Noncommutative Instantons and Reciprocity

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- We give a proof of one-to-one correspondence between moduli space of instantons and moduli space of ADHM data in noncommutative spaces.
- MH&Toshio Nakatsu(Setsunan), NC Instantons and Reciprocity, to appear, [cf. arXiv:1311.5227]
- MH&TN, work in progress.

1. Introduction

- Non-Commutative (NC) spaces are defined by noncommutativity of spatial coordinates:

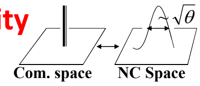
$$[x^\mu, x^\nu] = i\theta^{\mu\nu} \qquad \theta^{\mu\nu}\text{: NC parameter (real const.)}$$

(cf. CCR in QM : $[q,p]=i\hbar$)

(→ ``space-space uncertainty relation''→)

Resolution of singularity

(→ new physical objects)



Ex) Resolution of small instanton singularity

(→ U(1) instantons) [Nekrasov-Schwarz]

ASDYM eq. in 4-dim. with G=U(N)

- ASDYM eq. (real rep.) $\mu,\nu = 1,2,3,4$

$$F_{12} = -F_{34}, \quad F_{13} = -F_{42}, \quad F_{14} = -F_{23}.$$

$F_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu A_\nu - A_\nu A_\mu$ Field strength

A_μ: Gauge field (N×N anti-Hermitian)

$$(\Leftrightarrow F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0, \quad F_{z_1 z_2} = 0 \quad (cpx.\ rep.))$$

- There are two descripitions of NC extension:
  - Moyal-product formalism (deformation quantization)
  - Operator formalism (Connes' theory)

NC ASDYM eq. with G=U(N) in Moyal

- NC ASDYM eq. (real rep.)

$$F^*_{01} = -F^*_{23}, \quad F^*_{02} = -F^*_{31}, \quad F^*_{03} = -F^*_{12}$$

$$(F^*_{\mu\nu} := \partial_\mu A_\nu - \partial_\nu A_\mu + A_\mu * A_\nu - A_\nu * A_\mu)$$

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & \theta^1 & & \\ -\theta^1 & 0 & & O \\ & O & 0 & \theta^2 \\ & & -\theta^2 & 0 \end{pmatrix}$$

(Spell:All products are Moyal products.)

$$f(x)*g(x) := f(x)\exp\left(\frac{i}{2}\theta^{\mu\nu}\overleftarrow{\partial}_\mu\overrightarrow{\partial}_\nu\right)g(x)$$

$$= f(x)\cdot g(x) + i\frac{\theta^{\mu\nu}}{2}\partial_\mu f(x)\partial_\nu g(x) + O(\theta^2)$$

Under the spell, we can calculate :

$$[x^\mu, x^\nu]_* := x^\mu \cancel{x^\nu} + \frac{i}{2}\theta^{\mu\nu} - (x^\nu \cancel{x^\mu} - \frac{i}{2}\theta^{\mu\nu}) = i\theta^{\mu\nu}$$

Note: Coordinates and functions themselves are c-number-valued usual ones

G=U(N) NC ASDYM in operator formalism

- Take coordinates as operators (in 2dim):

$$[\hat{x},\hat{y}] = i\theta \xrightarrow{complex} [\hat{z},\hat{\bar{z}}] = 2\theta \xrightarrow{rescale} [\hat{a},\hat{a}^+] = 1$$

ann. op. cre op. acting on Fock space:

$$H = \oplus \mathbb{C}|n\rangle \quad n = 0,1,2,\ldots$$

Occupation number basis

field (infinite matrix):

$$\hat{F}(\hat{z},\hat{\bar{z}}) = \sum_{m,n}^{\infty} F_{mn}|m\rangle\langle n|$$

integration

$$2\pi\theta Tr_H \hat{F}(\hat{z},\hat{\bar{z}})$$

- NC ASDYM eq. (real rep.)

