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Entanglement Entropy and AdS/CFT

Part 1: EE in QFTs

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Out Line

Part 1: Entanglement Entropy (EE) in QFTs

➡ Definition, Properties, Calculations,
Cond-mat applications, …

Part 2: Holographic Entanglement Entropy (HEE)

➡ Holographic Calculations, Applications,….

Part 1 Contents

① Introduction

② Basic Properties of Entanglement Entropy (EE)

③ Calculations of EE in QFTs

Part 2 Contents

References (Review Articles)

(i) EE in QFT

Calabrese-Cardy, arXiv:0905.4013, J.Phys.A42:504005,2009.

Casini-Huerta, arXiv:0903.5284, J.Phys.A42:504007,2009.

(ii) Holographic EE

Nishioka-Ryu-TT, arXiv:0905.0932, J.Phys.A42:504008,2009.

(ii) EE and Black holes

Solodukhin, arXiv:1104.3712, Living Rev. Relativity 14, (2011), 8.

① Introduction

What is the entanglement entropy (EE) ?

A measure how much a given quantum state is quantum mechanically entangled (~complicated).

[.....We will explain more later, of course.]

Why interesting and useful ?

At present, it looks very difficult to observe EE in real experiments (→ a developing subject).

But, recently it is very common to calculate EE in `numerical experiments' of cond-mat systems.

➡ Classification of Quantum Phases

- EE = `Wilson loops' in quantum many-body systems

  ➡ A quantum order parameter

- The entanglement entropy (EE) is a helpful bridge between gravity (string) and cond-mat physics.

Gravity	⟺	Entanglement	⟺	Cond-mat. systems $\left|\Psi\right\rangle$
$g_{\mu\nu}$	AdS/CFT (Holography)	$S_A \approx \mathrm{Area}$		

Density matrix formalism

For a pure state, using the wave function $|\Psi\rangle$,
the density matrix is given by $\rho_{tot} = |\Psi\rangle\langle\Psi|$.

We can express the physical quantity as

$$\langle O \rangle = \mathrm{Tr}[O \cdot \rho_{tot}]. \qquad (\mathrm{Tr}[\rho_{tot}] = 1)$$

In a generic quantum system such as the one at finite temperature, it is not a pure state, but is a mixed state.

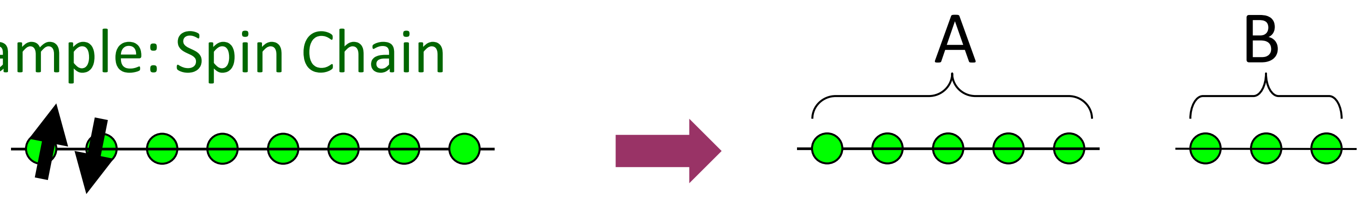
e.g. $\rho_{tot} = \dfrac{e^{-\beta H}}{\mathrm{Tr}[e^{-\beta H}]}$ for the canonical ensemble.

(1-1) Definition of entanglement entropy

Divide a quantum system into two parts A and B.
The total Hilbert space becomes factorized:

$$H_{tot} = H_A \otimes H_B \quad .$$

Example: Spin Chain



Define the reduced density matrix ρ_A for A by

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad ,$$

Finally, the entanglement entropy (EE) S_A is defined by

$$\boxed{S_A = -\mathrm{Tr}_A\, \rho_A \log \rho_A \quad .}$$ (von-Neumann entropy)

The Simplest Example: two spins (2 qubits)

$$\text{(i)} \quad |\Psi\rangle = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \otimes \left[|\uparrow\rangle_B + |\downarrow\rangle_B\right]$$

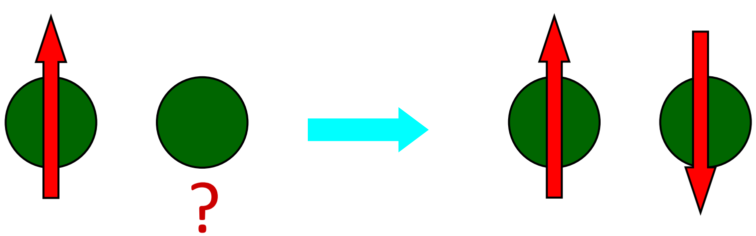
$$\Rightarrow \rho_{\mathrm{A}} = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A + |\downarrow\rangle_A\right] \cdot \left[\langle\uparrow|_A + \langle\downarrow|_A\right].$$

Not Entangled

$$\mathrm{S_A} = 0$$

$$\text{(ii)} \quad |\Psi\rangle = \left[|\uparrow\rangle_A \otimes |\downarrow\rangle_B + |\downarrow\rangle_A \otimes |\uparrow\rangle_{\mathrm{B}}\right] / \sqrt{2}$$

$$\Rightarrow \rho_A = \mathrm{Tr}_{\mathrm{B}}\left[|\Psi\rangle\langle\Psi|\right] = \frac{1}{2}\left[|\uparrow\rangle_A \langle\uparrow|_A + |\downarrow\rangle_A \langle\downarrow|_A\right].$$



Entangled

$$\mathrm{S_A} = \log 2$$

Note: The standard thermal entropy is obtained as a particular case of EE: i.e. A=total space.

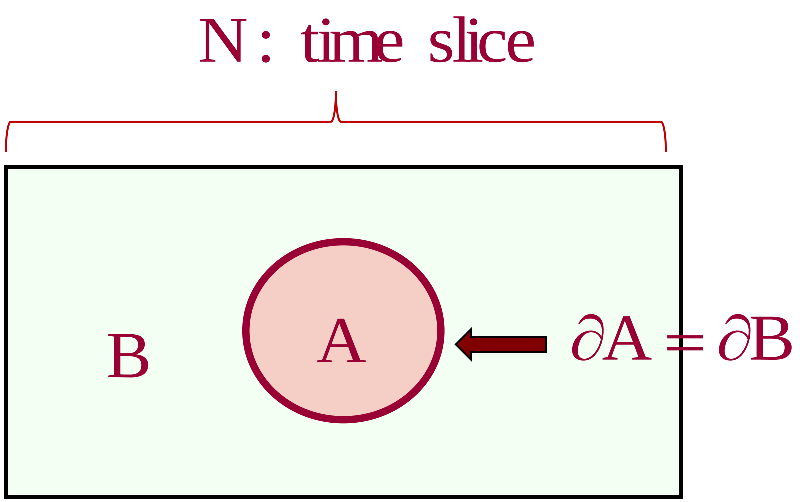
$$\rho = \frac{e^{-\beta H}}{Z}, \quad Z = \mathrm{Tr}[e^{-\beta H}].$$

$$\Rightarrow S = -\frac{\partial}{\partial n}\log[\mathrm{Tr}[\rho^n]]\Big|_{n\to 1} = -\frac{\partial}{\partial n}\left(\log[\mathrm{Tr}[e^{-\beta n H}]] - n\cdot\log Z\right)$$

$$= \beta\langle H\rangle + \log Z = \beta(E-F) = S_{thermal}.$$

EE in QFTs

In QFTs, the EE is defined geometrically (called geometric entropy).



$$H_{tot} = H_A \otimes H_B \quad .$$

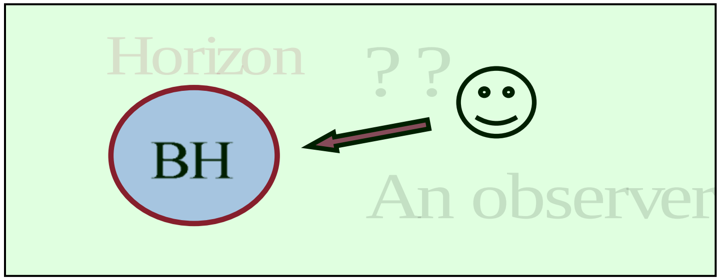
Historical origin: an analogy with black hole entropy

['t Hooft 85, Bombelli-Koul-Lee-Sorkin 86, Srednicki 93, …]

Because EE is defined by smearing out the Hilbert space for B,

E.E. ~ `Lost Information' hidden in B

This origin of entropy looks similar to the black hole entropy.



The boundary region ∂A ~ the event horizon ?

As we will explain, a complete answer to this historical question is found by considering the AdS/CFT correspondence !

(1-2) Basic Properties of EE

(i) If ρ_{tot} is a pure state (i.e. $\rho_{tot} = |\Psi\rangle\langle\Psi|$) and $H_{tot} = H_A \otimes H_B$,

then $S_A = S_B$. $\Rightarrow$ EE is not extensive !

[Proof]

This follows from the Schmidt decomposition:

$$|\Psi\rangle = \sum_{i=1}^{N} \lambda_i |a_i\rangle_A \otimes |b_i\rangle_B \ , \quad N \le \min\{|\ H_A\ |, |\ H_B\ |\}.$$

$$\Rightarrow \mathrm{Tr}[(\rho_A)^n] = \mathrm{Tr}[(\rho_B)^n],$$

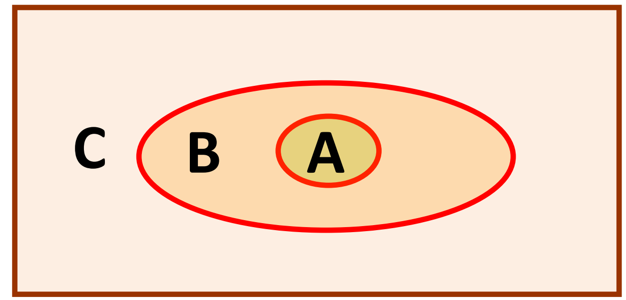
$$\Rightarrow\ S_A = -\frac{\partial}{\partial n}\mathrm{Tr}[(\rho_A)^n]\Big|_{n\to 1} = S_B\ .$$

(ii) Strong Subadditivity (SSA) [Lieb-Ruskai 73]

When $H_{tot} = H_A \otimes H_B \otimes H_C$, for any ρ_{tot} ,

$$S_{A+B} + S_{B+C} \geq S_{A+B+C} + S_B,$$

$$S_{A+B} + S_{B+C} \geq S_A + S_C.$$



Actually, these two inequalities are equivalent .

We can derive the following inequality from SSA:

$$|S_A - S_B| \leq S_{A\cup B} \leq S_A + S_B. \quad (\mathrm{Note: A \cap B \neq \phi \ \ in \ \ general})$$

Araki-Lieb inequality

Subadditivity

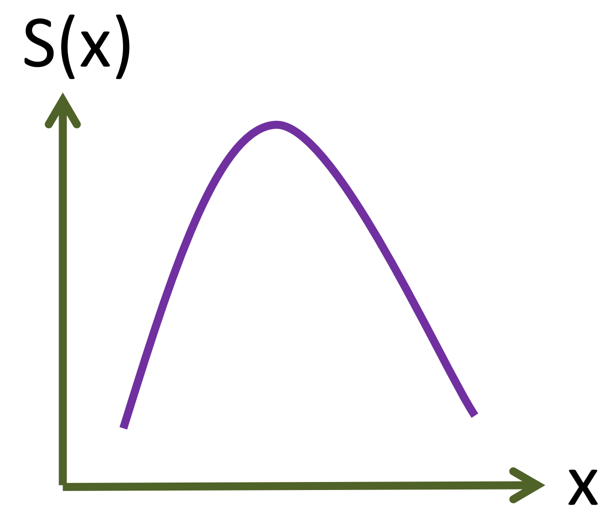
The strong subadditivity can also be regarded as the concavity of von-Neumann entropy.

Indeed, if we assume A,B,C are numbers, then

$$S(A+B)+S(B+C)\geq S(A+B+C)+S(B),$$

$$\Rightarrow 2\cdot S\left(\frac{x+y}{2}\right)\geq S(x)+S(y),$$

$$\Rightarrow \frac{d^2}{dx^2}S(x)\leq 0.$$



Mutual Information

We can define a positive quantity I(A,B) which measures an `entropic correlation’ between A and B:

$$I(A,B) = S_A + S_B - S_{A\cup B} \geq 0.$$

This is called the mutual information.

The strong subadditivity leads to the relation:

$$I(A,B+C) \geq I(A,B).$$

(iii) Area law [Bombelli-Koul-Lee-Sorkin 86, Srednicki 93]

EE in QFTs includes UV divergences.

Area Law

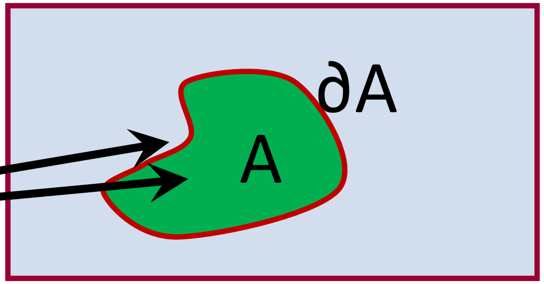
The leading divergent term of EE in a (d+1) dim. QFT is proportional to the area of the (d-1) dim. boundary ∂A:

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + (\text{subleading terms}),$$

where a is a UV cutoff (i.e. lattice spacing).

Intuitively, this property is understood like:

Most strongly entangled

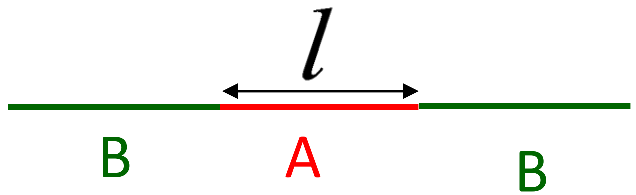


Comments on Area Law

- The area law can be applied for ground states or finite temperature systems. It is violated for highly excited states. (Note $S_A \leq \log(\dim H_A) \approx \mathrm{Vol}(A)$.)

- There are two exceptions:

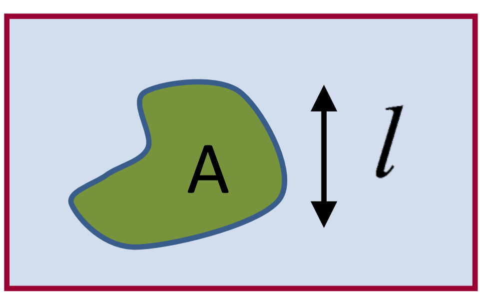
(a) 1+1 dim. CFT $S_A = \frac{c}{3} \log \frac{l}{a}.$



[Holzhey-Larsen-Wilczek 94, Calabrese-Cardy 04]

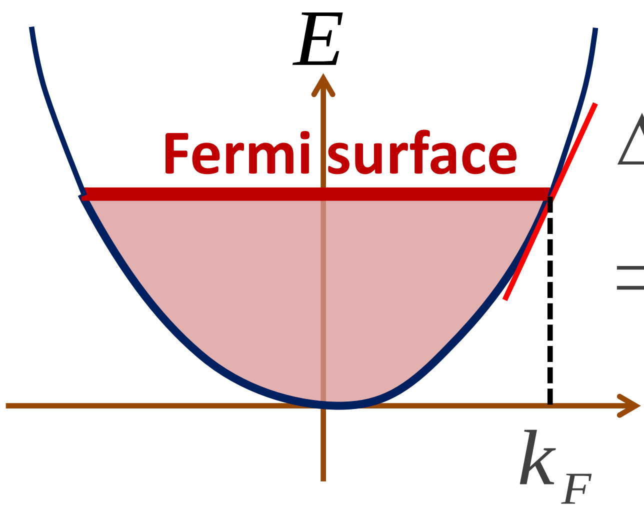
(b) QFT with Fermi surfaces ($k_F \sim a^{-1}$)

$$S_A \sim \left(\frac{l}{a}\right)^{d-1} \cdot \log \frac{l}{a} + \ldots$$



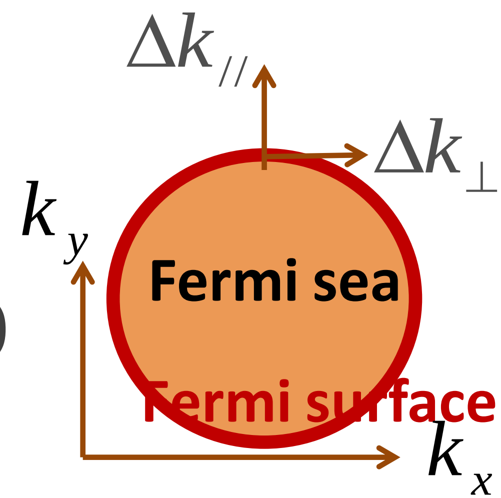
[Wolf 05, Gioev-Klich 05]

This logarithmic behavior of EE in the presence of Fermi surfaces can be understood if we note that we can approximate the excitations of Fermi liquids by **an infinite copies of 2 dim. CFTs**.



$$\Delta E \propto \Delta k_{//} + O(\Delta k_{\perp}^{2})$$

$$\Rightarrow \ \mathrm{FL} \approx \prod_{\vec{k}F} \mathrm{CFT}_2(\vec{k}_F)$$



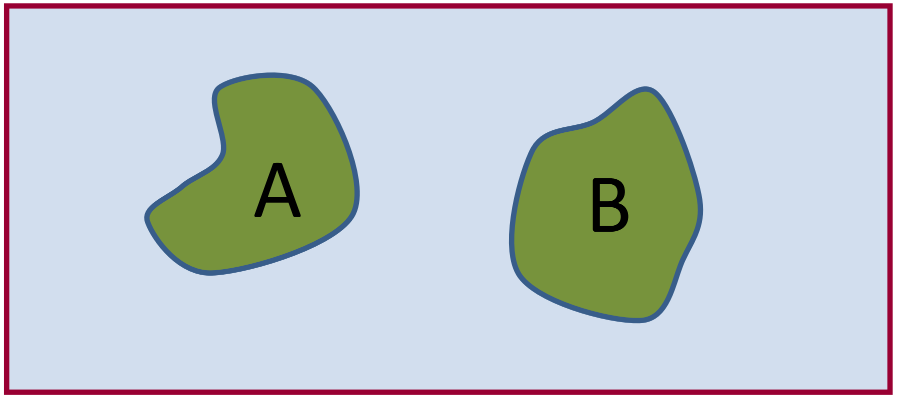
$$\Rightarrow \quad S_A \sim \underbrace{\left(\frac{l}{a}\right)^{d-1}}_{\#\,\text{of 2d CFTs}} \cdot \underbrace{\log\frac{l}{a}}_{\text{Each of 2d CFT}}.$$

- The proof of area law is available only for free field theories. [e.g. Plenio-Eisert-Dreissig-Cramer 04,05]

- The AdS/CFT predicts the area law for strongly interacting theories as long as the QFT has a UV fixed point.

