Extremal Surfaces in Asymptotic AdS Black Hole and Holographic Entanglement Entropy

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Based on T. Eguchi, and CH, [arXiv:15xx.xxxxx [hep-th]] work in progress

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Introduction:
A review of AdS$_{d+2}$/CFT$_{d+1}$ correspondence, entanglement entropy, and black hole entropy

AdS$_{d+2}$ (bulk space-time) ⟷ holographic dual **CFT$_{d+1}$** (boundary)

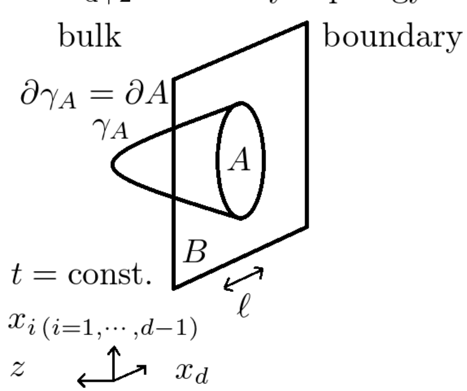
- holographic entanglement entropy:
  [Ryu-Takayanagi '06]

$$S_{A_{\rm HEE}} = \frac{\mathrm{Area}(\gamma_A)}{4G_{\rm N}},$$

γ_A : minimal surface in bulk space-time

cf. black hole entropy: $S_{\rm BH} = \dfrac{A}{4G_{\rm N}},$

e.g. AdS_{d+2} boundary topology : $\mathbb{R}^{1,d}$



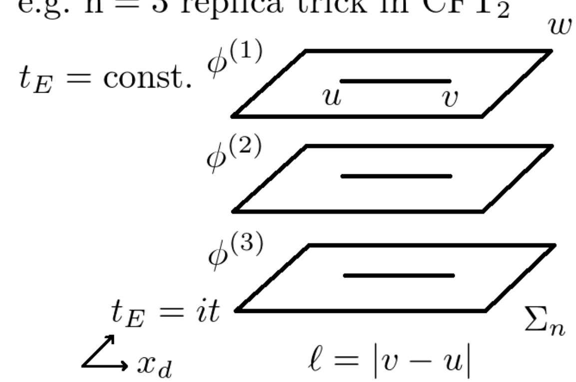
- entanglement entropy:

$\mathcal{H}_{\rm tot} = \mathcal{H}_A \otimes \mathcal{H}_B,$
$\rho_A = \mathrm{Tr}_{\mathcal{H}_B}\rho_{\rm tot}$

$$S_{A_{\rm EE}} = -\mathrm{Tr}_{\mathcal{H}_A}\rho_A \log\rho_A, \\ = -\lim_{n\to 1}\frac{\partial}{\partial n}\mathrm{Tr}_{\mathcal{H}_A}(\rho_A)^n,$$

(replica trick)

e.g. n = 3 replica trick in CFT_2



$z = f(w)$

conformal mapping

$\mathbb{R}^2$

$\mathbb{S}^1 \times \mathbb{R}^1$

$\mathbb{R}^1 \times \mathbb{S}^1$

$\mathbb{T}^2$

Results of holographic entanglement entropy as extremal (and/or minimal) surface:
-- In the case of BTZ black hole

- rotating BTZ black hole metric:

$$ds^2_{\text{BTZ black hole}} = -f(r)\mathrm{d}t^2 + f(r)^{-1}\mathrm{d}r^2 + r^2\left(\mathrm{d}\phi - \frac{J}{2r^2}\mathrm{d}t\right)^2,$$

$$= -\frac{(r^2-r_+^2)(r^2-r_-^2)}{R_{\text{AdS}}^2 r^2}\mathrm{d}t^2 + \frac{r^2 R_{\text{AdS}}^2}{(r^2-r_+^2)(r^2-r_-^2)}\mathrm{d}r^2 + r^2\left(\mathrm{d}\phi + \frac{r_+ r_-}{R_{\text{AdS}} r^2}\mathrm{d}t\right)^2,$$

$$f(r) = \frac{r^2}{R_{\text{AdS}}^2} - M + \left(\frac{J}{2r}\right)^2,$$

$$M = \frac{r_+^2 + r_-^2}{R_{\text{AdS}}^2} > 0, \quad J = \frac{2r_+ r_-}{R_{\text{AdS}}}, \quad M \geq \frac{J}{R_{\text{AdS}}},$$

-- non-rotating (t=const. and J=0) holographic entanglement entropy:
[Hubeny-Rangamani-Takayanagi '07]

$$S_A = \frac{\text{Area}(\gamma_A)}{4G_{\text{N}}^{(3)}} = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left|\frac{2\pi\Delta\phi_A}{\beta}\right|\right),$$

$$c = \frac{2R_{\text{AdS}}}{3G_{\text{N}}^{(3)}}, \quad \beta = \frac{2\pi}{\sqrt{M}},$$