$$\hat{F} = \sum_{m_1,m_2,n_1,n_2}^{\infty} F_{m_1,m_2,n_1,n_2}|m_1,m_2\rangle\langle n_1,n_2|$$

$$\hat{F}_{01} = -\hat{F}_{23}, \quad \hat{F}_{02} = -\hat{F}_{31}, \quad \hat{F}_{03} = -\hat{F}_{12}$$

$$\theta^{\mu\nu} = \begin{pmatrix} 0 & -\theta^1 & & \\ \theta^1 & 0 & & O \\ & O & 0 & -\theta^2 \\ & & \theta^2 & 0 \end{pmatrix} \quad \begin{matrix} \to H_1 \\ \to H_2 \end{matrix}$$

2. Atiyah-Drinfeld-Hitchin-Manin Construction

based on duality for the instanton moduli space

4dim. ASDYang-Mills eq. (Difficult)		ADHM eq. (≒0dim. ASDYM) (Easy)
$F_{z_1\bar{z}_1} + F_{z_2\bar{z}_2} = 0$, $F_{z_1 z_2} = 0$ N × N PDE	1:1	$[B_1,B_1^+]+[B_2,B_2^+]+II^+ - J^+J = 0$, $[B_1,B_2]+IJ=0$ k × k Matrix eqs.
Sol.= instantons (G=U(N), C_2 =k)	⟷	Sol.=ADHM data (G=`U(k)')
$A_\mu : N\times N$		$B_{1,2} : k\times k, \quad I : k\times N, \quad J : N\times k$
Gauge trf.: $A_\mu \mapsto g^{-1}A_\mu g + g^{-1}\partial_\mu g$, $g\in U(N)$		Gauge trf.: $B_{1,2}\mapsto \tilde{g}^{-1}B_{1,2}\tilde{g}, \quad \tilde{g}\in U(k)$, $I\mapsto \tilde{g}^{-1}I, \quad J\mapsto J\tilde{g}$

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

G → / ← F

4dim. ASD Yang-Mills eq. on the dual torus

$\widetilde F_{\xi_1\bar\xi_1}+\widetilde F_{\xi_2\bar\xi_2}=0$
$\widetilde F_{\xi_1\xi_2}=0$ k × k PDE

Sol.=instantons (G=U(N), C_2 =k)

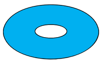
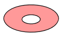
1:1 ↕

Sol.=the dual instantons (G=U(k), C_2=N)

$A_\mu : N\times N$

$\widetilde A_\mu : k\times k$

Define the maps F & G, & G∘F=id. & F∘G=id.

On a 4-torus

On the dual 4-torus

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

1:1

4dim. ASD Yang-Mills eq. on the dual torus

$\widetilde F_{\xi_1\bar\xi_1}+\widetilde F_{\xi_2\bar\xi_2}=0$
$\widetilde F_{\xi_1\xi_2}=0$ k × k PDE

Sol.=instantons (G=U(N), C_2 =k) ←→ Sol.=the dual instantons (G=U(k), C_2=N)

$A_\mu(x)=\langle V,\partial_\mu V\rangle_\xi$ ← map F (Dirac eq.)

N×N

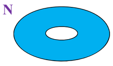
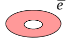
$\widetilde A_\mu(\xi) : k\times k$

$\nabla^+ V=\bar e^\mu\otimes(\frac{\partial}{\partial\xi^\mu}+\widetilde A_\mu - ix_\mu)V=0$

$\bar e^\mu := (i\sigma_a, 1_2)$

$V : 2k\times N$

Family index thm.

On a 4-torus $: x_\mu$

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm transformation

Beautiful duality between instanton moduli on 4-tori and instanton moduli on the dual tori

4dim. ASDYang-Mills eq. on a 4-torus

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

1:1

4dim. ASD Yang-Mills eq. on the dual torus

$\widetilde F_{\xi_1\bar\xi_1}+\widetilde F_{\xi_2\bar\xi_2}=0$
$\widetilde F_{\xi_1\xi_2}=0$ k × k PDE

Sol.=instantons (G=U(N), C_2 =k) ←→ Sol.=the dual instantons (G=U(k), C_2=N)

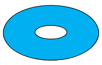
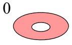
$A_\mu(x) : N\times N$ — map G (Dirac eq.) → $\widetilde A_\mu(\xi)=\langle\psi,\widetilde\partial_\mu\psi\rangle_\xi$

$\bar e_\mu D_\mu\psi=\bar e^\mu\otimes(\frac{\partial}{\partial x^\mu}+A_\mu - i\xi_\mu)\psi=0$

k×k

$\psi : 2N\times k$

Family index thm.