- The UV divergence cancels out in the mutual information.

$$\Rightarrow I(A,B) = S_A + S_B - S_{A\cup B} = \text{finite} \geq 0, \quad \text{if} \quad A \cup B = \phi.$$



• The area law resembles the Bekenstein-Hawking formula of black hole entropy:

$$S_{BH} = \frac{\mathrm{Area(horizon)}}{4G_N}.$$

Actually, the EE can be interpreted not as the total but as a partial (i.e. quantum corrections) contribution to the black hole entropy.   [Susskind-Uglm 94]

➡ A more complete understanding awaits the AdS/CFT !

(iv) Relation to Thermal Entropy

- At high temp., the finite part of EE is dominated by thermal entropy:

$$S_A \approx (\text{divergence}\ ) + S_{th}(A).$$

- If we set A=total space, B=empty, then we should get the total thermal entropy.

  More precisely, we have

$$\lim_{|B|\to 0}(S_A - S_B) = S_{th}.$$

(v) Renyi entropy and entanglement spectrum

Renyi entropy is defined by

$$S_A^{(n)} = \frac{\log \mathrm{Tr}[(\rho_A)^n]}{1-n}.$$

This is related to EE in the limit $\lim_{n\to 1} S_A^{(n)} = S_A$.

If we know $S_A^{(n)}$ for all n, we can obtain all eigenvalues of ρ_A . They are called the entanglement spectrum.

(1-3) Applications of EE to condensed matter physics

$S_A \approx$ Log[``Effective rank'' of density matrix for A]

$\Rightarrow$ This measures how much we can compress the quantum information of ρ_A .

Thus, EE estimates difficulties of computer simulations such as in DMRG etc. [Osborne-Nielsen 01, …]

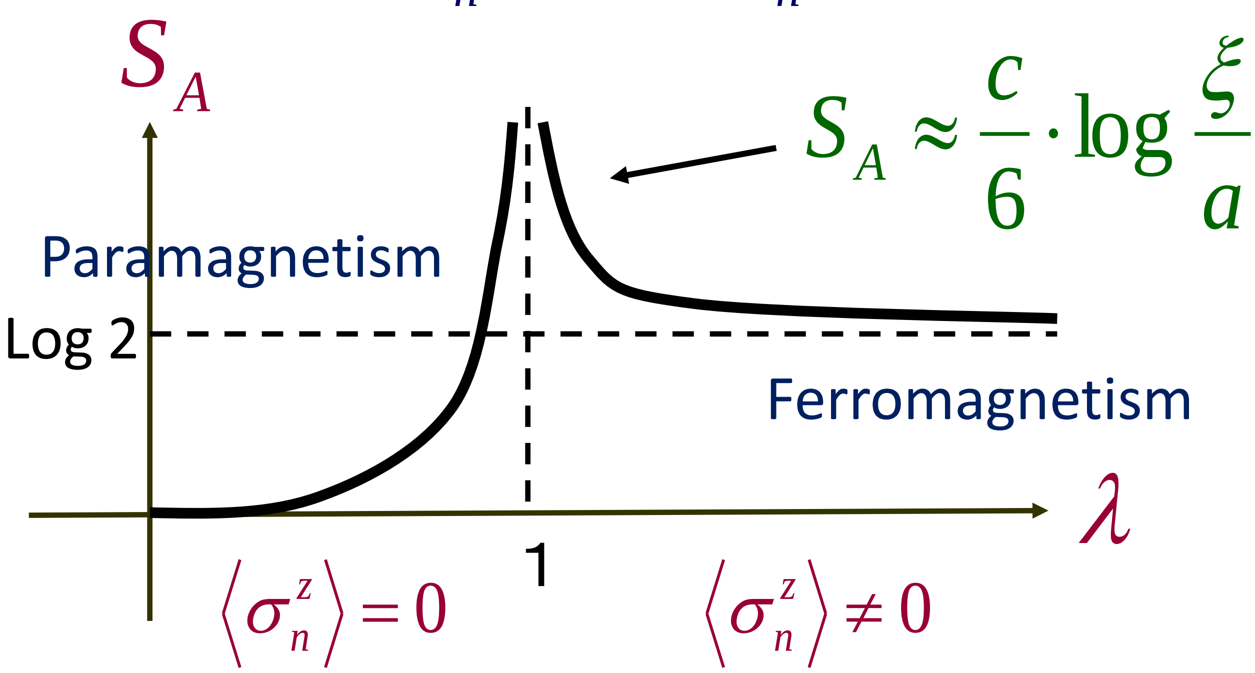
Especially, EE gets divergent at the quantum phase transition point (= quantum critical point).

$\Rightarrow$ EE = a quantum order parameter !

Ex. Quantum Ising spin chain

The Ising spin chain with a transverse magnetic field:

$$H = -\sum_n \sigma_n^x - \lambda \sum_n \sigma_n^z \sigma_{n+1}^z$$



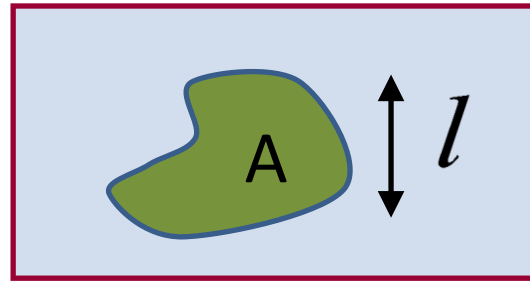
[Vidal-Latorre-Rico-Kitaev 02, Calabrese-Cardy 04]

Topological Entanglement Entropy

[Kitaev-Preskill 06, Levin-Wen 06]

In a 2+1 dim. mass gapped theory, EE behaves like

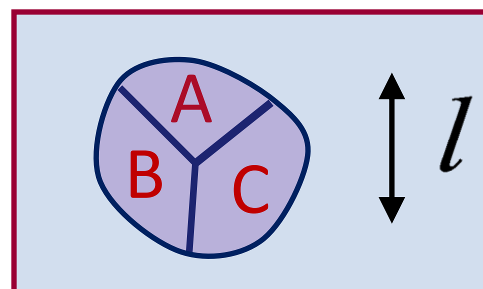
$$S_A = \gamma \cdot \frac{l}{a} + S_{top} \quad .$$



The finite part $S_{top} \equiv -\log D$ is invariant under smooth deformations of the subsystem A. ⇒ Topological !

- Top. EE offers us an order parameter of topological systems. (cf. ~~correlation functions~~)
- To eliminate divergences, equally we have

$$S_{top} = S_A + S_B + S_C - S_{A+B} - S_{B+C} - S_{C+A} + S_{A+B+C} \quad .$$



Summary

(1) EE is the entropy for an observer who is only accessible to the subsystem A and not to B.

➡ EE measures the **amount of quantum information**.

(2) EE is a sort of a `non-local version of correlation functions', which captures topological information. (cf. Wilson loops)

➡ EE can be a **quantum order parameter.**

(3) EE is proportional to the degrees of freedom.
It is non-vanishing even at zero temperature.

➡ EE is a **useful observable in numerical calculations** of quantum many-body systems.
Indeed, a practical numerical method to read off the central charge of a given spin chain is to look at EE.

③ Calculations of EE in QFTs

A basic method of calculating EE in QFTs is so called the **replica method**.

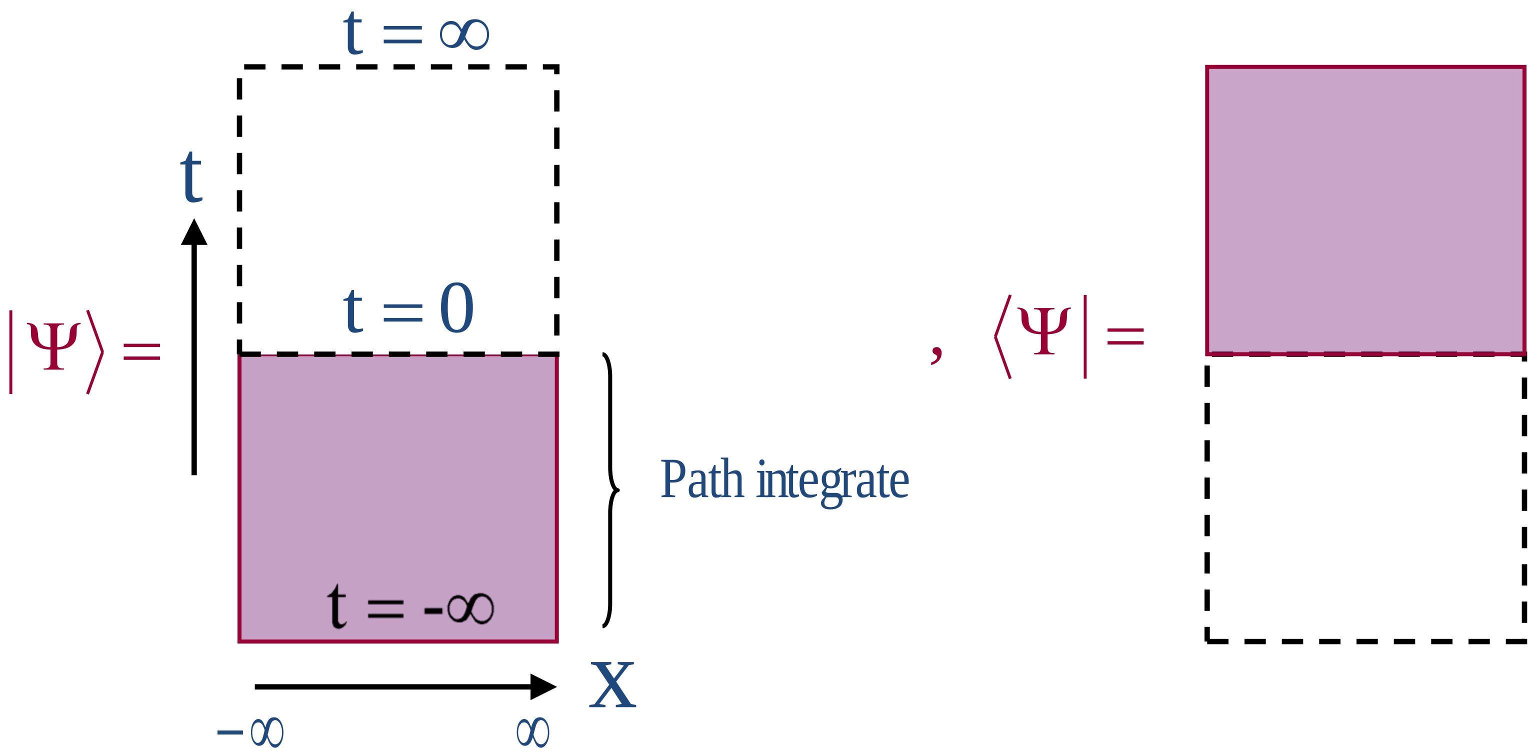
$$S_A = -\frac{\partial}{\partial n}\mathrm{Tr}_{\mathrm{A}}\,(\rho_A)^n\,|_{n=1} = -\frac{\partial}{\partial n}\,\log\mathrm{Tr}_{\mathrm{A}}\,(\rho_A)^n\,|_{n=1}\quad .$$

(3-1) 2d CFT

By using this, we can analytically compute the EE in 2d CFTs. [Holzhey-Larsen-Wilczek 94,…, Calabrese-Cardy 04]

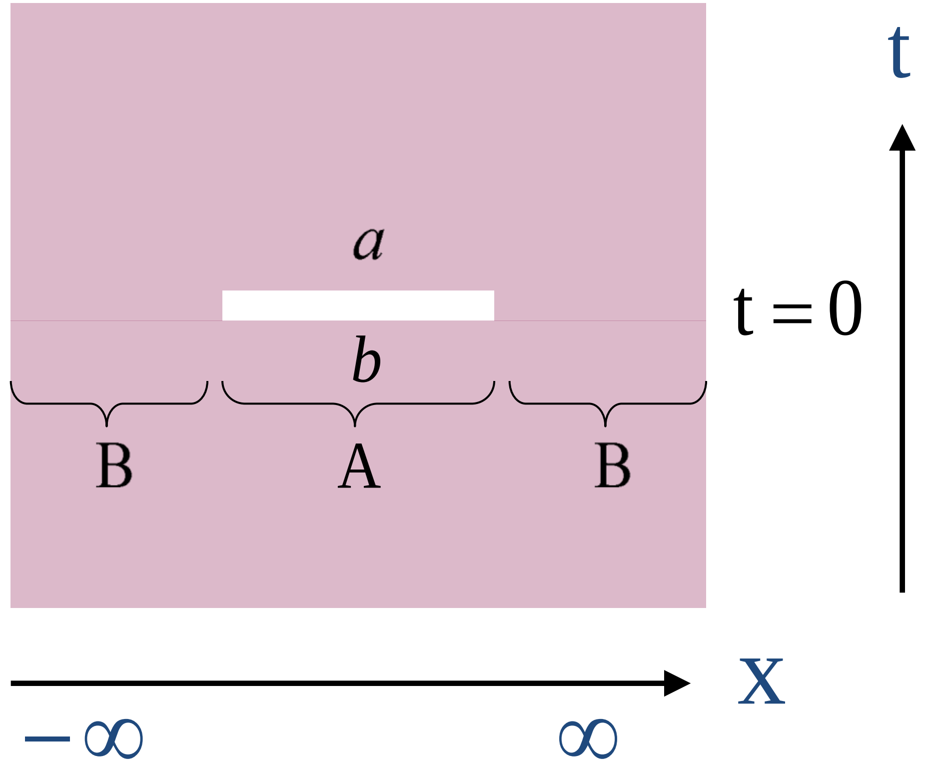
The replica method is also an important method to (often numerically) evaluate EE in more general QFTs.

In the path-integral formalism, the ground state wave function $|\Psi\rangle$ can be expressed in the path-integral formalism as follows:



Next we express $\rho_A = \mathrm{Tr}_B |\Psi\rangle\langle\Psi|$.

$[\rho_A]_{ab} =$

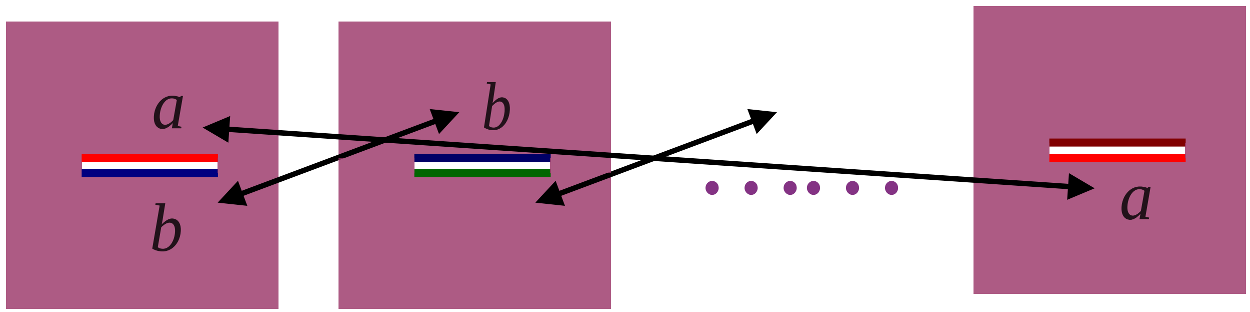


Finally, we obtain a path integral expression of the trace

$$\mathrm{Tr}\,(\rho_A)^n = [\rho_A]_{ab}[\rho_A]_{bc}\cdots[\rho_A]_{ka}$$ as follows:

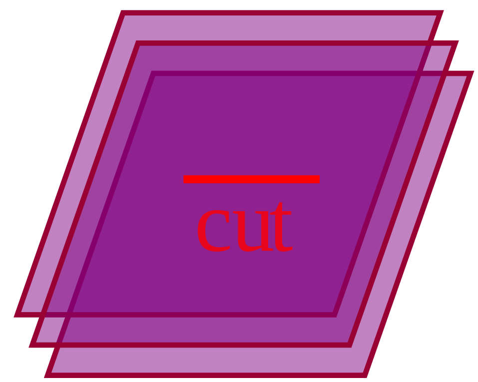
Glue each boundaries successive ly.

$\mathrm{Tr}\,(\rho_A)^n =$



$=$ a path integral over

n-sheeted Riemann surface Σ_n



n sheets

In this way, we obtain the following representation

$$\mathrm{Tr}\,(\rho_A)^n = \frac{Z_n}{(Z_1)^n} \ ,$$

where Z_n is the partition function on the n-sheeted Riemann surface Σ_n.

