The Interpolating Function

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References: MH,JHEP1412 019 (1408.2960),
MH-DPJ,NPB900 (2015) 533 (1504.02276),
AC-MH-ST and AC-MH, to appear,
MH, work in progress

based on collaborations with

Abhishek Chowdhury (HRI), Dileep P. Jatkar (HRI),
Somyadip Thakur (TIFR)

11th, Nov. YITP workshop 2015

Perturbative expansion

- ubiquitous
- does not often give satisfactory understanding of physics…
  (unless it has nice properties)
- even if it has nice property,
  higher order computation is usually hard task

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⇩

Approximation at finite values of parameters

How do we interpolate these two expansions?

Tool : Interpolating function

Single function consistent with the 2 pertubative expansions

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$$\frac{2g}{2+g}$$

(2-point Pade approximant)

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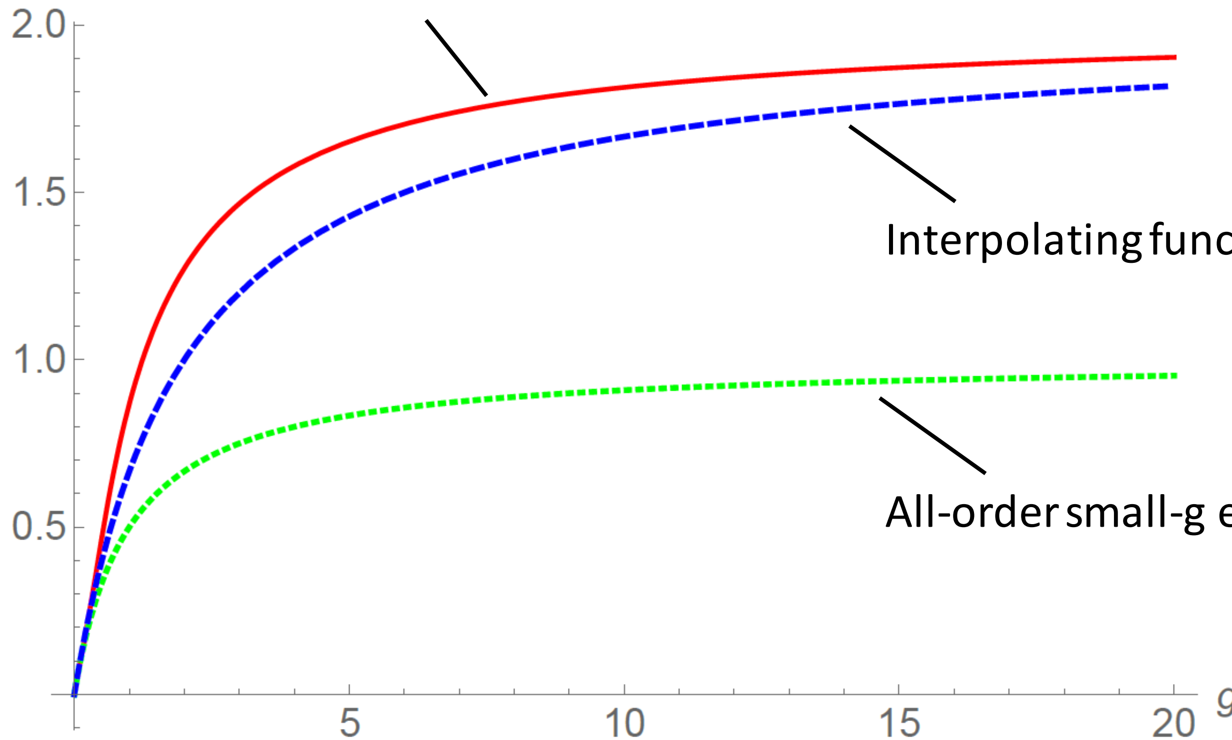

Outline

- introduce a class of interpolating functions [MH '14]

——— generalization of Pade and Sen's interpolating function

- Analytic property of interpolating function & dimensions of twist operators in planar ABJM

[Chowdhury-MH, to appear]

——— indirect evidence for a recent conjecture
on unknown function often called h(λ),
which appears in the context of integrability

Introduction to Interpolating function

Setup

Suppose that we know small-g and large-g expansions of a function F(g):

$$F(g) = \begin{cases} g^a(s_0 + s_1 g + s_2 g^2 + \cdots), \\ g^b(l_0 + l_1 g^{-1} + l_2 g^{-2} + \cdots), \end{cases}$$

Then we would like to find approximation of F(g) at finite g.