On a 4-torus $: x_\mu$

On the dual 4-torus $: \xi_\mu$

Fourier-Mukai-Nahm trf. (radii of the torus→∞)

Duality between instanton moduli on R^4 and instanton moduli on ``1pt.'' [cf. van Baal, hep-th/9512223]

4dim. ASDYang-Mills eq.

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

1:1

0dim. ASD Yang-Mills eq.

$\widetilde F_{\mu\nu} := \partial_\mu \widetilde A_\nu - \partial_\nu \widetilde A_\mu + [\widetilde A_\mu, \widetilde A_\nu]$ (derivative terms struck out)

$\widetilde F_{\xi_1\bar\xi_1}+\widetilde F_{\xi_2\bar\xi_2}=0$
$\widetilde F_{\xi_1\xi_2}=0$ Matrix eq. ! k × k ~~PDE~~

Sol.=instantons (G=U(N), C_2 =k) ←→ Sol.=``dual instantons'' (G=U(k), ``C_2=N'')

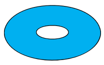
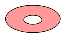
$A_\mu=V^+\partial_\mu V$ ← map F (0dim Dirac eq.)

$\widetilde A_\mu : k\times k$

$\nabla^+ V=\bar e^\mu\otimes(\frac{\partial}{\partial\xi^\mu}+\widetilde A_\mu - ix_\mu)V=0$ (derivative struck out) Matrix eq. !

$V : 2k\times N$

Linear alg.

On a 4-torus→R^4

On the dual 4-torus→1 pt.

Atiyah-Drinfeld-Hitchin-Manin (ADHM) Construction

based on the following duality

4dim. ASDYang-Mills eq.

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

G(4dim D.eq.) → / ← F(0dim D.eq.)

ADHM eq.. (≒0dim. ASDYM)

RHS is in fact $[z_1,\bar z_1]+[z_2,\bar z_2]$

$[B_1,B_1^+]+[B_2,B_2^+]+II^+-J^+J=0$
$[B_1,B_2]+IJ=0$
k × k matrix eq.

Sol.=instantons (G=U(N), C_2 =k)

1:1 ←→

Sol.=ADHM data (G=`U(k)')

$A_\mu : N\times N$

Proved in the same way as the Nahm trf.

$B_{1,2} : k\times k,$

$I : k\times N,\quad J : N\times k$

D-brane interpretation of ADHM Construction

[Witten, Douglas]

4dim. ASDYang-Mills eq. (G=U(N), C_2 =k)

$F_{z_1\bar z_1}+F_{z_2\bar z_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

G(4dim D.eq.) → / ← F(0dim D.eq.)

ADHM eq. (≒0dim. ASDYM) (G=`U(k)')

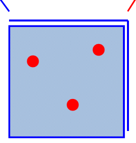
$[B_1,B_1^+]+[B_2,B_2^+]+II^+-J^+J=0$
$[B_1,B_2]+IJ=0$
k × k matrix eq.

SUSY trf. of gaugino

$\delta\psi\propto\varepsilon^\alpha\Gamma^{\mu\nu}F_{\mu\nu}$
$=\varepsilon^\alpha diag(\eta^{\mu\nu}_{SD}F_{\mu\nu},\eta^{\mu\nu}_{ASD}F_{\mu\nu})$

D-term conditions

0-0 strings ⇔ k×k: B
0-4 strings ⇔ k×N: I,J



N D4 branes

k D0 branes

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction
Ex.) Commutative BPST instanton (N=2, k=1)

4dim. ASDYang-Mills eq. | ADHM eq. (≒0dim. ASDYM)

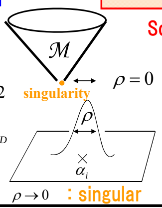
$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

$\mu_R=[B_1,B_1^\dagger]+[B_2,B_2^\dagger]+II^\dagger-J^\dagger J=0$
$\mu_C=[B_1,B_2]+IJ=0$
k × k matrix eq.