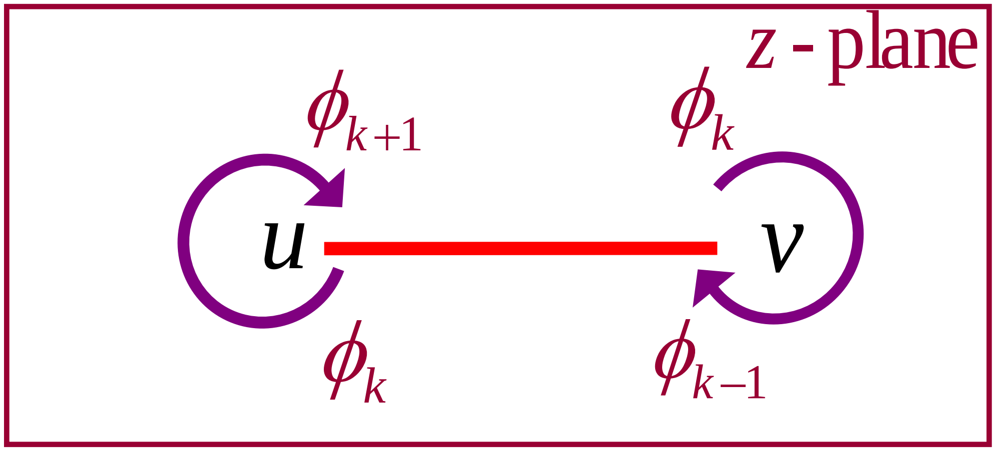
To evaluate Z_n, let us first consider the case where the CFT is defined by a complex free scalar field ϕ.

c=2

It is useful to introduce n replica fields $\phi_1, \phi_2, \cdots \phi_n$ on a complex plane $\Sigma_{n=1} = \mathrm{C}$.

Then we can obtain a CFT equivalent to the one on Σ_n by imposing the boundary condition

$$\phi_k(e^{2\pi i}(z-u)) = \phi_{k+1}(z-u), \qquad \phi_k(e^{2\pi i}(z-v)) = \phi_{k-1}(z-v),$$



By defining $\tilde{\phi}_k = \frac{1}{n}\sum_{k=0}^{n-1} e^{2\pi i k/n}\phi_k$, conditions are diagonalized

$$\tilde{\phi}_k(e^{2\pi i}(z-u)) = e^{2\pi i k/n}\tilde{\phi}_k(z-u), \quad \tilde{\phi}_k(e^{2\pi i}(z-v)) = e^{-2\pi i k/n}\tilde{\phi}_k(z-v),$$

Using the orbifold theoretic argument, these twisted boundary conditions are equivalent to the insertion of (ground state) twisted vertex operators at z=u and z=v.

This leads to

$$\mathrm{Tr}\left(\rho_A\right)^n = \prod_{k=0}^{n-1} \left\langle \sigma_{k/n}(u)\sigma_{-k/n}(v)\right\rangle \propto (u-v)^{-\frac{1}{3}(n-1/n)}.$$

$$\sigma_{k/n} : \text{ Twist operator s.t. } \phi \to e^{2\pi i k/n}\phi$$

$$\text{Conformal dim. : } \Delta(\sigma_{k/n}) = -\frac{1}{2}\left(\frac{k}{n}\right)^2 + \frac{1}{2}\frac{k}{n} \quad .$$

For general 2d CFTs with the central charge c , we can apply a similar analysis. In the end, we obtain

$$\mathrm{Tr}\left(\rho_A\right)^n \propto (u-v)^{-\frac{c}{6}(n-1/n)}.$$

In the end, we obtain

$$S_A = \frac{c}{3}\log\frac{l}{a} \qquad (l \equiv v-u).$$

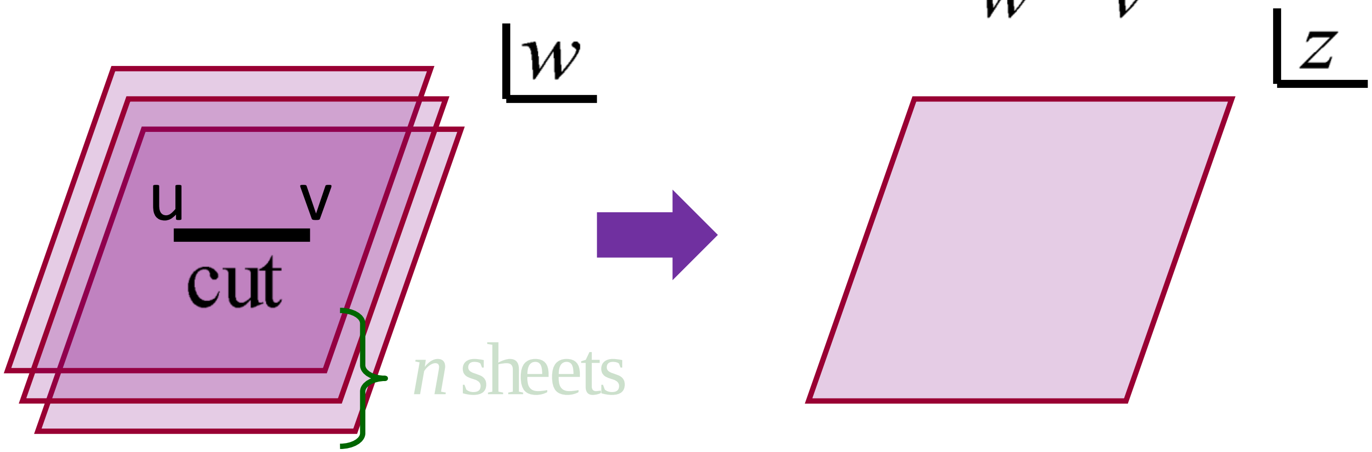
[Holzhey-Larsen-Wilczek 94]

Note: the UV cut off a is introduced such that

$S_A = 0$ at $l = a$.

General CFTs [Calabrese-Cardy 04]

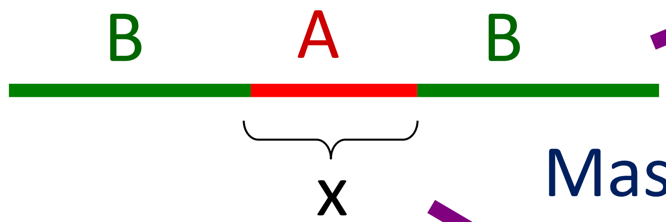
Consider the conformal map: $z^n = \dfrac{w-u}{w-v}$.



$$T(w) = \left(\frac{dz}{dw}\right)^2 \underbrace{T(z)}_{=0} + \frac{c}{12}\underbrace{\{z,w\}}_{\text{Schwarzian derivative}} = \frac{c(1-n^{-2})}{24} \cdot \frac{(v-u)^2}{(w-u)^2(w-v)^2}.$$

$$\Rightarrow \Delta_{\text{each sheet}} = \frac{c(1-n^{-2})}{24}, \qquad \Delta_{\text{tot}} = n\Delta_{\text{each sheet}} = \frac{c(n-1/n)}{24}.$$

More general results in 2d CFT [Calabrese-Cardy 04]



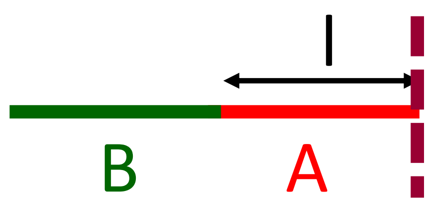
$$S_A = \frac{c}{3}\log\frac{x}{a}$$

Mass gap

$$S_A = \frac{c}{3}\log\frac{\xi}{a}$$

Finite Temp.

$$S_A = \frac{c}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi x}{\beta}\right)\right)$$

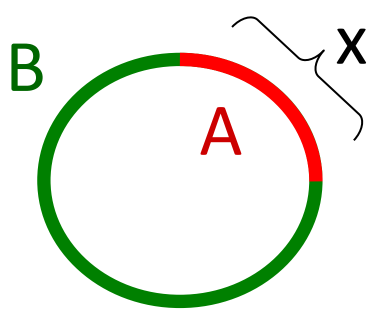


Boundary

$$S_A = \frac{c}{3}\log\left(\frac{l}{a}\right) + \underbrace{\log g}_{\text{Boundary Entropy}},$$

Boundary Entropy

[Affleck-Ludwig 91]



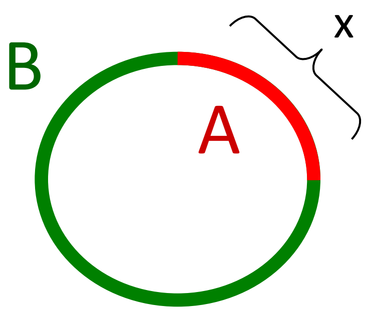
$$S_A = \frac{c}{3}\log\left(\frac{L}{\pi a}\sin\left(\frac{\pi x}{L}\right)\right)$$

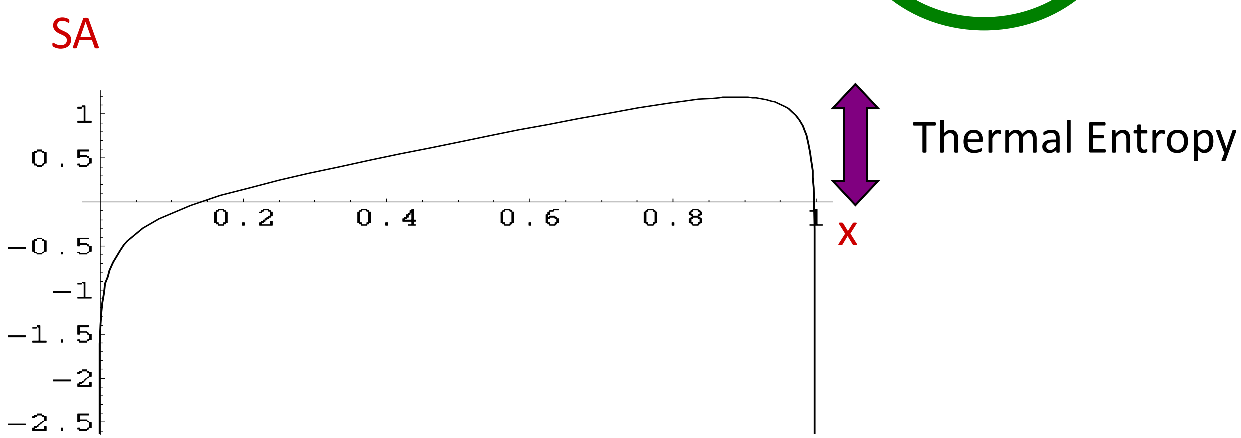
Finite size system at finite temp. (2D free fermion c=1)

[Azeyanagi-Nishioka-TT 07]

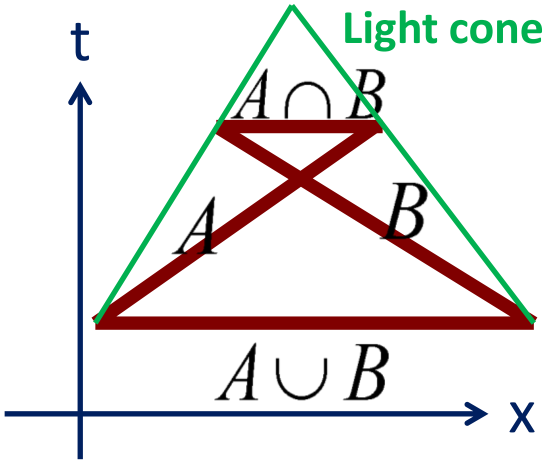
$$S_A = \frac{1}{3}\log\left(\frac{\beta}{\pi a}\sinh\left(\frac{\pi x}{\beta}\right)\right) + \frac{1}{3}\sum_{i=1}^{\infty}\log\left[\frac{(1-e^{2\pi x/\beta}e^{-2\pi m/\beta})(1-e^{-2\pi x/\beta}e^{-2\pi m/\beta})}{(1-e^{-2\pi m/\beta})^2}\right]$$

$$+2\sum_{m=1}^{\infty}\frac{(-1)^m}{m}\cdot\frac{\frac{\pi m x}{\beta}\cot\left(\frac{\pi m x}{\beta}\right)-1}{\sinh\left(\frac{\pi m}{\beta}\right)}\ .$$





Entropic C-theorem [Casini-Huerta 04]



Consider a relativistic QFT.

We have $S_A + S_B \geq S_{A\cup B} + S_{A\cap B}$,

$$l_A \cdot l_B = l_{A\cup B} \cdot l_{A\cap B} \quad .$$

We set $l_{A\cup B} = e^a, \quad l_{A\cap B} = e^b, \quad l_A = l_B = e^{(a+b)/2}$.

$$\Rightarrow 2 \cdot S\left(\frac{a+b}{2}\right) \geq S(a) + S(b),$$

$$\Leftrightarrow \frac{\partial^2 S(x)}{\partial x^2} = \frac{1}{3} \cdot \frac{\partial C(x)}{\partial x} \leq 0 \quad \text{(entropic c-theorem)}.$$

(3-2) Higher dimensional CFT

We can still apply the replica method:

$$S_A = -\frac{\partial}{\partial n}\log[\mathrm{Tr}\ (\rho_A)^n]\Big|_{n=1} = -\frac{\partial}{\partial n}\log\left[\frac{Z_n}{(Z_1)^n}\right]\Bigg|_{n=1} .$$

However, in general, there is no analytical way to calculate Z_n . (`Twist operators' get non-local !)
Thus in many cases, numerical calculations are needed.

➡ One motivation to explore the holographic analysis !

(3-3) EE in even dim. CFT and Central Charges

Consider the dependence of EE on the size l of the subsystem A. This is directly related to the Weyl anomaly:

$$l\frac{dS_A}{dl}=\frac{1}{2\pi}\lim_{n\to 1}\frac{\partial}{\partial n}\left\langle \int_{M_n} dx^{d+1}\sqrt{g}\,T^{\mu}_{\mu}(x)\right\rangle_{\Sigma_n} .$$

<u>2d CFT</u>

$$\left\langle T^{\mu}_{\mu}(x)\right\rangle=-\frac{c}{12}R, \qquad \chi(\Sigma_n)=\frac{1}{4\pi}\int_{\Sigma_n} dx^2\sqrt{g}R=2(1-n).$$

$$\Rightarrow l\frac{\partial S_A}{\partial l}=-\frac{1}{24\pi}\frac{\partial}{\partial n}\int_{\Sigma_n} dx^2\sqrt{g}R=\frac{c}{3} \quad ,$$

$$\Rightarrow S_A=\frac{c}{3}\log\frac{l}{a} \quad .$$

4d CFT (There are two central charges a and c)

$$\langle T^{\mu}_{\mu}(x)\rangle = \underbrace{-\frac{c_{CFT}}{8\pi}W^{\mu\nu\rho\sigma}W_{\mu\nu\rho\sigma}}_{(\text{Weyl curvature})^2} + \underbrace{\frac{a_{CFT}}{8\pi}\tilde{R}^{\mu\nu\rho\sigma}\tilde{R}_{\mu\nu\rho\sigma}}_{\text{Euler density}}.$$

By integrating w.r.t. the linear size l of A, we obtain

$$S_A = \gamma_1 \cdot \frac{\text{Area}(\partial A)}{a^2} + \gamma_2 \cdot \log\left(\frac{l}{a}\right) + \text{const.},$$

$$\gamma_2 = \underline{\frac{c_{CFT}}{6\pi}\int_{\partial A}(R + 2R_{ijij} - R_{ii})} - \frac{a_{CFT}}{2\pi}\int_{\partial A} R \quad ,$$ [Ryu-TT 06]

where i, j denotes the directions normal to ∂A.

We assumed that the extrinsic curvatures are vanishing.

Comments

- The full expression of the coefficient of log term is obtained as

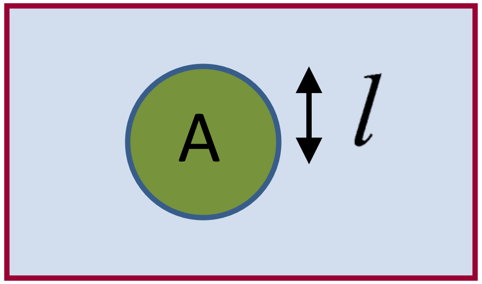
$$\gamma_2 = \frac{c_{CFT}}{2\pi}\int_{\partial A} dx^2 \left[C^{abcd}h_{ac}h_{bd} - \mathrm{Tr}[K^2] + \tfrac{1}{2}(\mathrm{Tr}[K])^2\right] - \frac{a_{CFT}}{2\pi}\int_{\partial A} dx^2 R \quad .$$

by employing the holographic EE [Solodukhin 08, Hung-Myers-Smolkin 11].

