = \epsilon^{\mu\nu\rho}\left(A_\mu \partial_\nu A_\rho + \frac{2i}{3} A_\mu A_\nu A_\rho\right) - \bar{\lambda}\lambda + 2D\sigma$$

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a saddle point value of Q-closed term

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The one-loop determinant of QV

$$Z = \int \mathcal{D}\Phi e^{-S[\Phi]-tQV[\Phi]}$$

Kapsutin-Willet-Yaakov

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$$QV[\Phi] = S_{\rm YM} + S_\psi$$

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Evaluation of multi-contour integrals

(Pasquetti, Hwang-Kim-Park, Taki)

Higgs branch localization

We add a Q-exact term

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$$\mathcal{L}_H = QV_H = -iQ\mathrm{Tr}\left[\frac{(\epsilon^\dagger\lambda - \bar{\epsilon}^\dagger\bar{\lambda})(\phi\bar{\phi} - \chi\mathbf{1}_N)}{4i}\right]$$

real number

$$tQV = t\int \sqrt{g}d^3x(\mathcal{L}_{\mathrm{YM}} + \mathcal{L}_\psi + \mathcal{L}_H)$$

The final result does not depend on both t and χ

The saddle point equations change !

In the limit $\chi \sim \infty$, the saddle points $(QV = 0)$ become the saddle points $(\theta \neq 0, \pi)$

$$F_{\mu\nu} = 0,\ D_\mu \sigma = 0,\ D + \frac{1}{Rf(\vartheta)}(\sigma + M) = 0,$$

$$(\sigma + M)\phi = 0,\ \phi\bar{\phi} - \chi \mathbf{1}_N = 0$$

root of Higgs branch

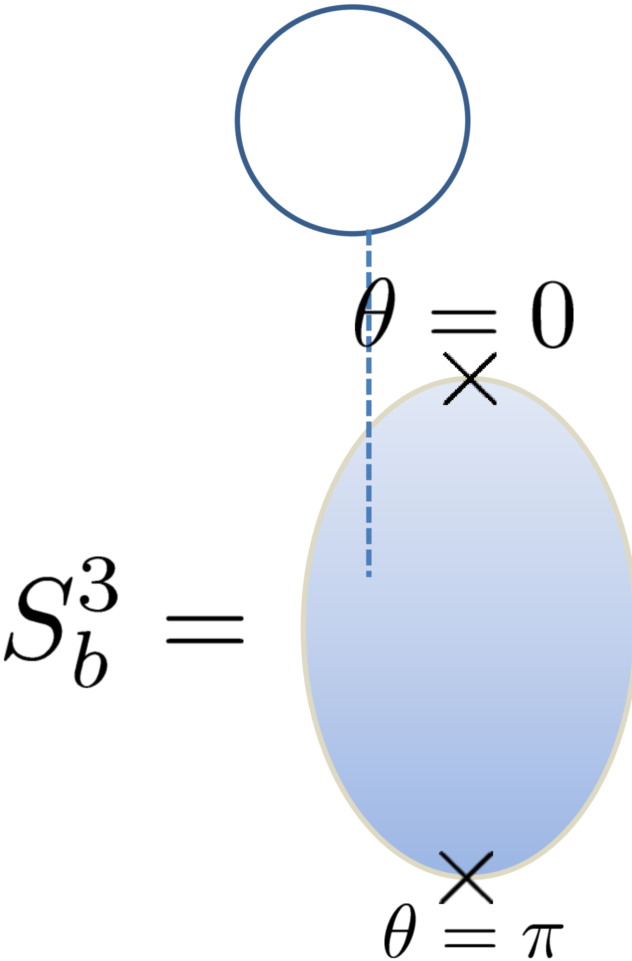
$$\sigma_i = -m_{l_i}, \quad \phi_{iA} = \sqrt{\chi}\delta_{l_i A}$$

$\sigma = \mathrm{diag}(\sigma_1, \cdots, \sigma_N)$ $\quad i = 1, \cdots, N$ $\{l_1, \cdots, l_N\} \subset \{1, \cdots, N_f\}$