BPST instanton (G=U(2), C_2 =1)

$$A_\mu=\frac{i(x-b)^\nu\eta^{ASD}_{\mu\nu}}{(z-\alpha)^2+\rho^2},\quad 2\times 2$$

$$F_{\mu\nu}=\frac{2i\rho^2}{((z-\alpha)^2+\rho^2)^2}\eta^{ASD}_{\mu\nu}$$



$\rho\to 0$: singular

Sol.=ADHM data (G=`U(1)')

$\rho=0$ position

$B_{1,2}=\alpha_{1,2},\quad 1\times 1$

$I=(\rho,0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$ size

ADHM(Atiyah-Drinfeld-Hitchin-Manin) construction
Ex.) NC BPST instanton (N=2, k=1)

NC ASDYang-Mills eq. | NC ADHM eq.

$F_{z_1\bar{z}_1}+F_{z_2\bar{z}_2}=0$
$F_{z_1 z_2}=0$ N × N PDE

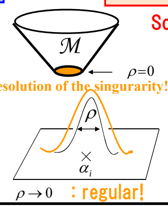
$\mu_R=[B_1,B_1^\dagger]+[B_2,B_2^\dagger]+II^\dagger-J^\dagger J=\zeta$
$\mu_C=[B_1,B_2]+IJ=0$
k × k marix eq.

NC BPST instanton (G=U(2), C_2 =1)

By calculation of TrFAF

$A_\mu, F_{\mu\nu}$: exact sol.

Do k×k ADHM data give Instanton number k in general ? (We prove this.)



$\rho\to 0$: regular!

Sol.: ADHM data (G=`U(1)')

position

$B_{1,2}=\alpha_{1,2},$

$I=(\sqrt{\rho^2+\zeta},0),\ J=\begin{pmatrix}0\\ \rho\end{pmatrix}$

size Fat by ζ!

3. Proof of the duality: (inst)<->(ADHM)

NC instanton —G→ / ←F— NC ADHM

$A_\mu: N\times N$

$B_{1,2}: k\times k,$
$I: k\times N,\quad J: N\times k$

(i) ASD (ASDYM eq.)
(ii) C_2=k
~~(iii) 𝔇^2 has inverse~~

(i) ASD (ADHM eq.)
(ii) matrix size = k, N
~~(iii) ∇^2 has inverse~~

(iii) is automatically satisfied in the noncommutative situation
[Maeda-Sako] [Nakajima] (For any θ [MH, Nakatsu])

Proof of the one-to-one ⇔ Define the maps F & G, & G∘F=id. & F∘G=id.

F: (ADHM)→(inst):ADHM construction

NC instanton ←0dim. Dirac eq.— NC ADHM

$\nabla^\dagger * V=0,\quad V^\dagger * V=1_N$

$A_\mu=V^\dagger *\partial_\mu V: N\times N$

$$\nabla=\begin{pmatrix}I^\dagger & J\\ \bar{z}_2-B_2^\dagger & -(z_1-B_1)\\ \bar{z}_1-B_1^\dagger & z_2-B_2\end{pmatrix}$$
0 dim Dirac op. $(N+2k)\times 2k$

$B_{1,2}: k\times k,$
$I: k\times N,\quad J: N\times k$

(i) ASD(ASDYM eq) [Nekasov-Schwarz]
(ii) C_2=k ←[MH Nakatsu],…

(i) ASD (ADHM eq.)
(ii) matrix size= k, N

We prove the NC version of the formula: cf.[Atiyah, Hori]

$$\int d^4x Tr_N F^*_{\mu\nu} * F^*_{\mu\nu}=-\int d^4x Tr_{2k}\Omega^*_{\mu\nu} *\Omega^*_{\mu\nu}$$

$$where\quad \Omega_{\mu\nu}:=\partial_\mu\omega-\partial_\nu\omega_\mu+[\omega_\mu,\omega_\nu]_*$$

$$\omega_\mu:=\tilde{\nabla}^\dagger *\partial_\mu\tilde{\nabla},\qquad \tilde{\nabla}:=\nabla*(\nabla^\dagger *\nabla)^{1/2}$$

2k×2k   (N+2k)×2k

F: (ADHM)→(inst):ADHM construction

NC instanton ←0dim. Dirac eq.— NC ADHM

$\nabla^\dagger * V=0,\quad V^\dagger * V=1_N$

$A_\mu=V^\dagger *\partial_\mu V: N\times N$

$$\nabla=\begin{pmatrix}I^\dagger & J\\ \bar{z}_2-B_2^\dagger & -(z_1-B_1)\\ \bar{z}_1-B_1^\dagger & z_2-B_2\end{pmatrix}$$
0 dim Dirac op. $(N+2k)\times 2k$