- When A is a round ball with the radius l ,

$$\frac{1}{4\pi}\int_{\Sigma_n} dx^2\sqrt{g}R = \chi(\partial A \cong S^2) = 2. \quad \Rightarrow \gamma_2 = -4a_{CFT} \ .$$

$$S_A = \gamma_1 \cdot \frac{l^2}{a^2} - 4a_{CFT}\cdot\log\left(\frac{l}{a}\right) + \mathrm{const.}$$



[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

- a_{CFT} is expected to satisfy the c-theorem. [Cardy 88, Myers-Sinha 10]

(3-4) EE in CFT and Thermal Entropy

[Casini-Huerta-Myers 11]

When A = a round ball, we can relate the EE in CFT to a thermal entropy in the de-Sitter space:

$$ds^2_{(d+1)} = -dt^2 + dr^2 + r^2 d\Omega^2_{(d-1)}.$$

Coordinate transformation

$$\begin{cases} t = R\dfrac{\cos\theta \sinh(\tau/R)}{1+\cos\theta\cosh(\tau/R)}, \\[2ex] r = R\dfrac{\sin\theta}{1+\cos\theta\cosh(\tau/R)} \end{cases}$$

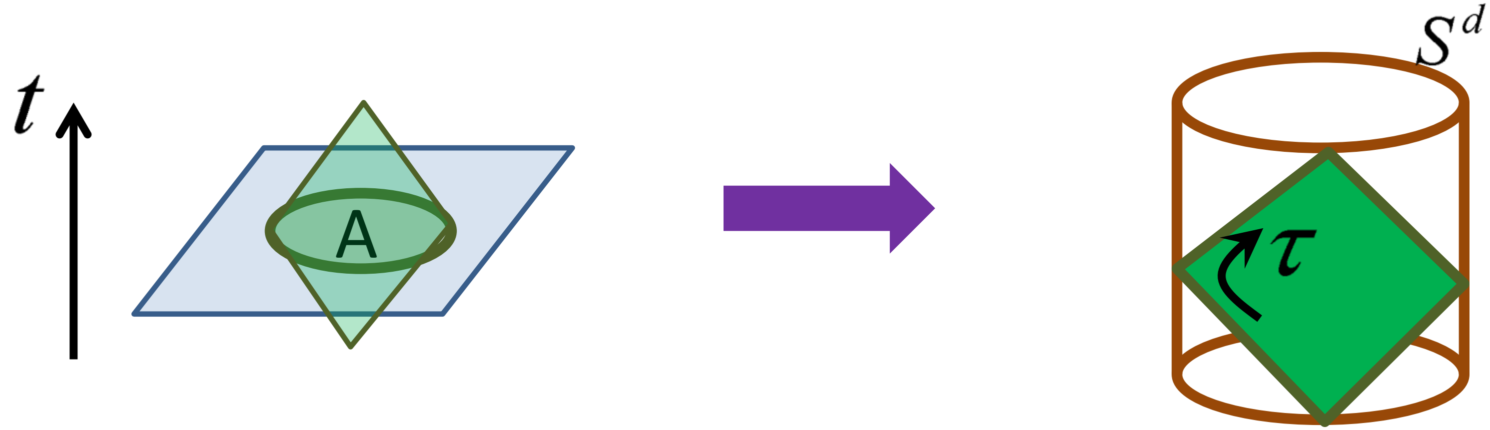
$$ds^2_{(d+1)} = \Lambda(\theta)^2 \underline{(-\cos^2\theta \cdot d\tau^2 + R^2(d\theta^2 + \sin^2\theta d\Omega^2_{(d-1)}))},$$

$$\Lambda(\theta) \equiv (1+\cos\theta\cosh(\tau/R))^{-1}.$$

de Sitter space (static cord.)

$$0 \le \theta \le \pi/2$$

$$\text{Note}: \ (t=0,\ r=R) \cong (\tau=0, \theta=\pi/2) \rightarrow \text{de Sitter horizon.}$$



de Sitter space

(Static coordinate)

$$\Rightarrow \quad S_A = S^{\text{Thermal}}_{\text{de Sitter}}.$$

Moreover, in odd dim. CFT, there is no conformal anomaly.

Thus, we have $\quad S_A = S_{thermal} = \beta(\underset{\overset{\parallel}{0}}{E} - F) = -\beta F,$

Therefore, $\quad S_A = \log Z(S^{d+1}).$

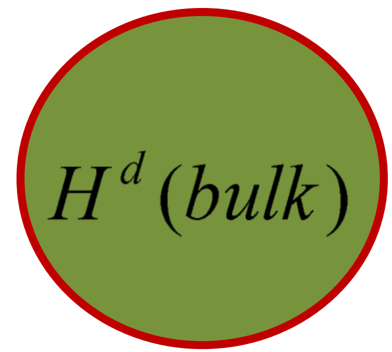
(Note: Euclidean de-Sitter = Sphere)

Comments

- We can also relate EE in CFT to a thermal entropy on $S^1 \times H^d$:

$$ds^2_{(d+1)} = -dt^2 + dr^2 + r^2 d\Omega^2_{(d-1)}.$$

$$\begin{cases} t = R\dfrac{\sinh(\tau/R)}{\cosh u + \cosh(\tau/R)} \\ r = R\dfrac{\sinh u}{\cosh u + \cosh(\tau/R)} \end{cases},$$



$S^{d-1}(edge)$

$$ds^2_{(d+1)} = \Lambda(\theta)^2(-d\tau^2 + R^2(du^2 + \sinh^2 u\, d\Omega^2_{(d-1)})),$$

$$\Lambda(\theta) \equiv (\cosh u + \cosh(\tau/R))^{-1}.$$

$$\text{Note}: \ (t=0,\ |r| \le R) \cong (\tau = 0, 0 \le u < \infty).$$

- In topological theories, this leads to `bulk-edge correspondence':

Entanglement spectrum in bulk = Physical spectrum on edge

[Li-Haldane 08, Swingle-Senthil 11]

$$\rho_A \approx e^{-H_{edge}}$$

Lectures at CERN Winter School on Supergravity, Strings, and Gauge Theory, 2012

Entanglement Entropy and AdS/CFT

Part 2: Holographic Entanglement Entropy

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Part2 Contents

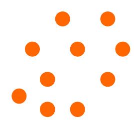
Recent applications

④ A Quick Introduction to Holography and AdS/CFT

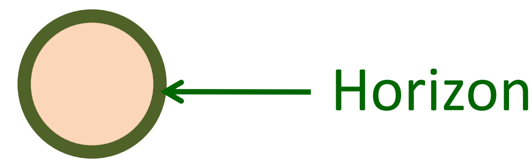
(4-1) What is ``Holography'' ?

In the presence of gravity,

A lot of massive objects
in a small region



Black Holes (BHs)

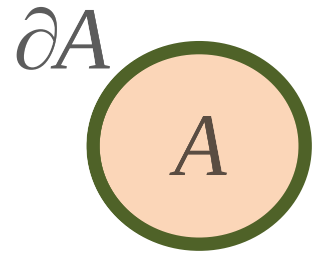


The information hidden inside BHs is measured by the Bekenstein-Hawking black hole entropy:

$$S_{BH} = \frac{\mathrm{Area(Horizon)}}{4G_N}$$

This consideration leads to the idea of entropy bound:

$$S(A) \leq \frac{\mathrm{Area}(\partial \mathrm{A})}{4G_N}$$



(S(A) = the entropy in a region A)

The degrees of freedom in gravity are proportional to the area instead of the volume !

cf. In non-gravitational theories, the entropy is proportional to volume.

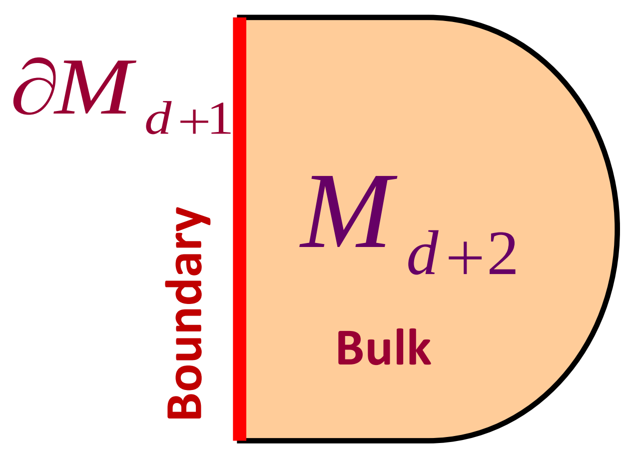
Motivated by this, holographic principle has been proposed ['t Hooft 93 and Susskind 94]:

Often, lives on the boundary of (d+2) dim. spacetime

Holographic Principle

(d+2) dimensional Quantum gravity ⟺ (d+1) dimensional Non-gravitational theory (e.g. QM, QFT, CFT, etc.)

Equivalent



(4-2) AdS/CFT Correspondence

The best established example of holography is
the AdS/CFT correspondence [1997 Maldacena]:

AdS/CFT

Gravity (String Theory) on AdS_{d+2} = CFT on R^{d+1}

Isometry of AdS_{d+2} = $SO(d+1,2)$ = Conformal Sym.

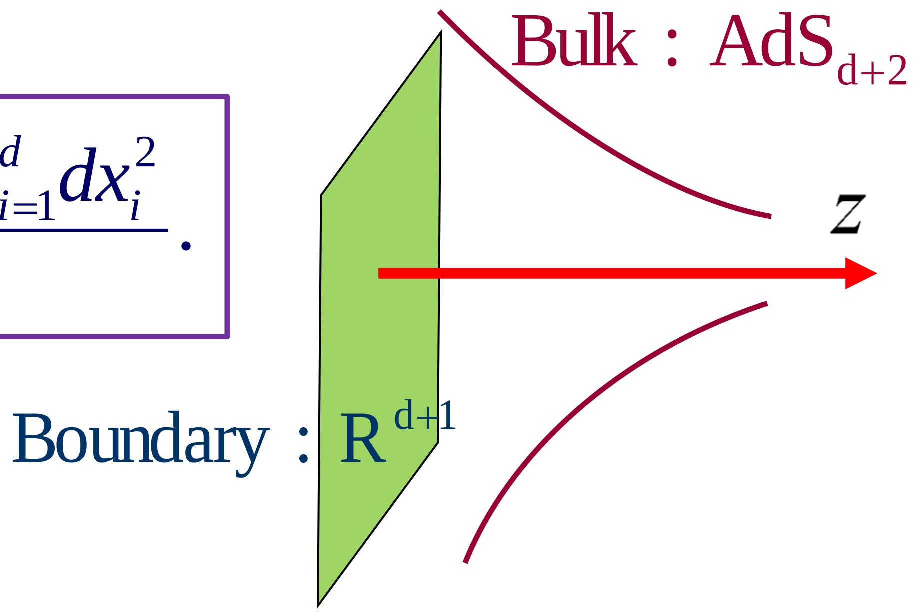
AdS spaces

They are homogeneous solutions to the vacuum Einstein equation with a negative cosmological constant:

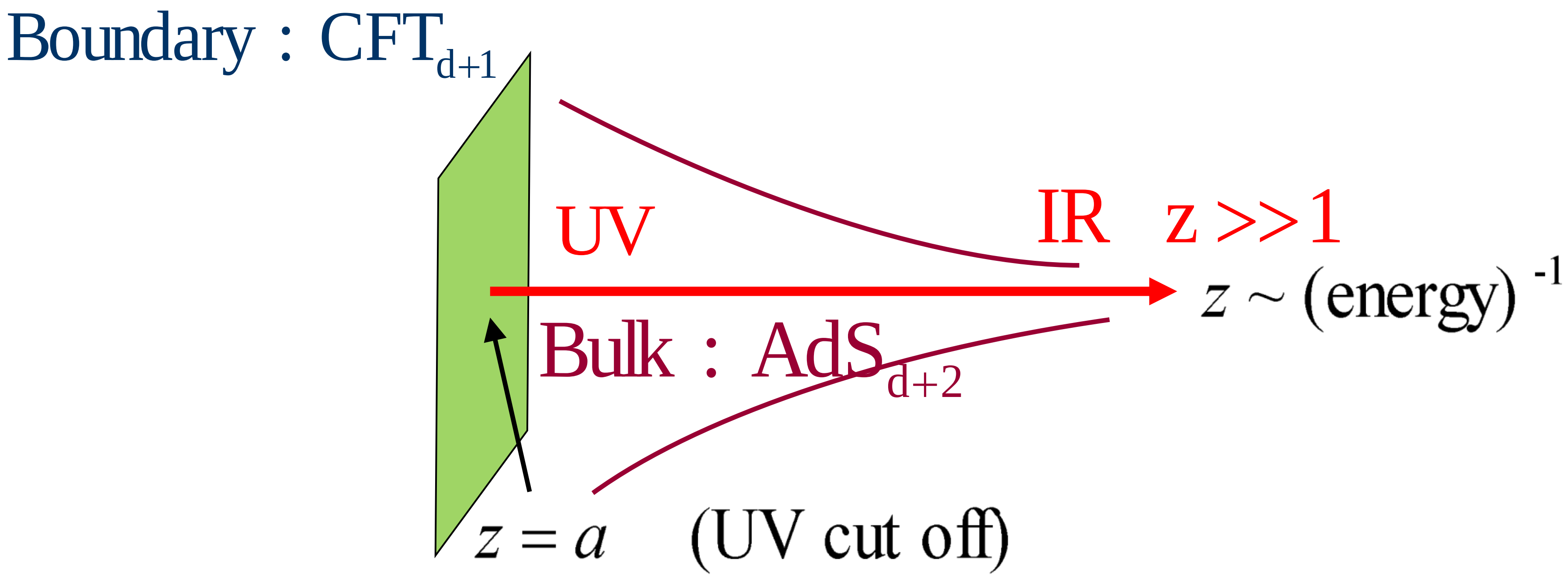
$$S_g = \frac{1}{16\pi G_N} \int dx^{d+2} \sqrt{-g} \left[R - 2\Lambda \right], \qquad \Lambda \equiv -\frac{(d+1)d}{2R^2}.$$

The metric of AdSd+2 (in Poincare coordinate) is given by

$$\boxed{ds^2_{AdS_{d+2}} = R^2 \frac{dz^2 - dx_0^2 + \Sigma_{i=1}^{d} dx_i^2}{z^2}.}$$



A Sketch of AdS/CFT



The radial direction z corresponds to the length scale in CFT under the RG flow.

Note: String (or M) theory is 10 (or 11) dim. ⇒ $AdS_p \times M^q$

CFT (conformal field theory)

⇒Typically SU(N) gauge theories in the large N limit.

e.g. Type IIB String on AdS5 $\times$ S^5

= N=4 SU(N) Super Yang-Mills in 4 dim.

↓

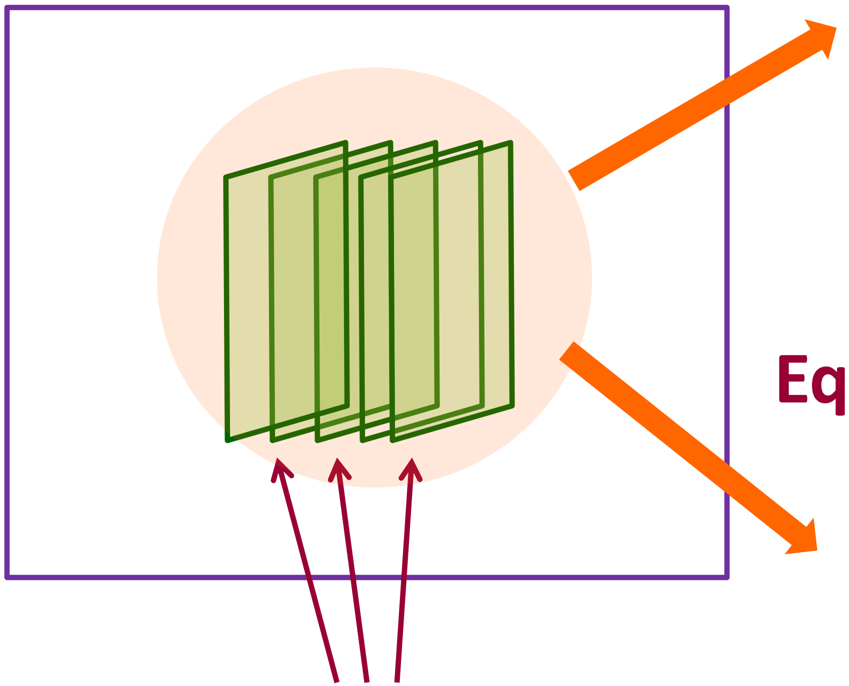
$$\text{Gauge field} \;+\; 6\ \text{Scalar fields} \;+ 4\ \text{Fermions}$$

$$(A_\mu) \quad (\phi_1, \phi_2, \phi_3, \phi_4, \phi_5, \phi_6) \quad (\psi_1, \psi_2, \psi_3, \psi_4)$$

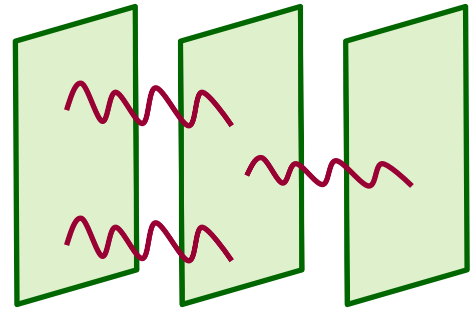
Symmetry of S^5 ⇔ SO(6) R symmetry

Discovery of AdS/CFT in String Theory ex. AdS5/CFT4

10 dim. type IIB string theory
with N **D3-branes**

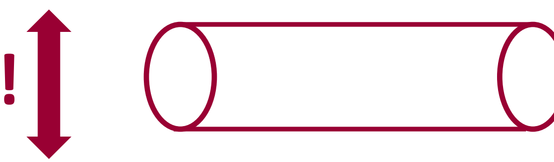


N D3-branes
= (3+1) dimensional sheets

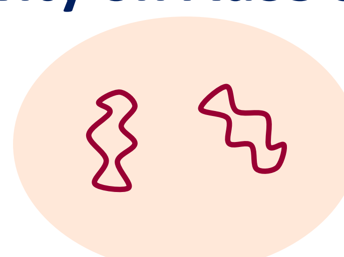


Open Strings between D-branes
→ SU(N) gauge theories

Equivalent !



Type IIB closed string on AdS5 × S5
→ Gravity on AdS5 spacetime



$$\text{IIB string on } AdS_5 \times S^5 \Leftrightarrow \text{4D } N=4 \text{ SU(N) SYM}$$

$$SO(2,4) = \text{4D conformal symmetry}$$

$$SO(6) = \text{R-symmetry of } N=4\,\text{SYM}$$

$$\frac{R_{AdS}}{l_{Planck}} \propto N^{1/4}$$

$$\frac{R}{l_{String}} = (Ng_{YM}^2)^{1/4} \equiv \lambda^{1/4}.$$

(i) small quantum gravity corrections = large N CFT

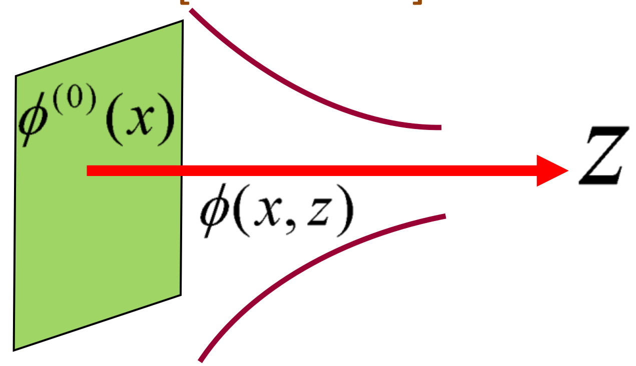
(ii) small stringy corrections = strong coupled CFT

In this lecture, we mainly ignore both of these corrections. Therefore we concentrate on strongly coupled large N CFT.