Results of holographic entanglement entropy as extremal (and/or minimal) surface:
-- In the case of BTZ black hole

(detail)

-- non-rotating (t=const. and J=0) holographic entanglement entropy:
[Hubeny-Rangamani-Takayanagi '07]

space-like geodesic equation:

$$\frac{\mathrm{d}r}{\mathrm{d}\phi} = \pm\frac{1}{R_{\mathrm{AdS}}} r\sqrt{r^2 - r_+^2}\sqrt{\frac{r^2}{r^{*2}} - 1},$$

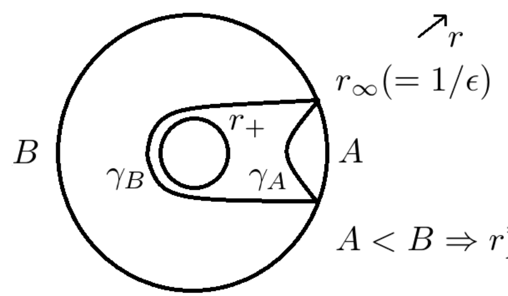
$$2\Delta\phi_A = \pm R_{\mathrm{AdS}} \int_{r_\infty}^{r_A^*} \frac{\mathrm{d}r}{r\sqrt{r^2 - r_+^2}\sqrt{\frac{r^2}{r_A^{*2}} - 1}},$$

$$2\Delta\phi_B = \pm R_{\mathrm{AdS}} \int_{r_\infty}^{r_B^*} \frac{\mathrm{d}r}{r\sqrt{r^2 - r_+^2}\sqrt{\frac{r^2}{r_B^{*2}} - 1}},$$

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_{\mathrm{N}}^{(3)}} = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left|\frac{2\pi\Delta\phi_A}{\beta}\right|\right),$$

$$S_B = \frac{\mathrm{Area}(\gamma_B)}{4G_{\mathrm{N}}^{(3)}} = \frac{c}{3}\log\left(\frac{\beta}{\pi\epsilon}\sinh\left|\frac{2\pi\Delta\phi_B}{\beta}\right|\right),$$

e.g. minimal surfaces in non-rotating BTZ black hole



$A < B \Rightarrow r_A^* > r_B^*$

$\Delta\phi_A + \Delta\phi_B = \pi,\ \gamma_A \neq \gamma_B \Rightarrow S_{A_{\mathrm{HEE}}} \neq S_{B_{\mathrm{HEE}}}$

$(2+1)\mathrm{dim.} \Rightarrow S_{\mathrm{BH}} \neq |S_{B_{\mathrm{HEE}}} - S_{A_{\mathrm{HEE}}}|$

Results of holographic entanglement entropy as extremal (and/or minimal) surface:
-- In the case of Schwarzschild-AdS_4 black hole

- Schwarzschild-AdS_4 black hole metric (AdS boundary topology $R^{1,d}$):

$$\mathrm{d}s^{2\ \text{AdS bounray tolpogy}\ \mathbb{R}^{1,2}}_{\text{Schwarzschild}-\text{AdS}_4\ \text{BH}} = R^2_{\text{AdS}} \frac{-f(z)\mathrm{d}t^2 + \frac{1}{f(z)}\mathrm{d}z^2 + (\mathrm{d}x^2 + \mathrm{d}y^2)}{z^2},$$

$$f(z) = 1 - \left(\frac{z}{z_H}\right)^3,$$

cf. AdS_4 space-time in Poincare AdS_4 coordinate:

$$\mathrm{d}s^2_{\text{Poincare AdS}_4} = R^2_{\text{AdS}} \frac{-\mathrm{d}t^2 + \mathrm{d}z^2 + (\mathrm{d}x^2 + \mathrm{d}y^2)}{z^2},$$

-- static (t=const. and translational invariant in y direction) holographic entanglement entropy:

$$S_{A_{\text{HEE}}} = \frac{\text{Area}(\gamma_A)}{4G_{\text{N}}} = \frac{R^2_{\text{AdS}} L}{4G_N} \int_{\epsilon_{\text{UV}}}^{z^*} \frac{\mathrm{d}z}{z^2 \sqrt{1 - \left(\frac{z}{z_H}\right)^3} \sqrt{-1 + \left(\frac{z^*}{z}\right)^4}} \left(\frac{z^*}{z}\right)^2 \sim \mathcal{O}(\epsilon^{-1}_{\text{UV}}),$$

Area low

Results of holographic entanglement entropy as extremal (and/or minimal) surface:
-- In the case of Schwarzschild-AdS_4 black hole

(detail)