$B_{1,2}: k\times k,$
$I: k\times N,\quad J: N\times k$

Then: $$C_2:=\frac{1}{16\pi^2}\int d^4x Tr_N F^*_{\mu\nu} * F^*_{\mu\nu}=\frac{-1}{16\pi^2}\int d^4x Tr_{2k}\Omega^*_{\mu\nu} *\Omega^*_{\mu\nu}$$

$$=\frac{1}{24\pi^2}\int_{S^3} Tr_k 1_k\cdot Tr_2(g^{-1}dg)^3=k,\qquad g:=\frac{x^\mu e_\mu}{r}$$

comes from the size of the ADHM data!

Interpretation in operator formalism would be insteresting. (The matrix g is a shift operator!)

G: (inst)→(ADHM): inverse construction

NC instanton —4dim. Dirac eq.→ NC ADHM

$\bar{e}_\mu D_\mu *\psi=0,\quad \int d^4x\psi^\dagger *\psi=1_k$

$A_\mu: N\times N$

$e^\mu D_\mu$: 4 dim Dirac op.

$B_{1,2}=\int d^4x z_{1,2} *\psi^\dagger *\psi: k\times k,$

$\psi\approx\frac{I^\dagger, J}{r^3}: N\times k$

(i) ASD (ASDYM eq.)
(ii) C_2=k

(i) ASD (ADHM eq.)
(ii) matrix size= k, N

[Maeda-Sako2009] proves existence of the Dirac zero-mode by a formal power expansion of θ, recursively.

commutative input

$\psi(x,\theta)=\psi^{(0)}+\theta\psi^{(1)}+\theta^2\psi^{(2)}+\cdots,$
$A(x,\theta)=A^{(0)}+\theta A^{(1)}+\theta^2 A^{(2)}+\cdots,$

G∘F=id : (ADHM)→(inst)→(ADHM)

NC ADHM —(0dim. Dirac eq. V)→ NC instanton —(4dim. Dirac eq. ψ)→ NC ADHM

$B_{1,2}, I, J$

$A_\mu = V^+ * \partial_\mu V$

$B'_{1,2} = B_{1,2}$?

$I' = I$?, $J' = J$?

ψ find in terms of the original B, I, J, V to give the new data

$$\psi = \psi(B, I, J, V)$$
$$= V^+ * Cf$$ The answer

$$\bar{e}_\mu D_\mu * \psi = 0, \quad \int d^4x \psi^+ * \psi = 1_k$$

$$B'_{1,2} = \int d^4x\, z_{1,2} * \psi^+ * \psi = \cdots = B_{1,2}$$
$$\psi \approx \frac{I'^+, J'}{r^3} = \cdots = \frac{I^+, J}{r^3}$$

are shown [Maeda-Sako]

F∘G=id : (inst)→(ADHM)→(inst)

NC instanton —(4dim. Dirac eq. ψ)→ NC ADHM —(0dim. Dirac eq. V)→ NC instanton

A_μ

$B_{1,2}, I, J$

$A'_\mu = A_\mu$?

V find in terms of the original instanton Aμ to give the new instanton

$$V = V(A_\mu, \psi)$$
$$D^2 * V = -4\psi^+ C$$:the answer

[Maeda-Sako] assume the existence of V.
[MH-Nakatsu] prove it

$$\nabla^+ * V = 0, \quad V^+ * V = 1_N$$
$$A'_\mu = V^+ * \partial_\mu V = \cdots = A_\mu$$

are shown (some existence proofs is also made by us)

Main result : We prove the ADHM duality in the formal power series of θ-expansion for arbitrary noncommutativity (including ζ=0).

NC instanton ←(1:1)→ NC ADHM

NC instanton	NC ADHM
(i) ASD (ASDYM eq.)	(i) ASD (ADHM eq.)
(ii) C_2=k	(ii) matrix size = k, N

- This is valid only in the region that the θ-expansions converge.
- We proceed to reveal the duality in operator formalism. (mostly completed [work in progress])