(4-3) Bulk to boundary relation

The basic principle in AdS/CFT to calculate physical quantities is the bulk to boundary relation [GKP-W 98]:

$$\boxed{Z_{Gravity}(M) = Z_{CFT}(\partial M).}$$



Gravity theories includes metric, scalar fields, gauge fields etc…

$$Z_{Gravity} = \int Dg_{\mu\nu} D\phi \; e^{-S(g(x,z),\phi(x,z))} \cong e^{-S(g,\phi)}\Big|_{\substack{\text{Equation}\\ \text{of motion}}}.$$

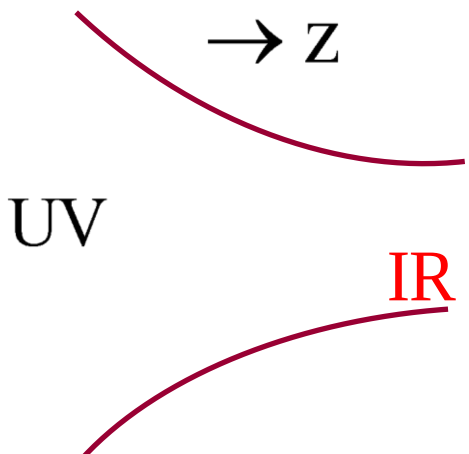
$$Z_{CFT} = \left\langle e^{\int dx^{d+1}[\delta g^{(0)}{}_{\mu\nu}(x)T^{\mu\nu}(x)+\phi^{(0)}(x)O(x)]} \right\rangle \Rightarrow \text{Correlation functions}$$

$$\langle O(x_1)O(x_2)\cdots O(x_n)\rangle$$

(4-4) Basic Deformations of AdS/CFT

AdS/CFT can be naturally generalized to the duality:

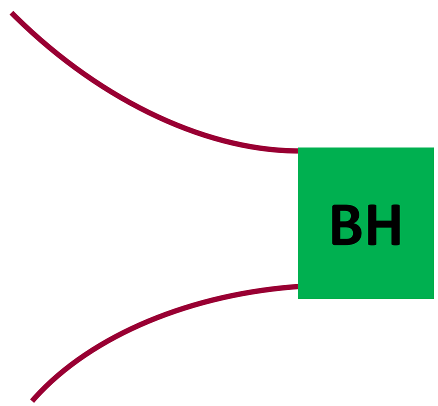
asymptotically AdS spaces ⇔ QFTs with UV fixed points .



Pure AdS

⬍

CFT at T=0

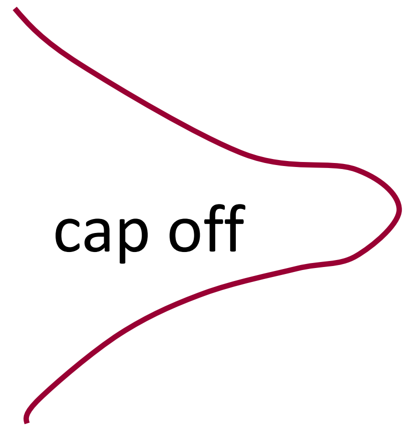


AdS BH

⬍

Finite temp. CFT

$S_{BH} \propto N^2$



AdS Soliton

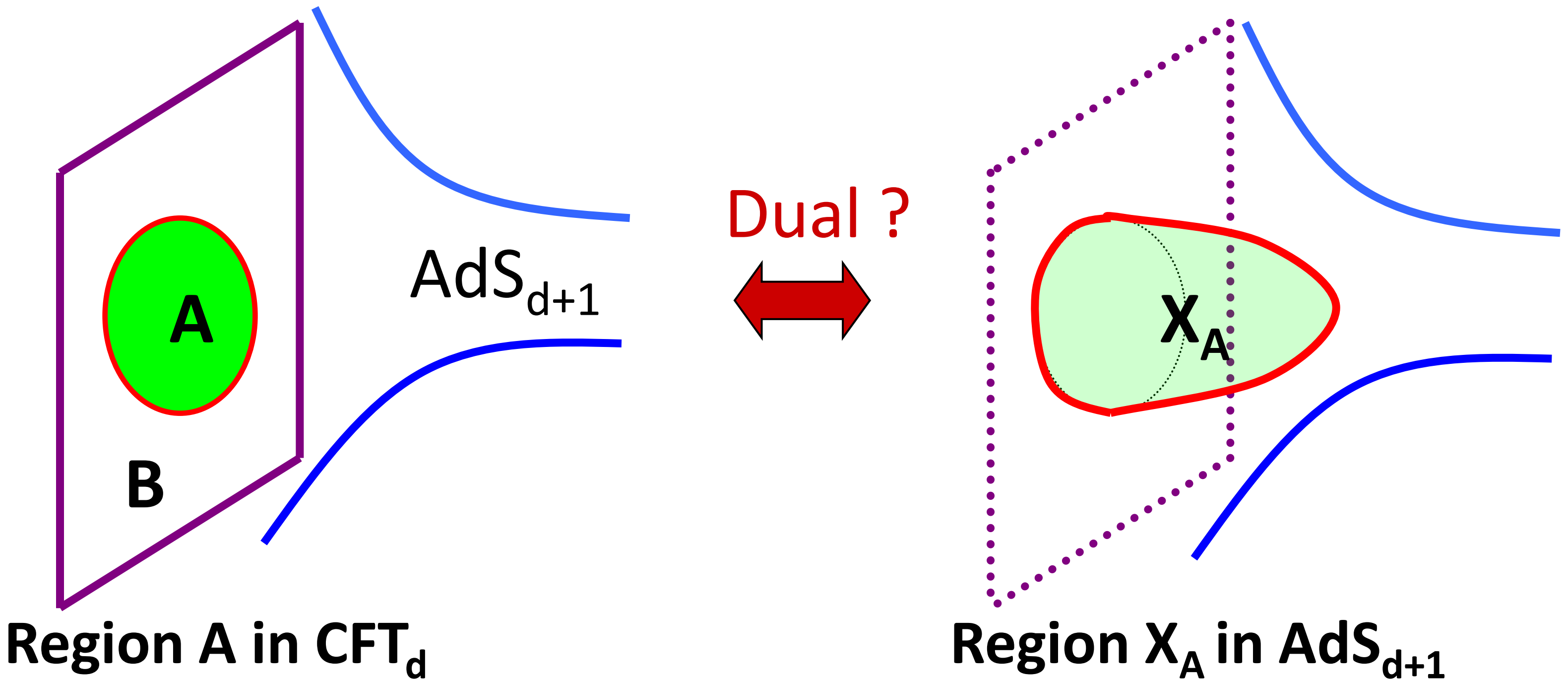
⬍

QFT with Mass gap

(confinement)

(4-5) Information in AdS ?

A Basic Question: Which region in the AdS does encode the `information in a certain region' of the CFT ?



➡ The entanglement entropy S_A provides us a definite measure of the amount of information !

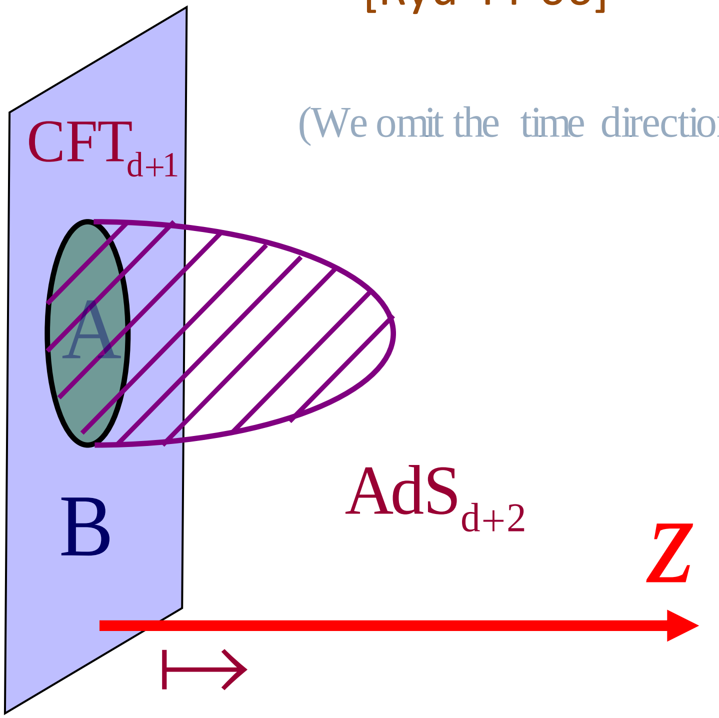
⑤ Holographic Entanglement Entropy (HEE)

(5-1) Holographic Entanglement Entropy Formula

[Ryu-TT 06]

(We omit the time direction.)

$$S_A = \frac{\mathrm{Area}(\gamma_A)}{4G_N}$$



γ_A is the minimal area surface (codim.=2) such that

$$\partial A = \partial \gamma_A \ \text{and} \ A \sim \gamma_A \ .$$

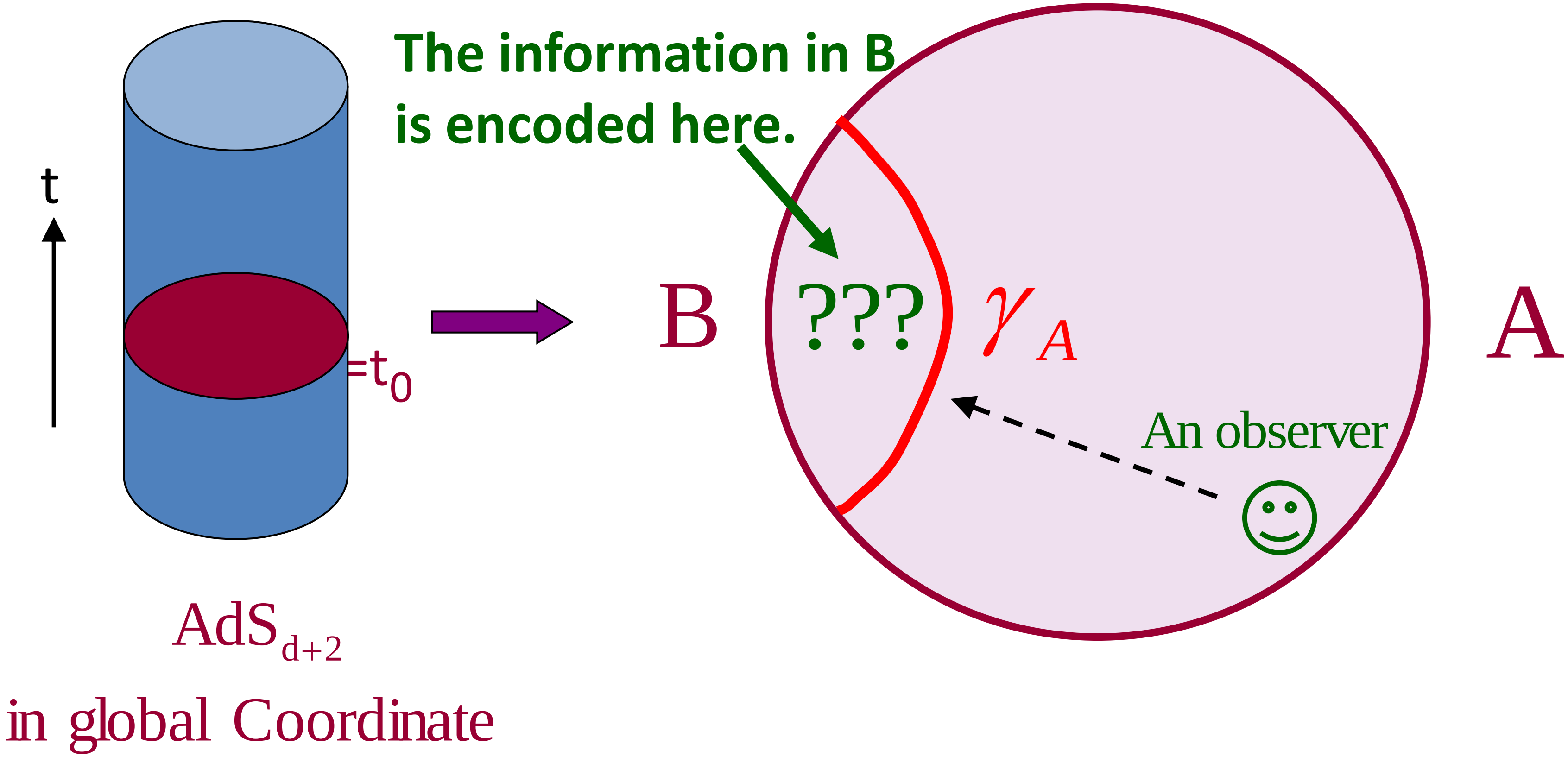
homologous

$z > a$ (UV cut off)

$$ds^2_{AdS} = R^2_{AdS}\frac{-dt^2 + \sum_{i=1}^{d-1} dx_i^2 + dz^2}{z^2} \ .$$

Motivation of this proposal

Here we employ the global coordinate of AdS space and take its time slice at $t=t_0$.



Leading divergence and Area law

For a generic choice of γ_A, a basic property of AdS gives

$$\mathrm{Area}(\gamma_A) \sim R^d \cdot \frac{\mathrm{Area}(\partial\gamma_A)}{a^{d-1}} + (\text{subleading terms}),$$

where R is the AdS radius.

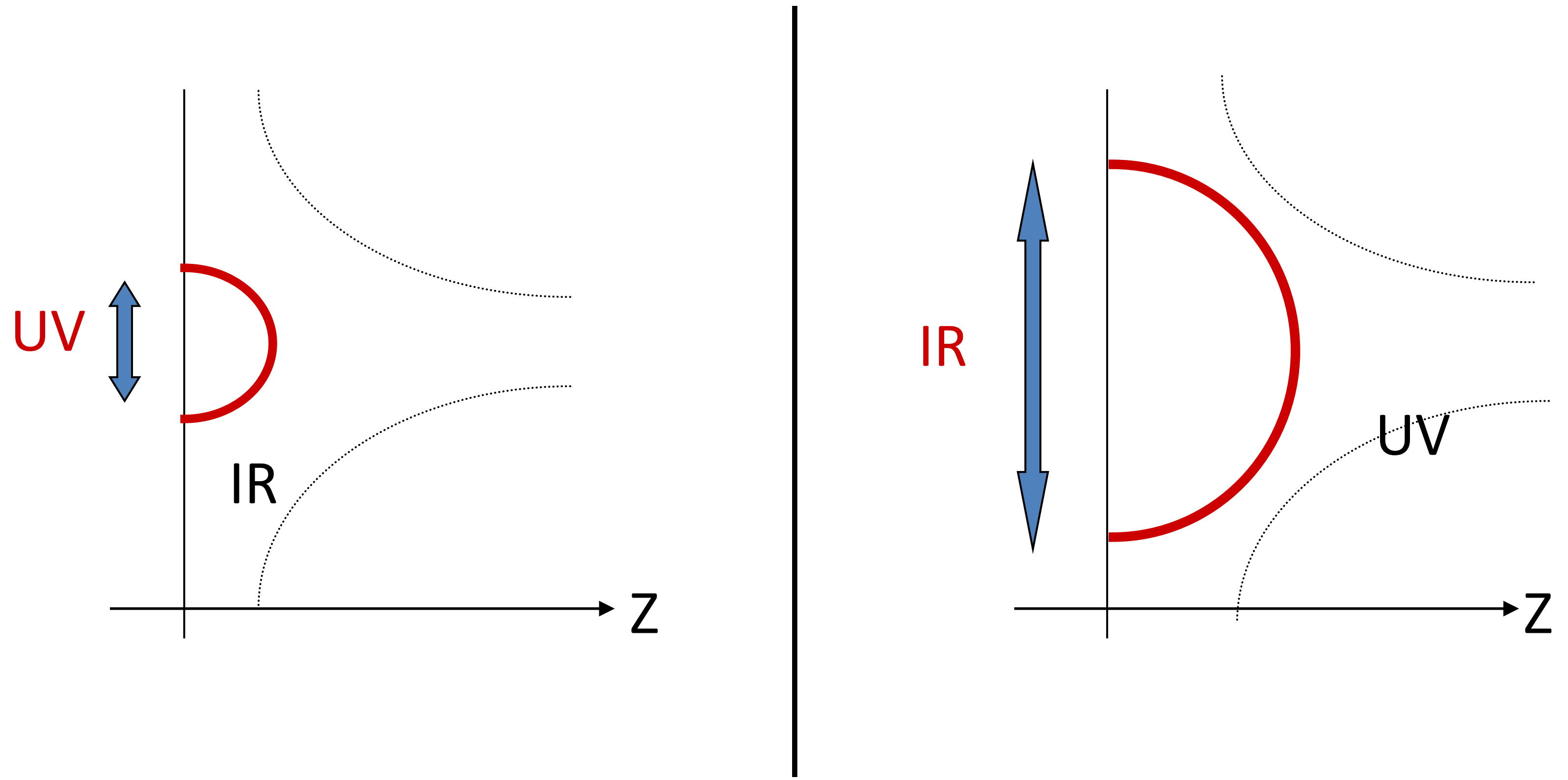
Because $\partial\gamma_A = \partial A$, we find

$$S_A \sim \frac{\mathrm{Area}(\partial A)}{a^{d-1}} + (\text{subleading terms}).$$

This agrees with the known area law relation in QFTs.

UV-IR duality

In the HEE calculation, the UV-IR duality is manifest:



Comments

- A complete proof of HEE formula is still missing, there has been many evidences and no counter examples. (We will explain some of them later.)