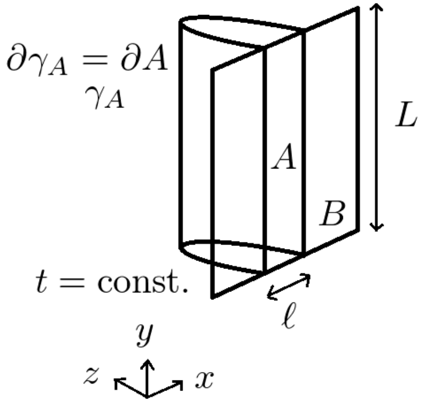
-- static (t=const. and translational invariant in y direction) holographic entanglement entropy:

$$\mathrm{Area}(\gamma_A) = R_{\mathrm{AdS}}^2 L \int_{-\frac{\ell}{2}}^{\frac{\ell}{2}} \frac{\mathrm{d}x}{z^2} \sqrt{1 + \frac{1}{f(z)} \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2},$$

$$0 = \delta \mathrm{Area}(\gamma_A) \quad \Rightarrow \quad \frac{\mathrm{d}^2 z}{\mathrm{d}x^2} - \frac{1}{2}\frac{f'}{f}\left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2 + \left[f + \left(\frac{\mathrm{d}z}{\mathrm{d}x}\right)^2\right]\frac{2}{z} = 0,$$

$$\frac{\mathrm{d}z}{\mathrm{d}x} = \sqrt{1 - \left(\frac{z}{z_H}\right)^3} \sqrt{-1 + \left(\frac{z^*}{z}\right)^4},$$

e.g. AdS_4 boundary topology : $\mathbb{R}^{1,2}$



$$S_{A_{\mathrm{HEE}}} = \frac{\mathrm{Area}(\gamma_A)}{4G_{\mathrm{N}}} = \frac{R_{\mathrm{AdS}}^2 L}{4G_N} \int_{\epsilon_{\mathrm{UV}}}^{z^*} \frac{\mathrm{d}z}{z^2 \sqrt{1 - \left(\frac{z}{z_H}\right)^3} \sqrt{-1 + \left(\frac{z^*}{z}\right)^4}} \left(\frac{z^*}{z}\right)^2 \sim \mathcal{O}(\epsilon_{\mathrm{UV}}^{-1}),$$

Area low

Discussions (form the view point of entanglement entropy and black hole entropy)

v.s. black hole entropy

- Static BTZ black hole:

$$S_{\mathrm{BH}} \neq |S_{B_{\mathrm{HEE}}} - S_{A_{\mathrm{HEE}}}|,$$

This results is coursed by (2+1) dimensional space-time specially.

- Time-dependent AdS_4 Black Hole:
  (under investigation)

$$S_{\mathrm{BH}} = |S_{B_{\mathrm{HEE}}} - S_{A_{\mathrm{HEE}}}|,$$

In higher than (2+1) dimensional space-time, above results is expected.

v.s. entanglement entropy

Because there is Hawking radiation from black hole in bulk space-time, total system density operator on boundary interpret as mixed state:

$$\rho_{\mathrm{tot}} \neq |\Psi\rangle\langle\Psi|,$$

Then, generally:

$$S_{A_{\mathrm{EE}}} \neq S_{B_{\mathrm{EE}}},$$

This property was confirmed in holographic entanglement entropy:

$$S_{A_{\mathrm{HEE}}} \neq S_{B_{\mathrm{HEE}}},$$

$$\text{cf. pure state } : \rho_{\mathrm{tot}} = |\Psi\rangle\langle\Psi|, \quad \Rightarrow \quad S_{A_{\mathrm{EE}}} = S_{B_{\mathrm{EE}}}.$$

Summary and future directions

Summary:

- We calculated holographic entanglement entropy, which is defined by minimal surface in bulk space-time, in the case of BTZ black hole and Schwartzshild-AdS_4 black hole, and reconsidered these results from the viewpoint of entanglement entropy and black hole entropy.
- Holographic entanglement entropy is important physical quantity in a study for AdS_{d+2}/CFT_{d+1} correspondence.

Future directions:

- To manifest degree of freedom on boundary in conformal field theory which is holographic dual with asymptotic AdS black hole in bulk space-time.
- To study in the case of other asymptotic AdS black holes which have not only mass but also charge and/or angular momentum.
- To expand to covariant holographic entanglement entropy.

References

[1] J. M. Maldacena, ``The Large N Limit of Superconformal Field Theories and Supergravity,'' *Adv.Theor.Math.Phys.* **2** (1998) 231-252 [arXiv:hep-th/9711200].

[2] S. Ryu, and T. Takayanagi, ``Holographic Derivation of Entanglement Entropy from AdS/CFT,'' *Phys.Rev.Lett.* **96** (2006) 181602 [arXiv:hep-th/0603001].

[3] V. E. Hubeny, M. Rangamani, and T. Takayanagi,``A Covariant Holographic Entanglement Entropy Proposal,'' JHEP0707 (2007) 062 [arXiv:0705.0016 [hep-th]].

[4] A. Lewkowycz, and J. Maldacena, ``Generalized gravitational entropy,'' JHEP08 (2013) 090 [arXiv:1304.4926 [hep-th]].