$A = 1, \cdots, N_f$

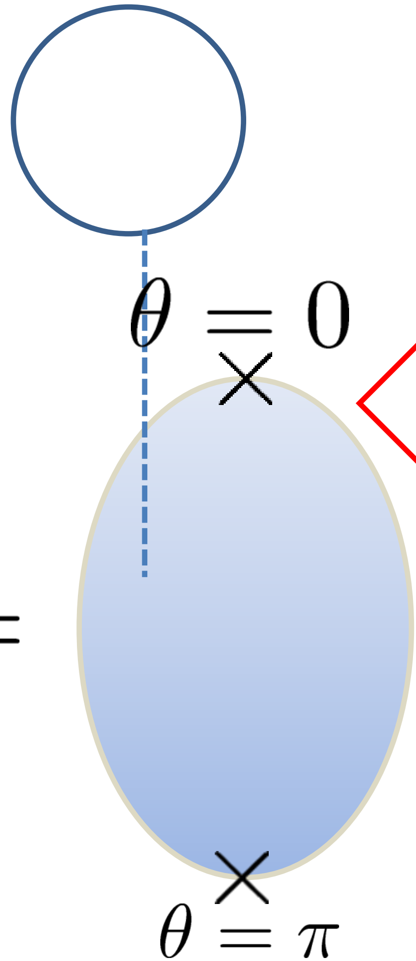
The saddle points are ${}_{N_f}C_N$ discrete points

the saddle point equations $(\theta = 0, \pi)$



the saddle point equations ($\theta = 0, \pi$)

$S_b^3 =$

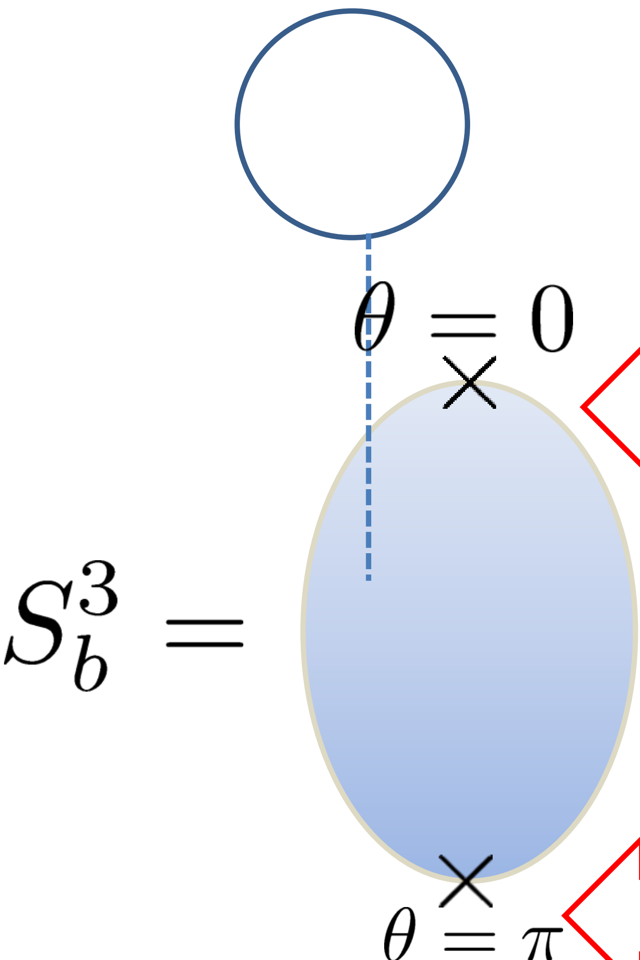


Vortex eq on the root of Higgs branch

$$F_{23} - \frac{1}{2}(\phi\bar{\phi} - \chi\mathbf{1}) = 0,\ \ D_2\phi + iD_3\phi = 0,$$

$$D_\mu\sigma = 0,\ \ (\sigma + M)\phi = 0,\ \cdots$$

the saddle point equations ($\theta = 0, \pi$)