- If backgrounds are time-dependent, we need to employ extremal surfaces in the Lorentzian spacetime instead of minimal surfaces. If there are several extremal surfaces we should choose the one with the smallest area. [Hubeny-Rangamani-TT 07]

- In the presence of black hole horizons, the minimal surfaces wraps the horizon as the subsystem A grows enough large.
  $\Rightarrow$ Reduced to the Bekenstein-Hawking entropy, consistently.

(5-2) HEE from AdS3/CFT2

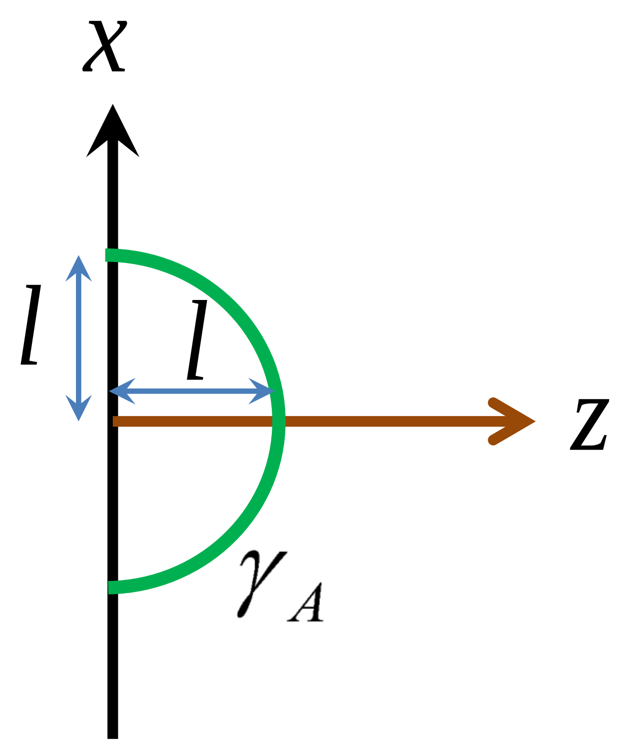
In AdS3/CFT2, the HEE is given by the geodesic length in the AdS3:

$$ds^2 = R^2 \cdot \frac{dz^2 - dt^2 + dx^2}{z^2}.$$

This is explicitly evaluated as follows:

$$x = \sqrt{l^2 - z^2} \quad \Rightarrow \quad ds^2_{circle} = \frac{l^2 dz^2}{z^2 \sqrt{l^2 - z^2}}.$$

$$L(\gamma_A) = 2R \int_a^l dz \frac{l}{z\sqrt{l^2 - z^2}} = 2R \log \frac{2l}{a}.$$



Finally, the HEE is found to be

$$S_A = \frac{L(\gamma_A)}{4G_N^{(3)}} = \frac{2R}{4G_N^{(3)}} \log\left(\frac{2l}{a}\right) = \frac{c}{3}\log\left(\frac{2l}{a}\right),$$

where we employed the famous relation

$$c = \frac{3R}{2G_N^{(3)}}.$$

[Brown-Henneaux 86]

In this way, HEE reproduces the 2 dim. CFT result.

Finite temperature CFT

Consider a 2d CFT in the high temp. phase $\frac{l}{\beta} >> 1$.

⇒ The dual gravity background is the BTZ black hole:

$$ds^2 = -(r^2 - r_H^2)dt^2 + \frac{R^2}{r^2 - r_H^2}dr^2 + r^2 d\phi^2,$$

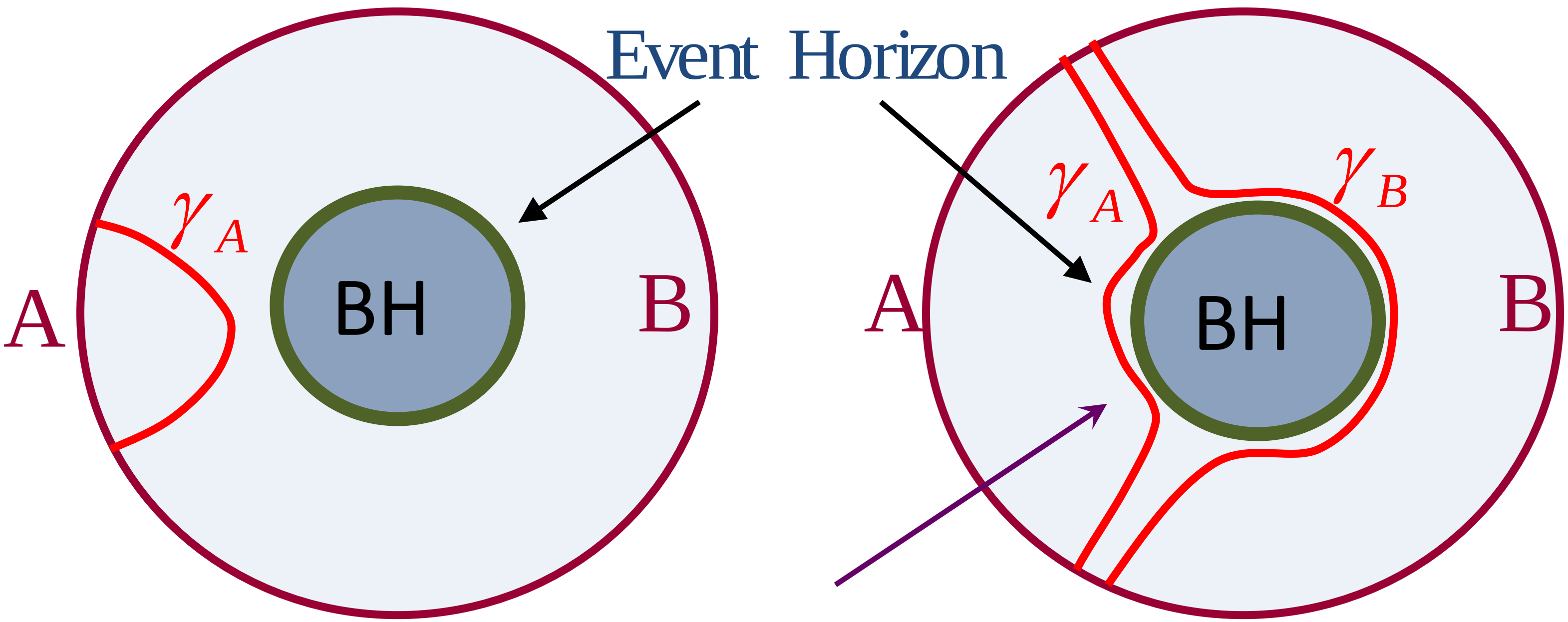
$$\text{where} \quad \phi \sim \phi + 2\pi, \qquad \frac{L}{\beta} = \frac{r_H}{R} >> 1.$$

$$\Rightarrow \quad S_A = \frac{c}{3}\log\left(\frac{\beta}{a}\sinh\left(\frac{\pi l}{\beta}\right)\right) .$$

agrees with the 2d CFT result.

Geometric Interpretation

(i) Small A (ii) Large A



When A is large (i.e. high temperature), γ_A wraps a part of horizon. This leads to the thermal contribution $S_A \approx (\pi/3) c\, l T$ to the entanglement entropy.

Note: $S_A \neq S_B$ due to the BH.

Disconnected Subsystem and Phase Transition

$$A = A_1 \cup A_2$$

[Headrick 10]

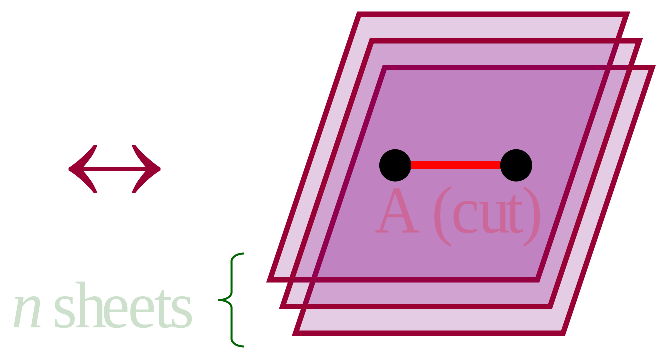
This is consistent with the CFT calculations done in [Calabrese-Cardy-Tonni 09] .

(5-3) Heuristic Understanding of HEE Formula

Let us try to derive the HEE from the bulk-boundary relation of AdS/CFT. ⇒ We employ the replica method.

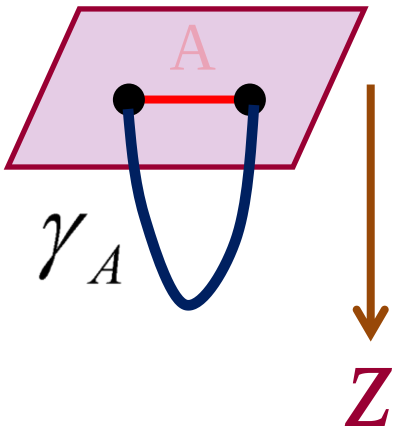
In the CFT side, the (negative) deficit angle $2\pi(1-n)$ is localized on ∂A:

$$\mathrm{Tr}_A[\rho_A^n] \quad \leftrightarrow$$



Assumption : The AdS dual is given by extending the deficit angle into the bulk AdS. [Fursaev 06]

$\Rightarrow$ The curvature is delta functionally localized on the deficit angle surface:



$$R = 4\pi(n-1)\cdot\delta(\gamma_A) + \ldots.$$

$$\Downarrow$$

$$S_{gravity} = \frac{1}{16\pi G_N}\int dx^{d+2}\sqrt{g}R + \ldots \;\rightarrow\; \frac{\mathrm{Area}(\gamma_A)}{4G_N}\cdot(n-1).$$

$$\Downarrow$$

$$S_A = -\frac{\partial}{\partial n}\log \mathrm{tr}_A \rho_A^n = -\frac{\partial}{\partial n}\log\left(\frac{Z_n}{(Z_1)^n}\right) = \frac{\mathrm{Area}(\gamma_A)}{4G_N}.$$

$$\delta S_{gravity} = 0 \;\rightarrow\; \gamma_A = \text{minimal surface !}$$

However, this argument is not completely correct because the assumption can easily fail. [Headrick 10]

⇒ Indeed, $\mathrm{tr}_A \rho_A^n$ does not agree with CFT results for n=2,3,.. due to back-reactions to make the geometry smooth.

HEE formula ⬄ The absence of backreaction in the `n→1 limit'

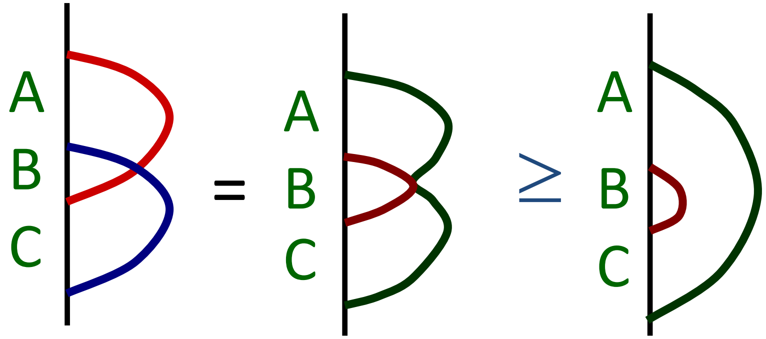
(not proven at present)

In particular, when ∂A = a round sphere, there is a direct proof of HEE formula by [Casini-Huerta-Myers 11].

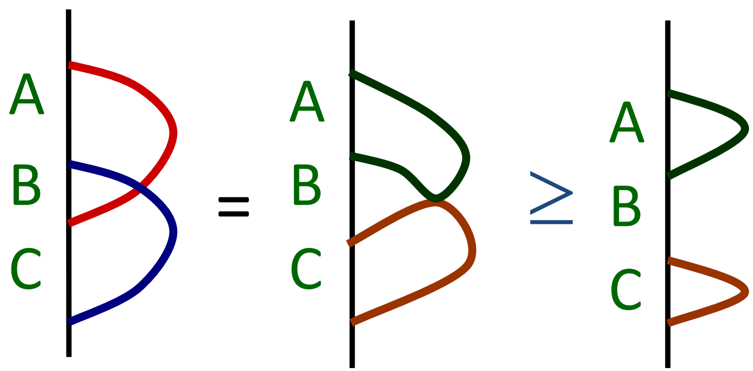
(5-4) Holographic Strong Subadditivity

The holographic proof of SSA inequality is very quick !

[Headrick-TT 07]



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A+B+C}+S_{B}$$



$$\Rightarrow S_{A+B}+S_{B+C}\geq S_{A}+S_{C}$$

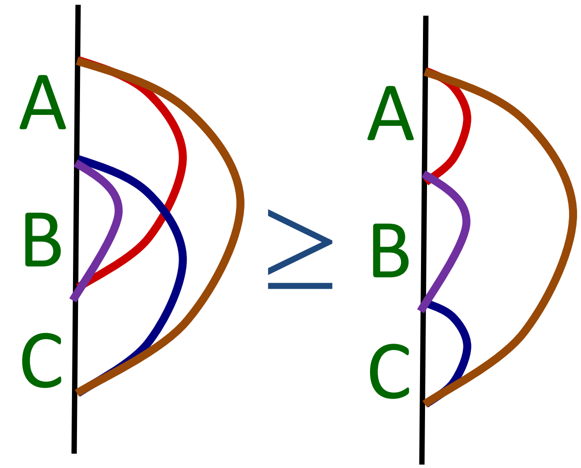
Note: This proof can be applied if $S_A = \underset{\gamma_A}{\mathrm{Min}}\left[F(\gamma_A)\right]$,
for any functional F.

$\Rightarrow$ higher derivative corrections

Tripartite Information [Hayden-Headrick-Maloney 11]

Recently, the holographic entanglement entropy is shown to have a special property called ***monogamy***.

$$S_{AB} + S_{BC} + S_{AC} \geq S_A + S_B + S_C + S_{ABC}$$
$$\Leftrightarrow I(A:B) + I(A:C) \leq I(A:BC)$$



Comments:

(i) HEE argues that *this is true for large N gauge theories*.

(ii) This property is not always true for QFTs.

(iii) This shows that HEE satisfies the *Cadney-Linden-Winter inequality*.

(iv) In 2+1 dim. gapped theories, this means that top. EE is non-negative.

(v) This property is also confirmed in time-dependent examples.

[Balasubramanian-Bernamonti-Copland-Craps-Galli 11, Allais-Tonni 11]

(5-5) Higher derivative corrections to HEE

Consider stringy corrections but ignore loop corrections in AdS.

(⇔deviations from strongly coupled limit, but still large N in CFT)

⇒ A precise formula was found for Lovelock gravities.

[Hung-Myers-Smolkin 11, de Boer-Kulaxizi-Parnachev 11]

Ex. Gauss-Bonnet Gravity

$$S_{GBG} = -\frac{1}{16G_N}\int dx^{d+2}\sqrt{g}\left[R-2\Lambda+\lambda R_{AdS}^2 L_{GB}\right]$$

$$L_{GB} \equiv R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

$$\Rightarrow \quad S_A = \underset{\gamma_A}{\mathrm{Min}}\left[\frac{1}{4G_N}\int_{\gamma_A} dx^d\sqrt{h}\,(1+2\lambda R_{AdS}^2 R)\right].$$

[But for general higher derivative theories, this is hard !]

⇒ However, HEE formula is not known in more general cases.

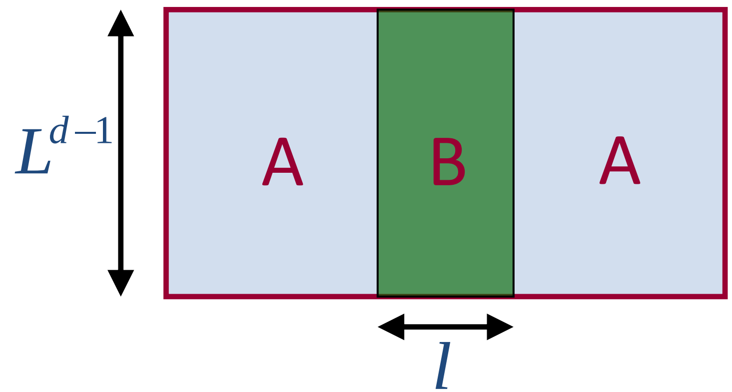
⑥ Aspects of HEE

(6-1) HEE in Higher dim.

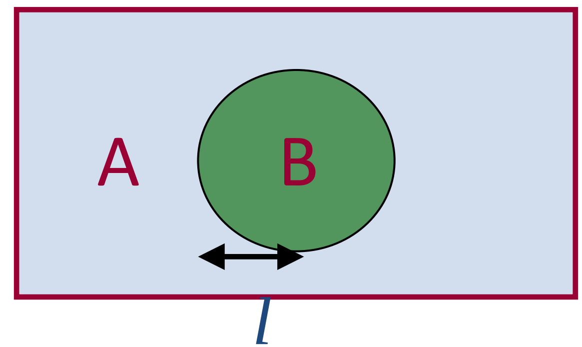
Consider the HEE in the Poincare metric dual to a CFT on $R^{1,d}$.

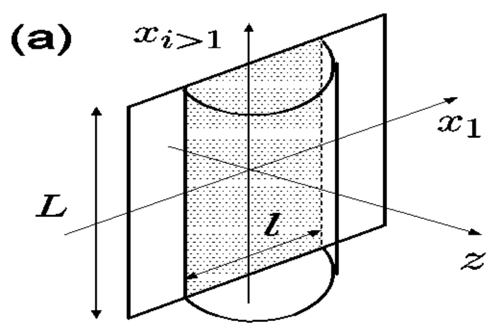
We concentrate on the following two examples.