(When we have expansions around $g=g_1$ and $g=g_2$,
changing the variable as $x=(g-g_1)/(g-g_2)$ gives small-x and large-x expansions)

(Two-point) Pade approximant

$$\mathcal{P}_{m,n}(g) = s_0 g^a \frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k},$$

$$\left(p = \frac{m+n+1+(b-a)}{2} \in \mathbf{Z}, \quad q = \frac{m+n+1-(b-a)}{2} \in \mathbf{Z} \right)$$

The coefficients are determined to reproduce the small-g exp. up to $\mathcal{O}(g^{a+m+1})$ and large-g exp. up to $\mathcal{O}(g^{b-n-1})$

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Some properties:

- can construct only for (b-a)∈**Z** (although avoidable by a change of variable)

  ((b-a) :even → (m+n): odd, (b-a) :odd → (m+n): even)

- No branch cut

Fractional Power of Polynomial (FPP)

[Sen '13]

$$F_{m,n}(g) = s_0 g^a \left[1 + \sum_{k=1}^{m} c_k g^k + \sum_{k=0}^{n} d_k g^{m+n+1-k} \right]^{\frac{b-a}{m+n+1}}$$

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Some properties:

- can construct for arbitrary (a,b,m,n)
- Type of branch cut is uniquely determined by (a,b,m,n)

Fractional Power of Rational function (FPR)

$$F^{(\alpha)}_{m,n}(g) = s_0 g^a \left[\frac{1+\sum_{k=1}^{p} c_k g^k}{1+\sum_{k=1}^{q} d_k g^k}\right]^{\alpha},$$

[MH'14]

$$\left(p = \frac{1}{2}\left(m+n+1+\frac{b-a}{\alpha}\right) \in \mathbf{Z}, \quad q = \frac{1}{2}\left(m+n+1-\frac{b-a}{\alpha}\right) \in \mathbf{Z} \right)$$

$$\alpha = \begin{cases} \frac{a-b}{2\ell+1} & \text{for } m+n : \text{even} \\ \frac{a-b}{2\ell} & \text{for } m+n : \text{odd} \end{cases}, \quad \text{with } \ell \in \mathbb{Z}.$$

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There are many cases where FPR gives very precise approximation.

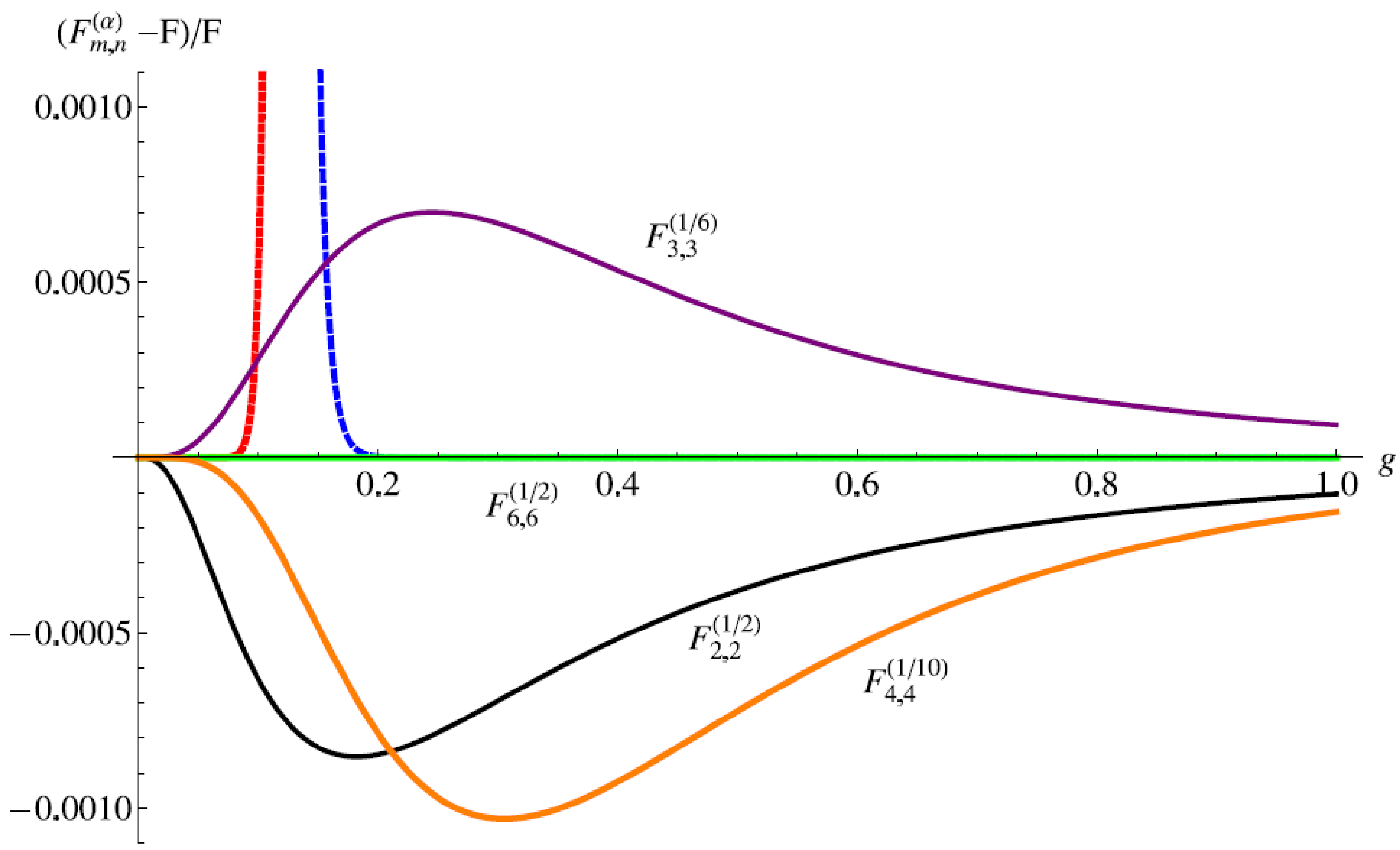
(although there are also many unsuccessful cases)