Vortex eq on the root of Higgs branch

$$F_{23} - \frac{1}{2}(\phi\bar{\phi} - \chi \mathbf{1}) = 0, \;\; D_2\phi + iD_3\phi = 0,$$
$$D_\mu \sigma = 0, \;\; (\sigma + M)\phi = 0, \;\cdots$$

Anti-vortex eq on the root of Higgs branch

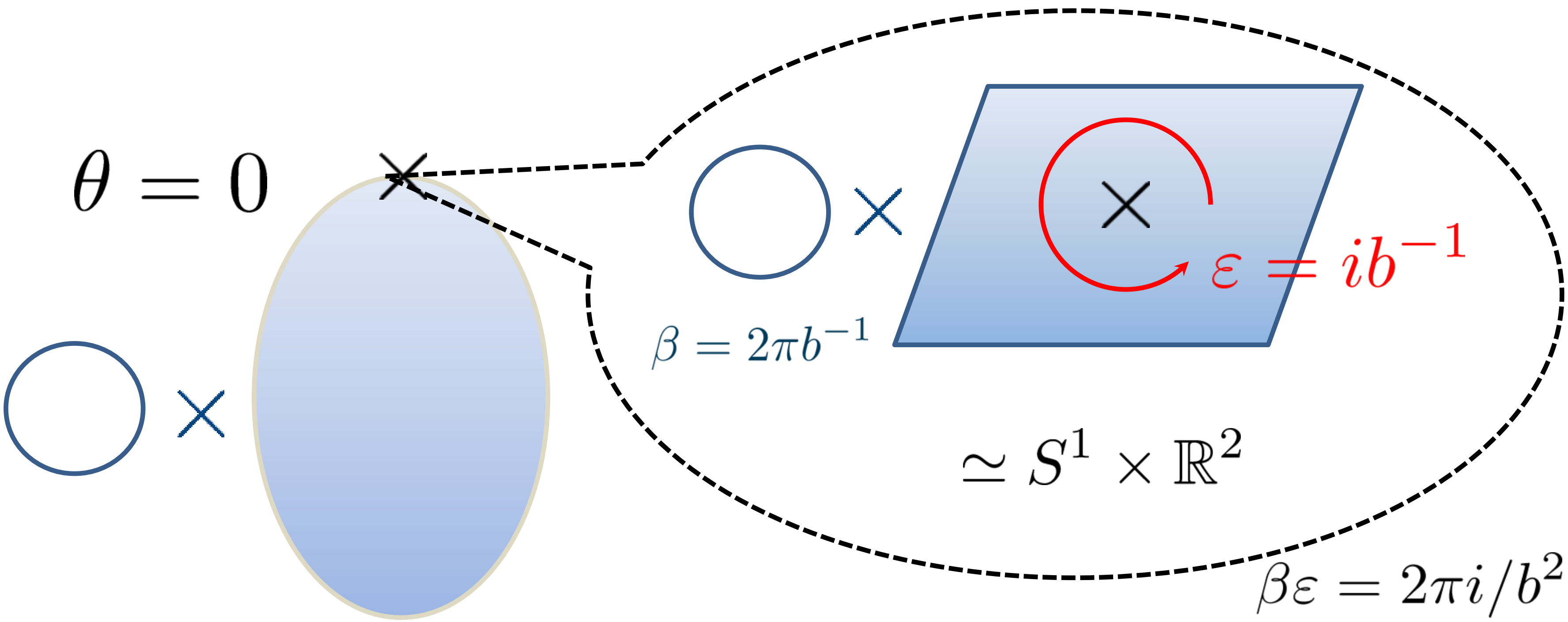
$$-F_{31} - \frac{1}{2}(\phi\bar{\phi} - \chi \mathbf{1}_N) = 0, \;\; D_1\phi + iD_3\phi = 0$$
$$D_\mu \sigma = 0, \quad (\sigma + M)\phi = 0, \cdots$$

In the limit $\chi \to \infty$, (the size of vortex) $\sim 1/\sqrt{\chi} \to 0$.