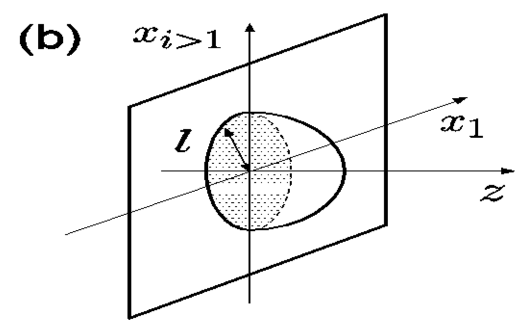
(a) Straight Belt



(b) Circular disk







Entanglement Entropy for (a) Infinite Strip from AdS

$$S_A = \frac{R^d}{2(d-1)G_N^{(d+2)}} \left[\underbrace{\left(\frac{L}{a}\right)^{d-1}} - C \cdot \underbrace{\left(\frac{L}{l}\right)^{d-1}} \right],$$

where $C = 2^{d-1}\pi^{d/2} \left(\Gamma\left(\frac{d+1}{2d}\right) \Big/ \Gamma\left(\frac{1}{2d}\right) \right)^d .$

Area law divergence

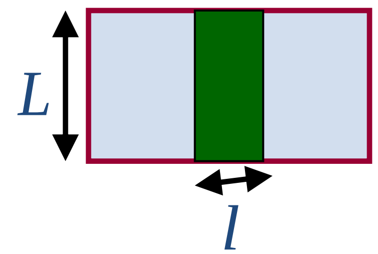
This term is finite and does not depend on the UV cutoff.

d=1 (i.e. AdS3) case:

$$S_A = \frac{R}{2G_N^{(3)}} \log \frac{l}{a} = \frac{c}{3} \log \frac{l}{a}.$$

Agrees with 2d CFT results
[Holzhey-Larsen-Wilczek 94 ;
 Calabrese-Cardy 04]

Basic Example of AdS5/CFT4



$$\mathrm{AdS}_5 \times \mathrm{S}^5 \Leftrightarrow N = 4\ \mathrm{SU(N)\ SYM}$$

$$CFT: \quad S_A^{freeCFT} = K \cdot \frac{N^2 L^2}{a^2} - 0.087 \cdot \frac{N^2 L^2}{l^2}.$$

$$Gravity: \quad S_A^{AdS} = K' \cdot \frac{N^2 L^2}{a^2} - 0.051 \cdot \frac{N^2 L^2}{l^2}.$$

The order one deviation is expected since the AdS result corresponds to the strongly coupled Yang-Mills.

[cf. 4/3 in thermal entropy, Gubser-Klebanov-Peet 96]

Entanglement Entropy for (b) Circular Disk from AdS

[Ryu-TT 06]

$$S_A = \frac{\pi^{d/2} R^d}{2G_N^{(d+2)}\Gamma(d/2)} \left[\underline{p_1 \left(\frac{l}{a}\right)^{d-1}} + p_3 \left(\frac{l}{a}\right)^{d-3} + \cdots \right.$$

$$\left. \cdots + \begin{cases} p_{d-1}\left(\frac{l}{a}\right) + \underline{p_d} & (\text{if } d = \text{even}) \\ p_{d-2}\left(\frac{l}{a}\right)^2 + \underline{q \log\left(\frac{l}{a}\right)} & (\text{if } d = \text{odd}) \end{cases} \right],$$

Area law divergence

$$\text{where} \quad p_1 = (d-1)^{-1},\ p_3 = -(d-2)/[2(d-3)], \ldots.$$

$$\ldots,\quad q = (-1)^{(d-1)/2}(d-2)!!/(d-1)!! \quad .$$

A universal quantity which characterizes odd dim. CFT

⇒ Satisfy 'C-theorem'

[Myers-Sinha 10; closely related to F-theorem Jafferis-Klebanov-Pufu-Safdi 11]

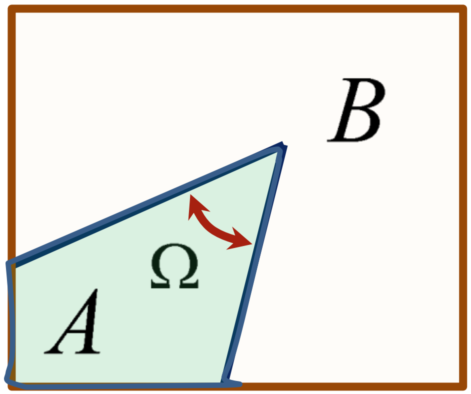
Conformal Anomaly (central charge)

2d CFT c/3・log(l/a)

4d CFT -4a・log(l/a)

[Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

HEE with a Cusp in 2+1 dim CFTs

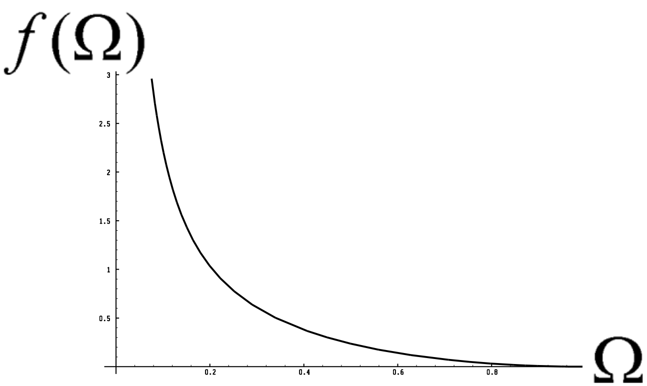


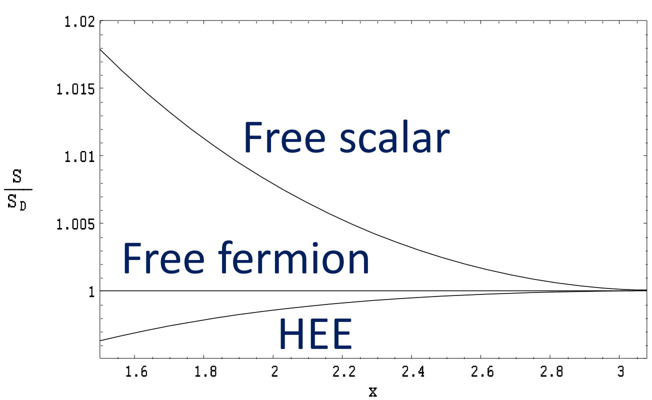
$$S_A = \gamma \cdot \frac{\partial A}{a} + f(\Omega)\log a + (\text{finite}) \ .$$

$$S_A = S_B \ \Rightarrow \ f(2\pi - \Omega) = f(\Omega),$$

$$\text{SSA} \ \Rightarrow \ f''(\Omega) \geq 0.$$

[Casini-Huerta 06,08, Hirata-TT 06]





AdS/CFT result :

$$f(\Omega) = \frac{R^2}{2G_N}\int_0^\infty dz \left[1 - \sqrt{\frac{z^2+\beta^2+1}{z^2+2\beta^2+1}}\right].$$

$$\Omega = \int_0^\infty dz \frac{2\beta\sqrt{1+\beta^2}}{(z^2+\beta^2)\sqrt{(z^2+\beta^2+1)(z^2+2\beta^2+1)}}.$$

[Hirata-TT 06]

- In spite of a heuristic argument [Fursaev, 06] , there has been no complete proof. However, there have been many evidences and no counter examples so far.

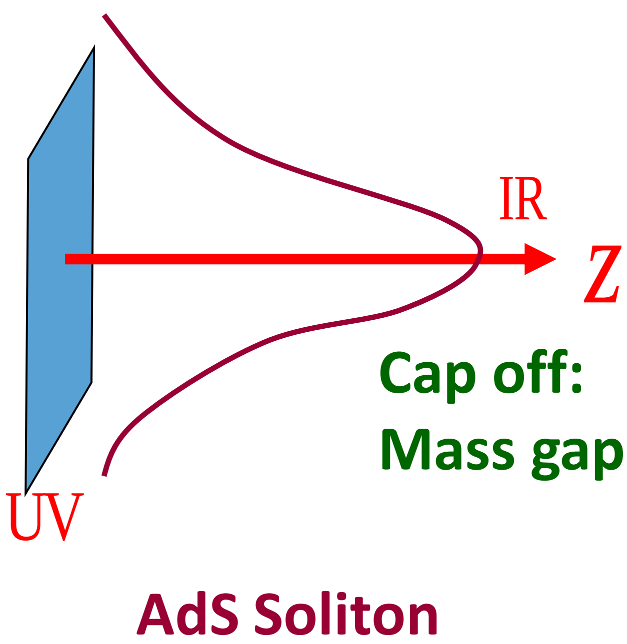
[A Partial List of Evidences]

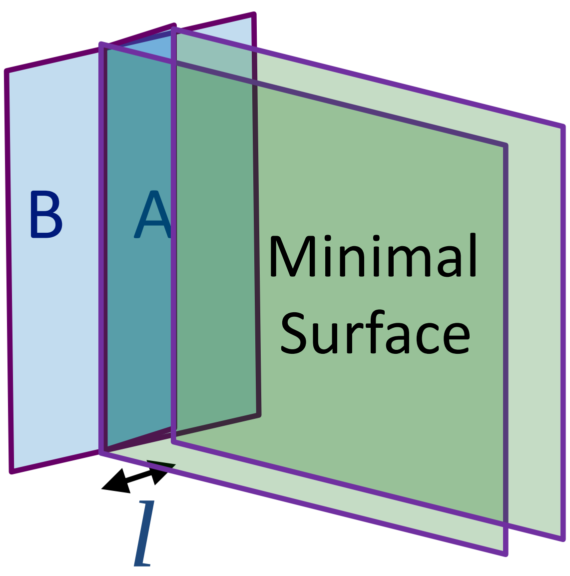
- Area law follows straightforwardly [Ryu-TT 06]
- Agreements with analytical 2d CFT results for AdS3 [Ryu-TT 06]
- Holographic proof of strong subadditivity [Headrick-TT 07]
- Consistency of 2d CFT results for disconnected subsystems [Calabrese-Cardy-Tonni 09] with our holographic formula [Headrick 10]
- Agreement on the coefficient of log term in 4d CFT (~a+c) [Ryu-TT 06, Solodukhin 08,10, Lohmayer-Neuberger-Schwimmer-Theisen 09, Dowker 10, Casini-Huerta, 10, Myers-Sinha 10, Casini-Hueta-Myers 11]

(6-2) Confinement/deconfinement Phase Transitions

Here we study a confinement/deconfinement phase transition to see if the HEE can be an order parameter. One of the simplest gravity duals of confining gauge theories is the AdS soliton.

The AdS5 soliton ⇔ (2+1) dim. pure SU(N) gauge theory.



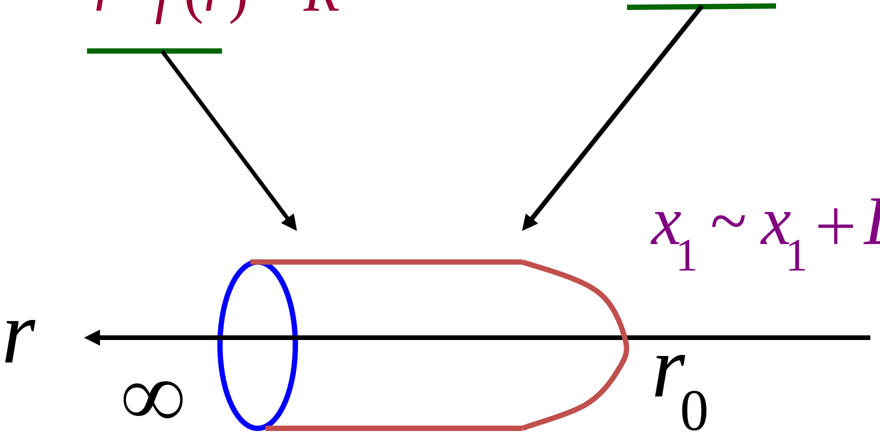


The metric of AdS soliton is given by the double Wick rotation of the AdS black hole solution.

$$ds^2_{\text{AdS BH}} = \frac{R^2 dr^2}{r^2 f(r)} + \frac{r^2}{R^2}\left(-f(r)dt^2 + dx_1^2 + dx_2^2 + dx_3^2\right),$$

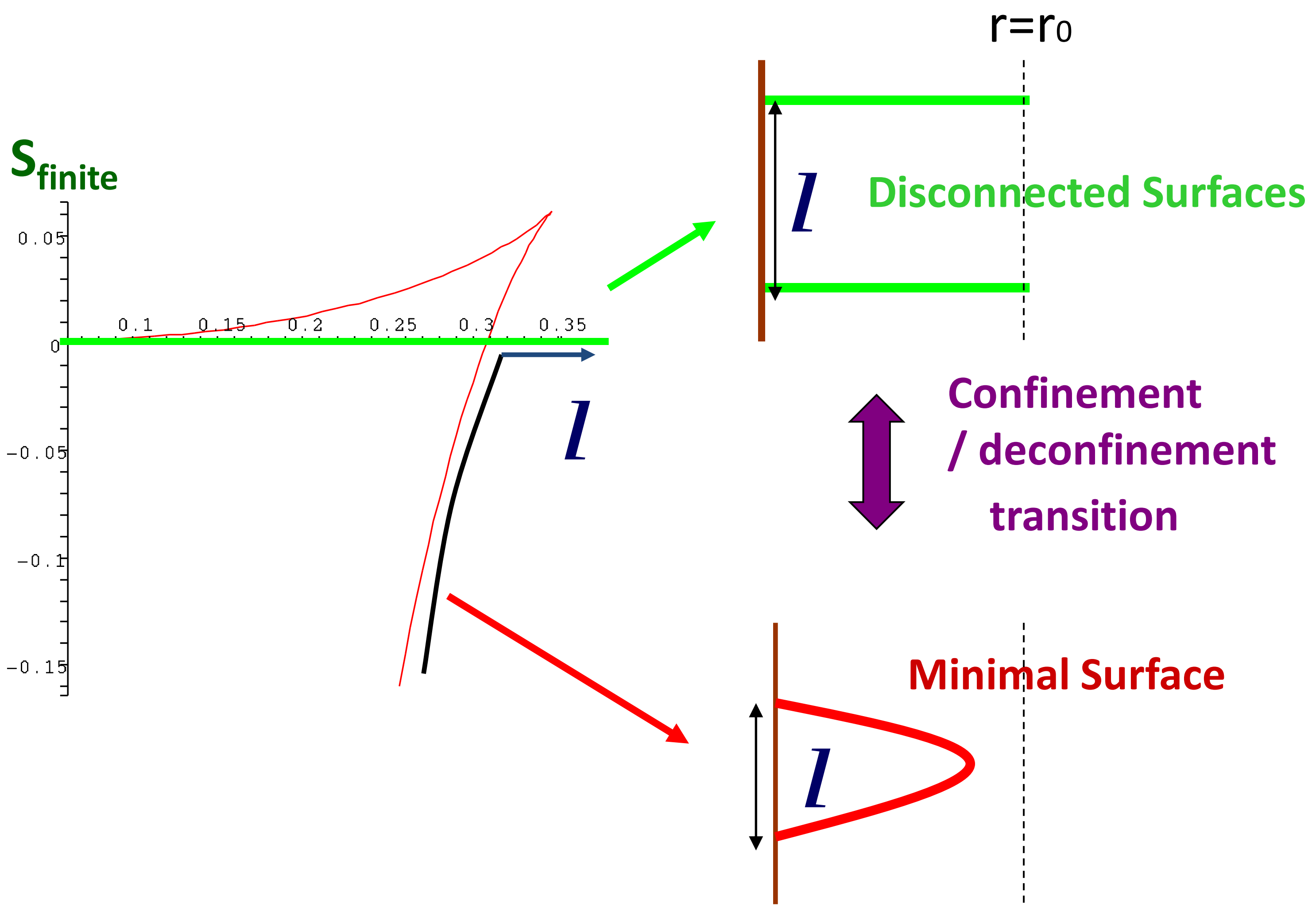
$$f(r) \equiv 1 - \frac{r_0^4}{r^4},$$

$$ds^2_{\text{AdS Soliton}} = \frac{R^2 dr^2}{r^2 f(r)} + \frac{r^2}{R^2}\left(-dt^2 + f(r)dx_1^2 + dx_2^2 + dx_3^2\right),$$



In the holographic calculation, two different surfaces compete and this leads to the phase transition.