Ex.) Partition function of 0d ϕ^4 theory

[Sen'13,MH'14]

$$F(g) = \int_{-\infty}^{\infty} dx\ e^{-\frac{x^2}{2} - g^2 x^4} = \frac{e^{\frac{1}{32g^2}}}{2\sqrt{2}g} K_{\frac{1}{4}}\left(\frac{1}{32g^2}\right),$$

$$F(g) = F_{m,n}^{(\alpha)}(g) + \mathcal{O}(g^{a+m+1}, g^{b-n-1})$$

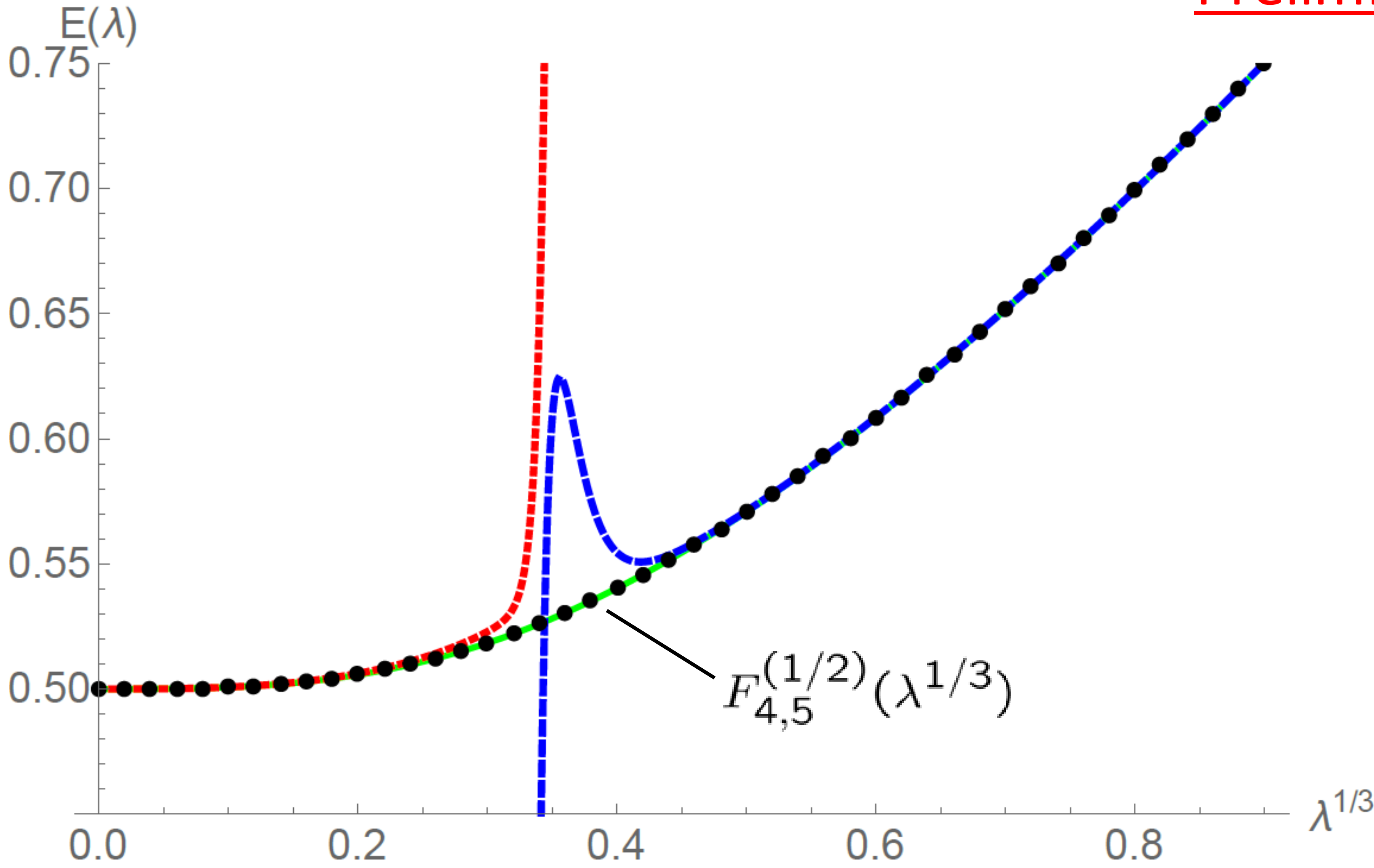


Ex.) Grand state Energy in anharmonic oscillator

[MH, work in progress]