Thus, point like (anti-)vortices sit on north (south) pole of the squashed 2-sphere.

From the point vortices, S^3_b can be regarded as $S^1_\beta \times \mathbb{R}^2_\varepsilon$.

Then we can use vorex partition functions on $S^1_\beta \times \mathbb{R}^2_\varepsilon$.



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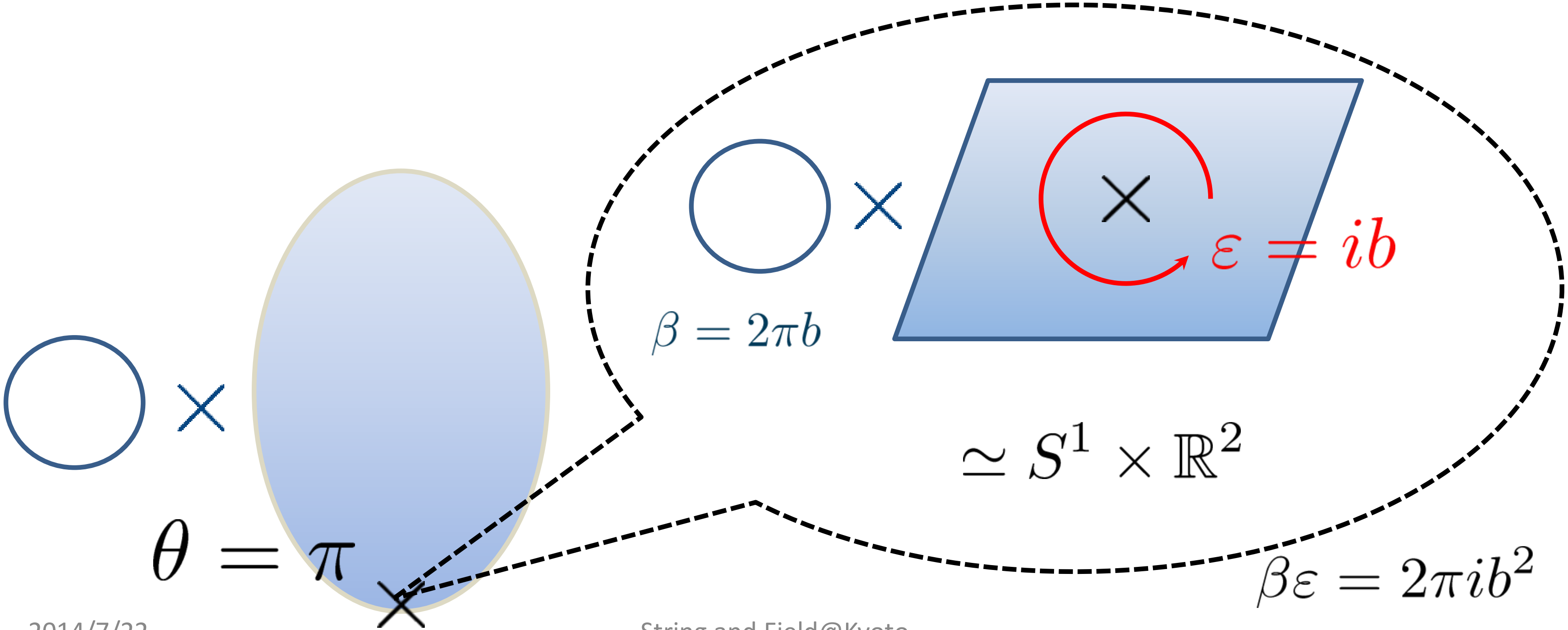
From the point vortices, S_b^3 can be regarded as $S_\beta^1 \times \mathbb{R}_\varepsilon^2$.
Then we can use vorex partition functions on $S_\beta^1 \times \mathbb{R}_\varepsilon^2$.



$$Z(S^3_b) = \sum_{\{l\}} e^{-S_{cl}} \, Z^{\text{1-loop}}_{\text{Higgs}} \, Z^{\{l\}}_V \, \bar{Z}^{\{l\}}_V$$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop}} Z_V^{\{l\}} \bar{Z}_V^{\{l\}}$$