$$\left[-\frac{d^2}{dx^2}+\frac{1}{4}x^2+\frac{1}{4}\lambda x^4\right]\psi(x)=E(\lambda)\psi(x)$$

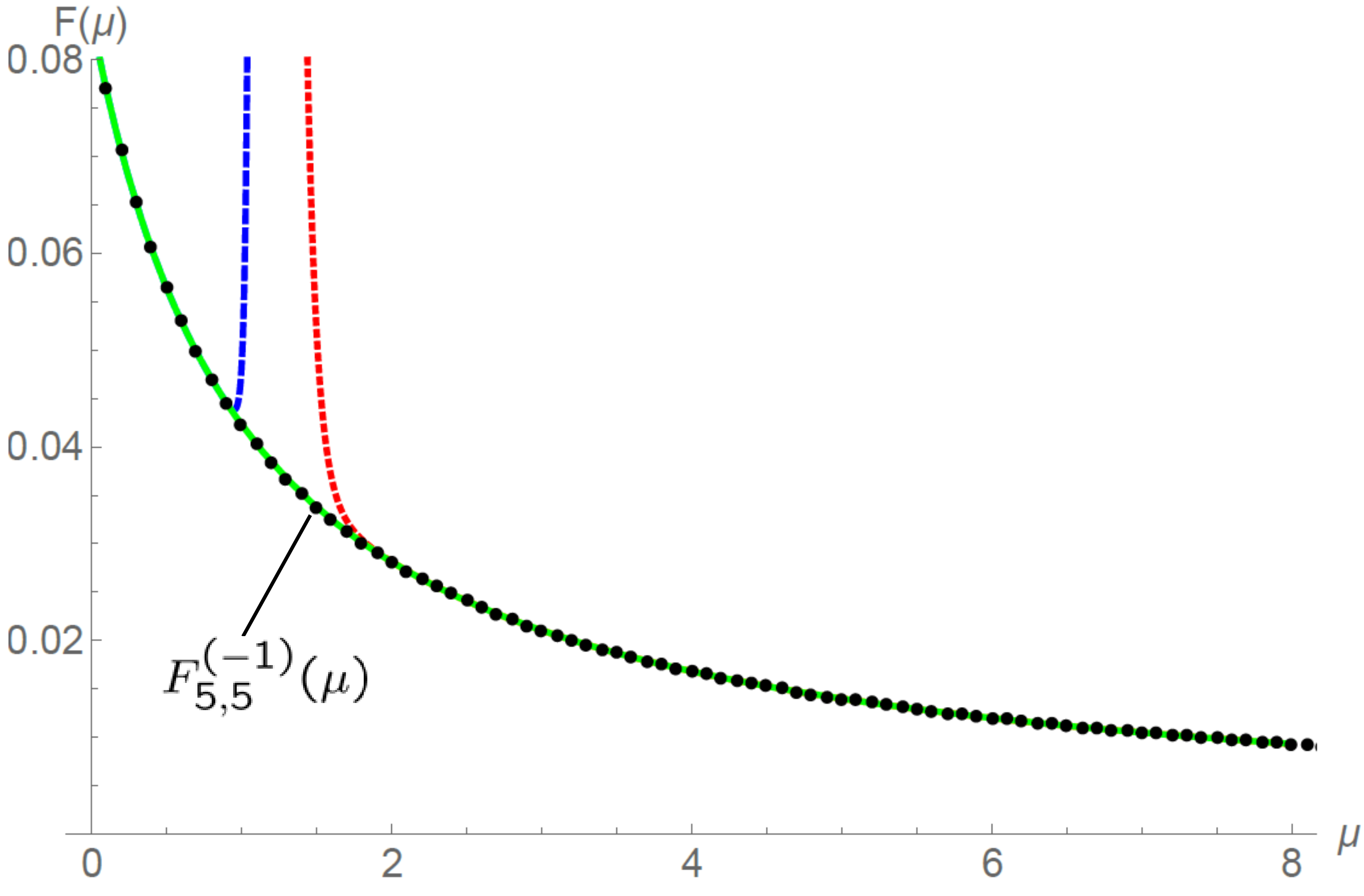
Preliminary



Ex.) Free energy of c=1 non-critical string

[MH '14]

μ: cosmological constant

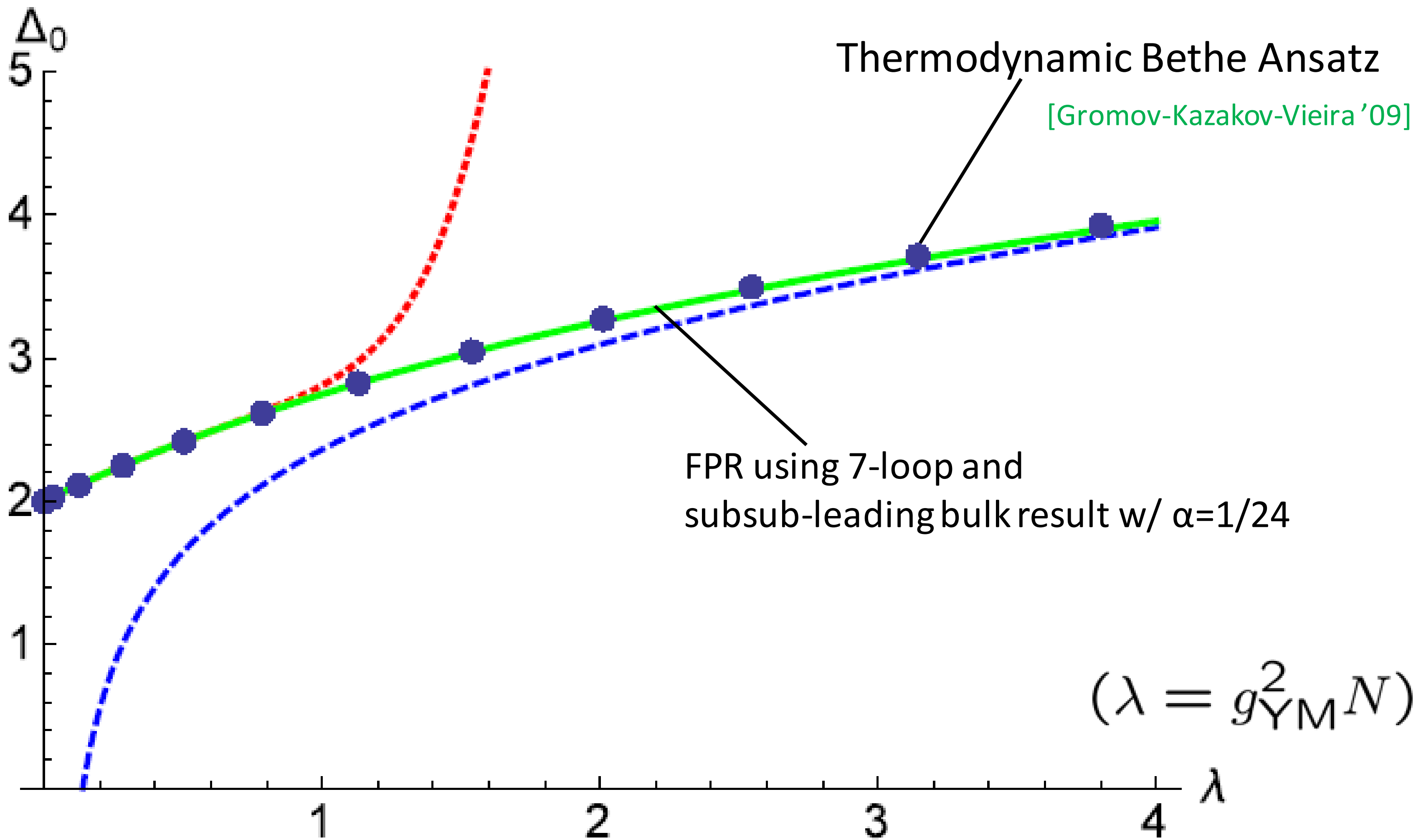


Ex.)Dimension of Konishi op. in planar $\mathcal{N}=4$ SYM

[Chowdhury-MH-Thakur, to appear]

$$\mathcal{O}_{\text{Konishi}} = \text{Tr}\phi^I\phi^I$$

Preliminary



Analytic property of interpolating function
&
Twist operators in planar ABJM

[Chowdhury-MH, to appear]

Twist-operators in ABJM

ABJM theory:

[Aharony-Bergman-Jafferis-Maldacena '08]

3d $\mathcal{N} = 6$ U(N)$_k$ x U(N)$_{-k}$ (k: CS level)

superconformal Chern-Simons theory

$$\mathcal{O}_{L,S} = \mathrm{Tr}\left[D_+^S (Y^1 Y_4^\dagger)^L\right]$$

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$$\Delta_{L,S}(k,N) = L + S + \gamma_{L,S}(k,N)$$

Here we focus on the planar limit:

$$\Delta_{L,S}(k,N) = \Delta_{L,S}^{(0)}(\lambda) + \mathcal{O}(N^{-2})$$

$$\lambda = \frac{N}{k}$$

Dressed coupling constant h(λ)

In the context of integrability analysis,

the dimension is described in terms of an unknown function h(λ).