The smmation over the saddle points

$$\{l\} = \{l_1, \cdots l_N\} \subset \{1, \cdots, N_f\}$$

$$\sigma_1 = -m_{l_1}, \cdots, \sigma_N = -m_{l_N}$$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop}} Z_V^{\{l\}} \bar{Z}_V^l$$

The saddle point value of FI-term $e^{-2\pi\zeta m_i}$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop},\{l\}} Z_V^{\{l\}} \bar{Z}_V^{\{l\}}$$

The one-loop determinant of QV around a saddle point labeled by $\{l\}$

$$Z_{\text{vec}}^{(1-\text{loop}),\{l\}} = \prod_{i<j} \sinh \pi b(m_{l_i} - m_{l_j}) \sinh \pi b^{-1}(m_{l_i} - m_{l_j}),$$

$$Z_{\text{chi}}^{(1-\text{loop}),\{l\}} = \prod_{A \neq \{l_i\}} \prod_{i=1}^{N} s_b \left(\frac{iQ}{2} + m_{l_i} - m_A \right).$$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop},\{l\}} Z_V^{\{l\}} \bar{Z}_V^{\{l\}}$$

The contribution from 3d vortex PF Z_V^l at north poles

$$Z_V^{\{l\}} = \sum_{\{k\}} \frac{e^{-2\pi\zeta b^{-1} k}}{\prod_{i,j}^{N} \prod_{n=1}^{k_i} 2\sinh \pi i b^{-2}(m_{l_j,l_i} + (n-i-k_j)) \prod_{l=1}^{k_i} 2\sinh \pi b^{-1}(m_{j,l_i} + lib^{-1})}$$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop},\{l\}} Z_V^{\{l\}} \bar{Z}_V^{\{l\}}$$

The contribution from 3d anti-vortex PF $\bar{Z}_V^l$ at south pole

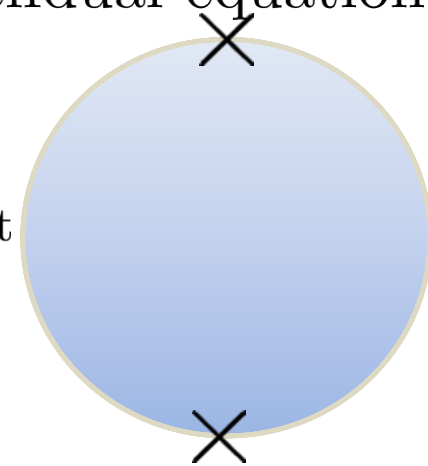
$$\bar{Z}_V^{\{l\}} = \sum_{\{k\}} \frac{e^{-2\pi\zeta bk}}{\prod_{i,j}^N \prod_{n=1}^{k_i} 2\sinh \pi i b^2 (m_{l_j,l_i} + (n - i - k_j)) \prod_{l=1}^{k_i} 2\sinh \pi b (m_{j,l_i} + lib)}$$

$$Z(S_b^3) = \sum_{\{l\}} e^{-S_{cl}} Z_{\text{Higgs}}^{\text{1-loop},\{l\}} Z_V^{\{l\}} \bar{Z}_V^{\{l\}}$$

This structure is similar to the PF of 4d $\mathcal{N}=2$ super YM theory on S^4

$$Z(S^4) = \int d^N a e^{-S(a)} Z^{\text{1-loop}}(a) Z_{\text{inst}} \bar{Z}_{\text{inst}}$$

(Pestun 2007)



Vortex partition funtion on $S^1_\beta \times \mathbb{R}^2_\varepsilon$

Before we explain the vortex PF, recall derivation of 5d instanton PF

$$Z^{\text{5d}}_{\text{inst}} = 1 + \sum_{k=1}^{\infty} e^{-\tau k} Z^{\text{5d}}_{k\text{-inst}}$$

instanton number

$$k = \frac{1}{8\pi^2}\int_{\mathbb{R}^4} F \wedge F$$

5-dim $G = U(N)$ k-instanton PF on $S^1 \times \mathbb{R}^4$ $(Z^{\text{5d}}_{k\text{-inst}})$

$\simeq$ PF of $G = U(k)$ SUSY gauged QM on S^1

SUSY gauged QM

matter contents: B_1, B_2, I, J(ADHM data)$\cdots$

$$(\text{D-term}) = \textstyle\sum_{i=1}^{2}[B_i, B_i^\dagger] + II^\dagger - J^\dagger J - \zeta_{\mathbb{R}}\mathbf{1}_k$$

$$(\text{F-term}) = B_1 B_2 + IJ$$

$$Z^{\text{5d}}_{k\text{-inst}} = \int \mathcal{D}B_i \mathcal{D}I \mathcal{D}J \cdots e^{-\int_{S^1} \mathcal{L}_{QM}}$$

$$= \sum_{\text{saddle pts}} Z^{\text{1-loop}}_{\text{SUSY QM}}$$

{The saddle points }

=The fixed points in k-instanton moduli space by equivariant $U(1)^N_a \times U(1)_{\varepsilon_1} \times U(1)_{\varepsilon_2}$-action

=The N-tuple Young diagrams $\{Y_i\}$ with the total number of box $k = \sum_{i=1}^N |Y_i|$

Vortex partition funtion on $S^1_\beta \times \mathbb{R}^2_\varepsilon$

$$Z_V^{3\text{d},\{l\}} = 1 + \sum_{k=1}^{\infty} e^{-\zeta k} Z^{3d,\{l\}}_{k\text{-vortex}}$$

vortex number

$$k = \frac{1}{2\pi}\int_{\mathbb{R}^2} F_{12}$$

3-dim $G = U(N)$ k-vortex PF on $S^1 \times \mathbb{R}^2$ $(Z^{3\text{d}}_{k\text{-vortex}})$
$\simeq G = U(k)$ PF of SUSY QM on S^1

SUSY gauged QM

matter contents: $B, I, J \cdots$

$$(\text{D-term}) = [B, B^\dagger] + II^\dagger - J^\dagger J - \chi \mathbf{1}_k$$

$$\mathcal{M}^{k\text{-vortex}}_{N,N_f} \simeq \{(B,I,J)|(\text{D-term})\}/U(k)$$

(Hanay-Tong 2002)

$$Z^{3\text{d},\{l\}}_{k\text{-vortex}} = \int \mathcal{D}B\mathcal{D}I\mathcal{D}J \cdots e^{-\int_{S^1} \mathcal{L}_{QM}}$$
$$= \sum_{\text{saddle pts}} Z^{\text{1-loop}}_{\text{SUSY QM}}$$

(fixed points)

=The fixed points in k-vortex moduli space by $U(1)_m^{N_f-1} \times U(1)_\varepsilon$-equivariant action

=The N-tuple non-negative integers k_i (1d Young diagrams) with the total number $k = \sum_{i=1} k_i$

Factorization of 4d $\mathcal{N}=1$ superconformal index

Y. Y 1403.0891
Peelaers 1403.2711

We consider $G=U(N)(SU(N))$ SYM theory +
N_f- fundamental and anti-fundamental chiral multiplets

If we consider SCI and evaluate the contour integrals, we find that the factorization of SCI only occur when traceless condition and anomaly free R-charge assignments ($R=1-N/N_f$)are satisfied.

This means that the factorization only occurs for $G=SU(N)$

There is a problem in Higgs branch localization for 4d SCI.

χ-term is necessary for Higgs branch localization to work well .
But we cannot introduce the χ-term for $G=SU(N)$ case(traceless).

Summary

We developed Higgs branch localization of 3d $\mathcal{N}=2$ theories
(In this talk we only mention on S_b^3 case,
we also performed Higgs branch localization on $S^1 \times S^2$ (SCI).)

We directly derived the vortex and anti-vortex factorization by constructing Q-exact term whose saddle points admit vortex (anti-vortex) eq at north (south) pole.

4d SCI has similar factorization.