[Giombi-Gaiotto-Yin]

$h(\lambda) \propto$ (Central charge of SU(2|2) sub-superconfomal algebra)

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Recently, exact form of h(λ) has been conjectured as [Gromov-Sizov]

$$\lambda = \frac{\sinh(2\pi h)}{2\pi} \, {}_3F_2\left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}; 1, \frac{3}{2}; -\sinh^2(2\pi h)\right).$$

Comparison with TBA for (L,S)=(1,2)

For both weak and strong coupling expansions, we use results obtained by integrability technique $\longrightarrow$ Work in $h(\lambda)$ rather than λ

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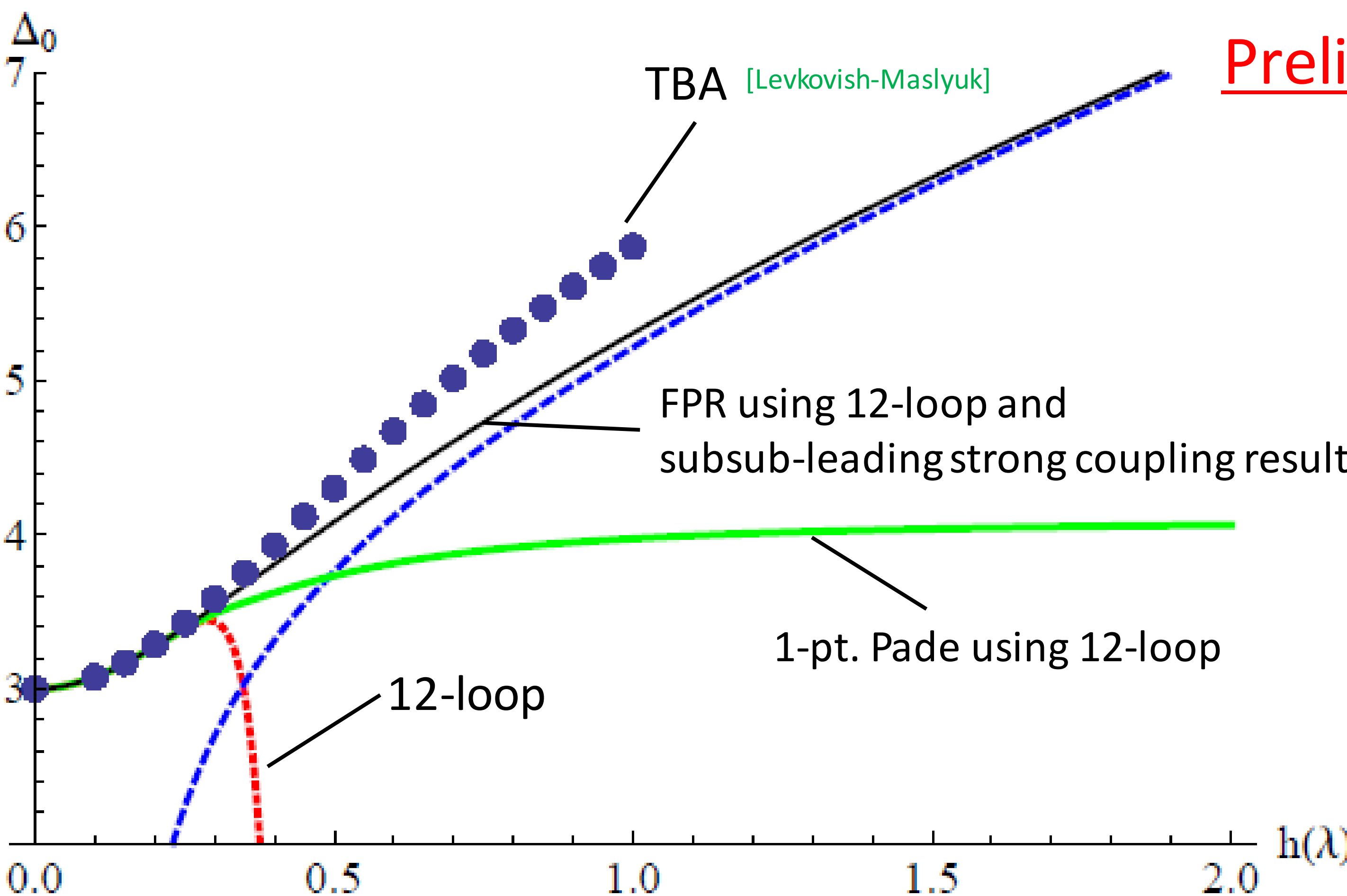
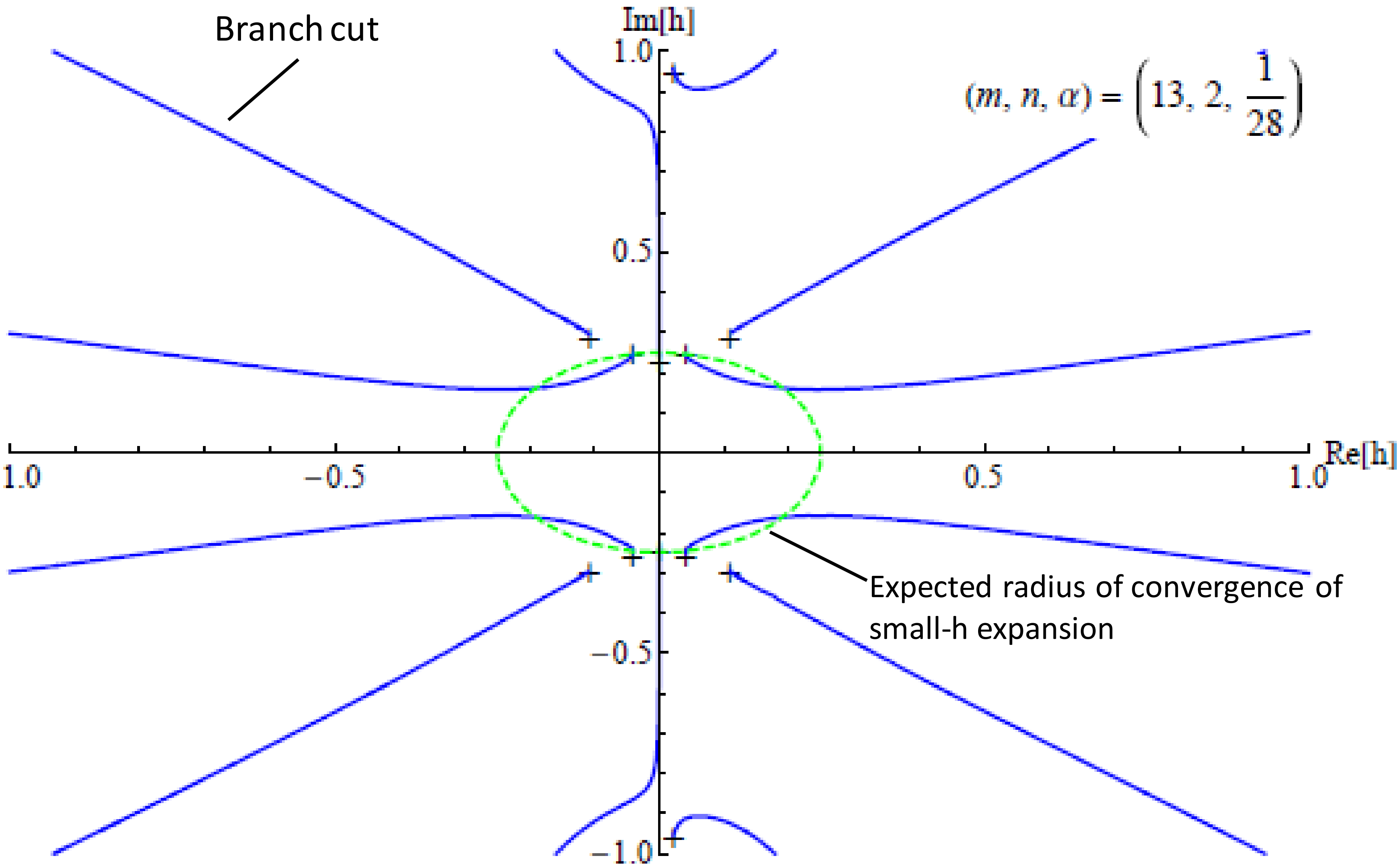
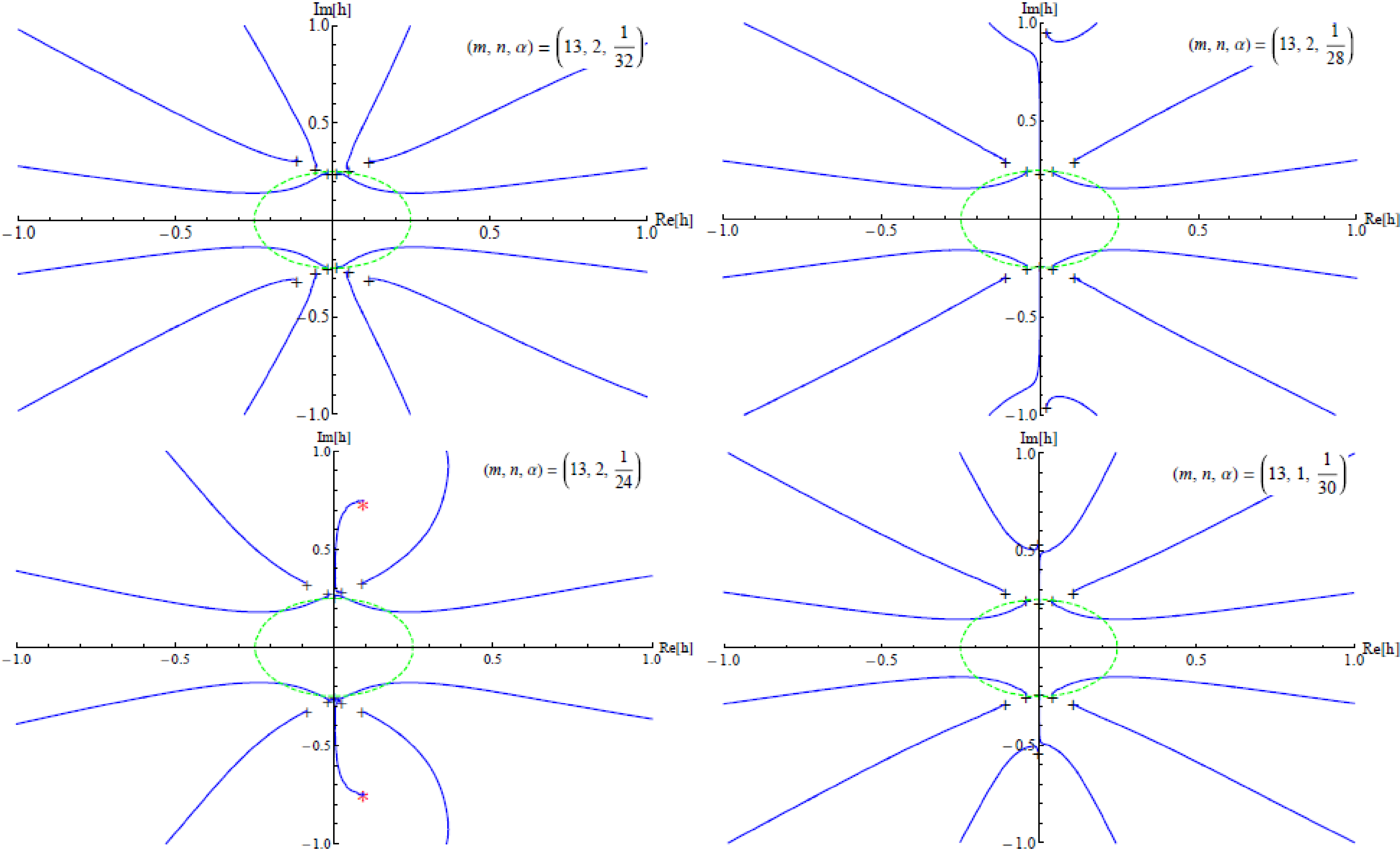


(Similar results hold for many other interpolating functions)

Analytic property of FPR for (L,S)=(1,2)

Preliminary





Many interpolating functions have singularities around $h=\pm i/4$!!

Similar results hold also for other (L,S).

Physical Interpretation

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⟶ Natural to think this as the prediction.

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If the conjecture $\lambda = \frac{\sinh(2\pi h)}{2\pi} {}_3F_2\left(\frac{1}{2},\frac{1}{2},\frac{1}{2};1,\frac{3}{2};-\sinh^2(2\pi h)\right).$ is correct,

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Surprisingly,

this is exactly the same as the critical point of S^3 free energy of ABJM

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Indirect evidence for the conjecture on h(λ)

Summary

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- We have introduced a class of interpolating functions (FPR)

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which includes Pade and FPP as the special cases

- Analytic property of FPR gives physical information on the dimensions of twist operators in the planar ABJM

—— indirect evidence for the recent conjecture on the dressed coupling constant h(λ)

Results on which I didn't talk (due to time)

- We can construct many interpolating functions

  ⟶ Which does give the best approximation?

  ⟶ Criterion for the best approximation [MH '14]

- Analytic property of interpolating function and Stokes Phenomena [MH-Jatkar '15]

- Interpolating function in Borel plane? [MH-Jatkar '15]

  —— Naïve idea is failed.

- Comparison with resurgence approach [MH-Jatkar '15]

- S-duality invariant interpolating function for twist op. in N=4 SYM [Chowdhury-MH-Thakur, to appear]

$$F_m^{(s,\alpha)}(\tau) = \left[\frac{\sum_{k=1}^{p} c_k E_{s+k}(\tau)}{\sum_{k=1}^{q} d_k E_{s+k}(\tau)} \right]^{\alpha}$$

[Generalization of Beem- Rastelli-Sen-van Rees, Alday-Bissi]

⟶ Compare with conformal bootstrap and draw conformal manifold

- Applications to 2d Ising model, lattice YM, W-loop in N=4 SYM etc...

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