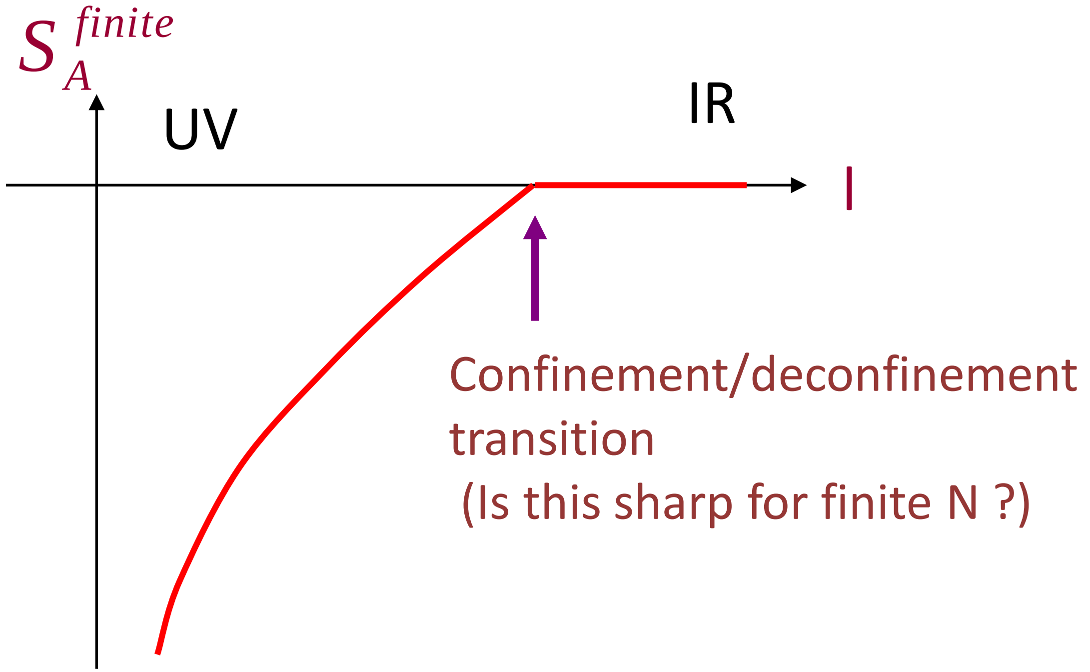
[Nishioka-TT 06', Klebanov-Kutasov-Murugan 07']



In summary, we find the following behavior

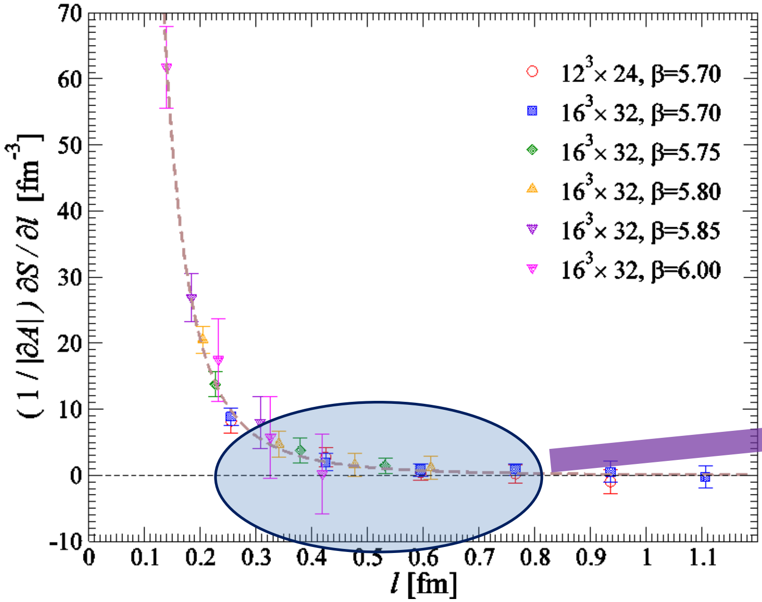
$$S_A^{finite} \approx -N^2 \cdot \frac{L^2}{l^2} \qquad (l \to 0: \text{ Asymptotic Free}),$$

$$S_A^{finite} \approx const. \qquad (l \to \infty : \text{Confined})$$

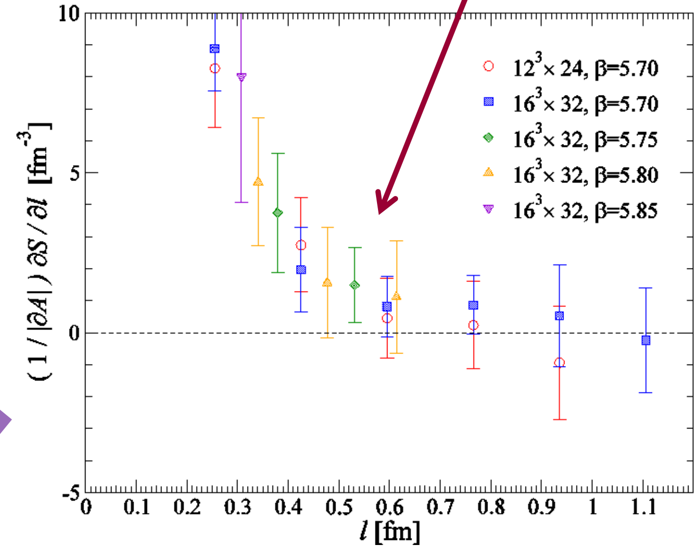


Lattice Results for 4D Pure YM

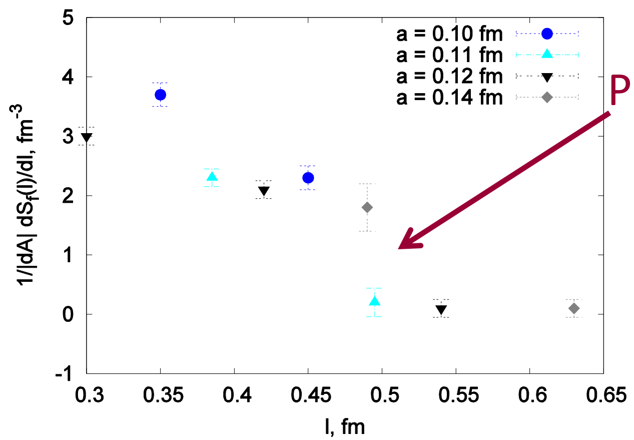
[4d SU(3): Nakagawa-Nakamura-Motoki-Zakharov 0911.2596]



Phase Transition



[4d SU(2): Buividovich-Polikarpov 0802.4247]



Phase Transition

[See for other calculations of EE in lattice gauge theory: Velytsky 0801.4111, 0809.4502; Buividovich-Polikarpov 0806.3376, 0811.3824]

Twisted AdS Soliton

Next we consider the twisted AdS Soliton
dual to the N=4 4D Yang-Mills with twisted boundary conditions. In general, supersymmetries are broken.

$$\Rightarrow \quad \text{Twisted Circle}: \ (z, x_1) \sim (z \cdot e^{2\pi i \varsigma}, x_1 + \mathrm{L})$$

$$\text{Scherk Schwarz}: \ \varsigma = 0, \qquad \text{pure AdS}: \ \varsigma = 1 \ .$$

The dual metric can be obtained from the double Wick rotation of the rotating 3-brane solution.

The metric of the twisted AdS Soliton

$$ds^2 = \frac{1}{\sqrt{f}}(-dt^2 + h\ d\chi^2 + dx_1^2 + dx_2^2) + \sqrt{f}\Big[\frac{dr^2}{\tilde{h}} - \frac{2lr_0^4 \cosh\alpha}{r^4 \Delta f} \sin^2\theta d\chi d\phi \\ + r^2(\Delta d\theta^2 + \tilde{\Delta}\sin^2\theta d\phi^2 + \cos^2\theta d\Omega_3^2)\Big], \tag{1}$$

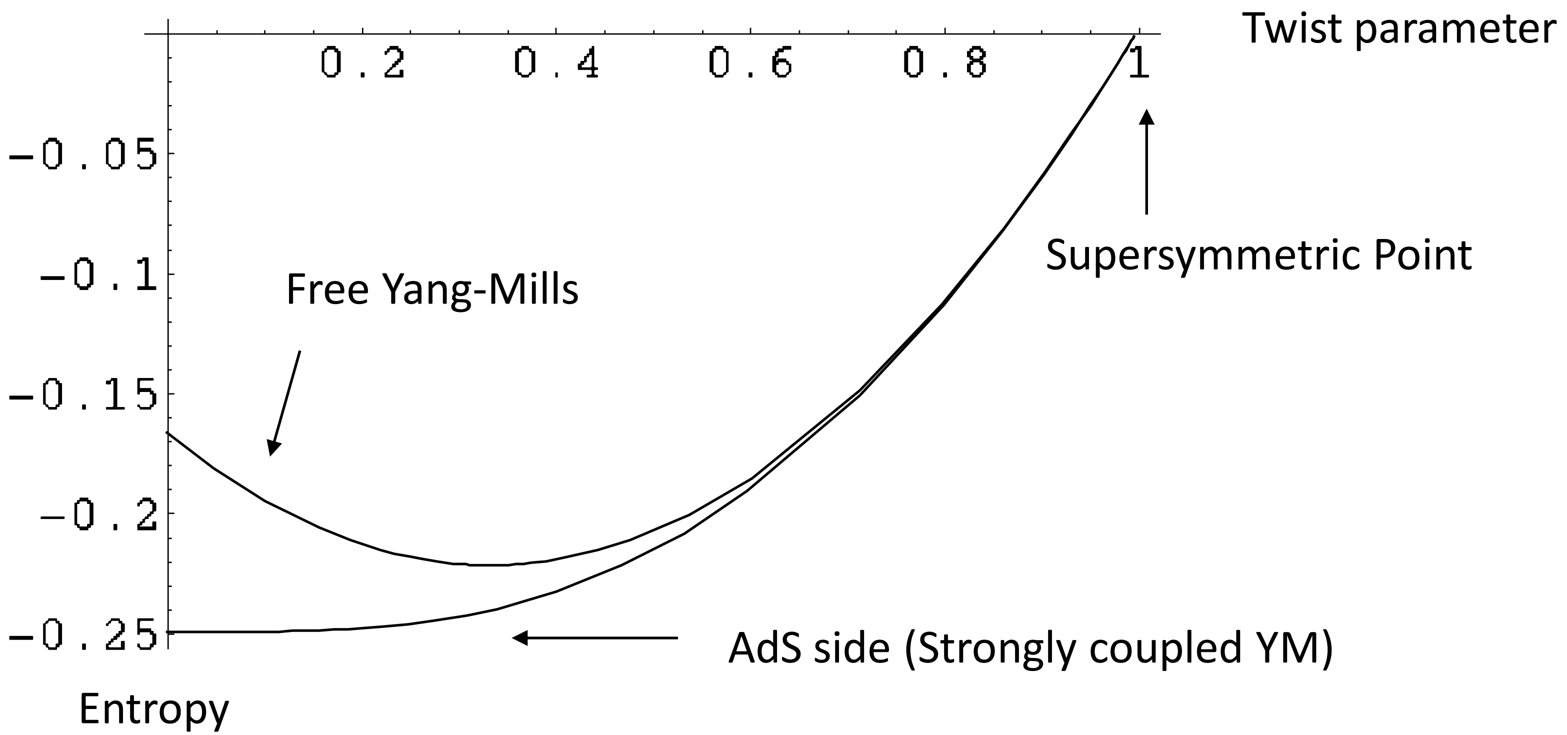
where f, h, $\tilde{h}$, Δ and $\tilde{\Delta}$ are defined as follows

$$f = 1 + \frac{r_0^4 \sinh^2\alpha}{r^4 \Delta}, \quad \Delta = 1 - \frac{l^2\cos^2\theta}{r^2}, \quad \tilde{\Delta} = 1 - \frac{l^2}{r^2} - \frac{r_0^4 l^2 \sin^2\theta}{r^6 \Delta f}, \\ h = 1 - \frac{r_0^4}{r^4 \Delta}, \quad \tilde{h} = \frac{1 - \frac{l^2}{r^2} - \frac{r_0^4}{r^4}}{\Delta}. \tag{2}$$

The parameter l before the double Wick rotation is proportional to the angular momentum of the black brane solution. The allowed lowest value r_H of r is given by the solution to $\tilde{h}(r) = 0$

$$r_H^2 = \frac{l^2}{2} + \sqrt{r_0^4 + \frac{l^4}{4}} \quad (> l^2). \tag{3}$$

The entanglement entropies computed in the free Yang-Mills and the AdS gravity agree nicely!



This is another evidence for our holographic formula.

⑦ HEE and Thermalization

(7-1) Time Evolution of HEE

Consider the following time-dependent setup of AdS/CFT:

Black hole formation in AdS ⇔ Thermalization in CFT

Explicit examples:

GR analysis in AdS: Chesler-Yaffe 08, Bhattacharyya-Minwalla 09,…

Probe D-brane (apparent BH on D-branes): Das-Nishioka-TT 10,…

Note: The thermalization under a sudden change of Hamiltonian is called **quantum quench** and has been intensively studied in condensed matter physics. [Calabrese-Cardy 05-10]

An Entropy Puzzle

(i) Von-Neumann entropy remains vanishing under a unitary evolutions of a pure state.

$$\rho_{tot}(t) = U(t,t_0)\,|\Psi_0\rangle\langle\Psi_0|\,U(t,t_0)^{-1}$$

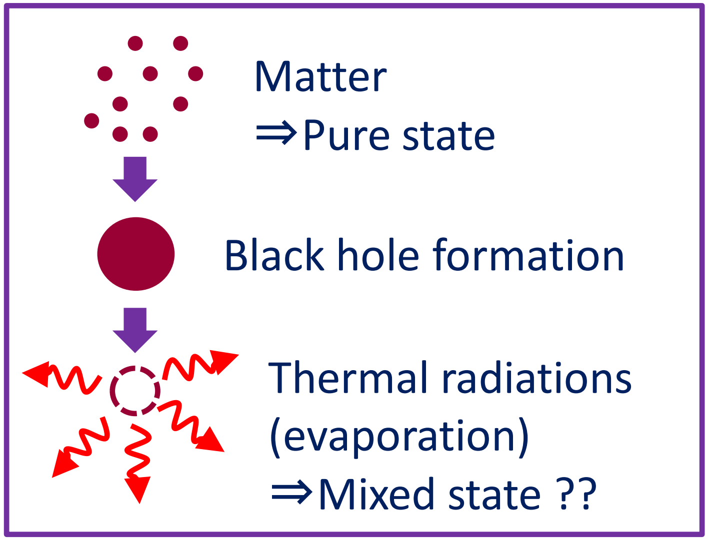
$$\Rightarrow \quad S(t) = -Tr\,\rho_{tot}(t)\,\log\rho_{tot}(t) = S(t_0).$$

(ii) In the gravity dual, its holographic dual inevitably includes a black hole at late time and thus the entropy looks non-vanishing !

➡ Clearly, (i) and (ii) contradicts !

cf. the black hole information paradox

$\Rightarrow$ we need to include quantum corrections.



Resolution of the Puzzle via Entanglement Entropy

[Arrastia-Aparicio-Lopez 10, Ugajin-TT 10]

Upshot: The non-vanishing entropy appears only after coarse-graining. The von-Neumann entropy itself is vanishing even in the presence of black holes in AdS.

First, notice that the (thermal) entropy for the total system can be found from the entanglement entropy via the formula

$$S_{tot} = \lim_{|B|\to 0}(S_A - S_B).$$

This is indeed vanishing if we assume the pure state relation

$$S_A = S_B.$$

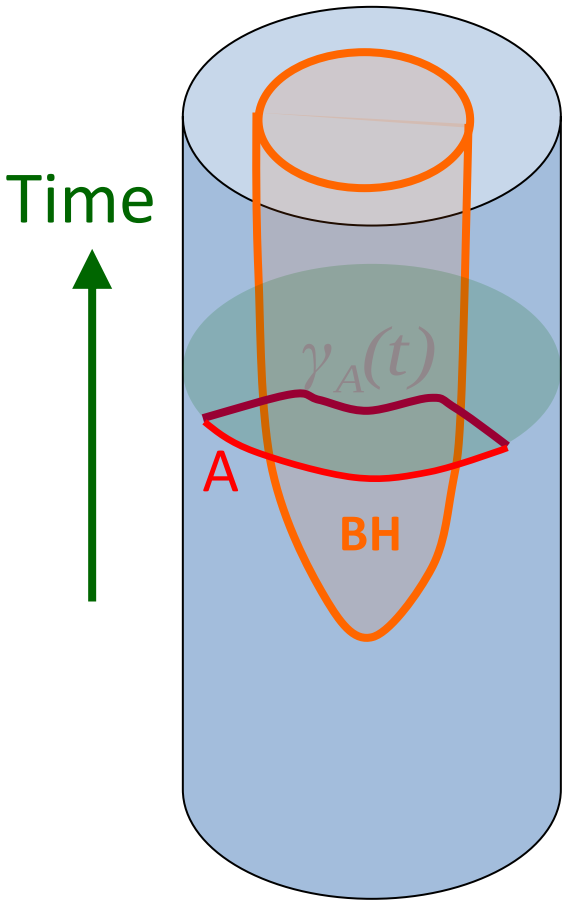
Indeed, we can holographically show this as follows:

$$S_A(t) = \min\left[\frac{\mathrm{Area}(\gamma_A(t))}{4G_N}\right],$$

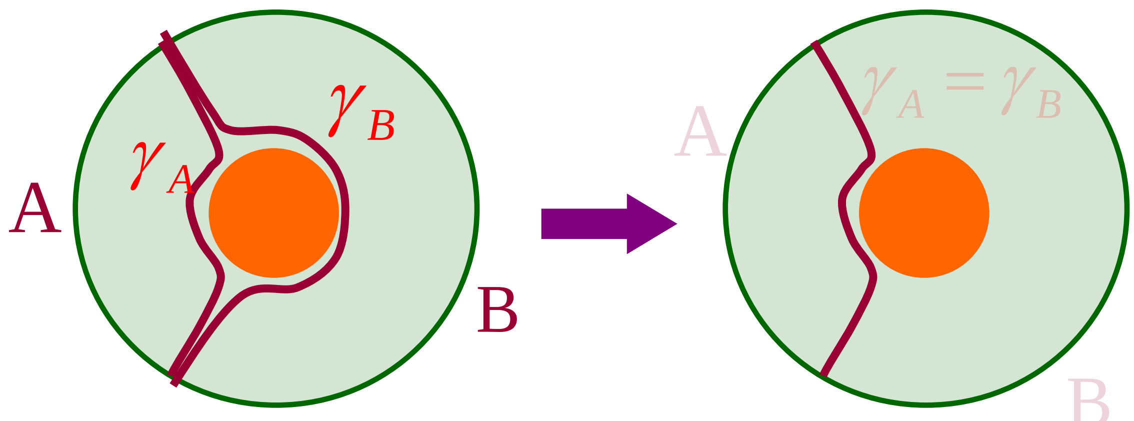
$$\gamma_A(t) = \text{extremal surfaces homotopic to } A(t)$$

$$\text{such that } \partial\gamma_A(t) = \partial A(t).$$

[Hubeny-Rangamani-TT 07]



Black hole formation in global AdS_{d+2}



Continuous deformation

Therefore, if the initial state does not include BHs, then always we have $S_A=S_B$ and thus $S_{tot}=0$.

⇒ In such a pure state system, the total entropy is not useful to detect the BH formation.

⇒ Instead, the entanglement entropy S_A can be used to probe the BH formation as it is a coarse-grained entropy.

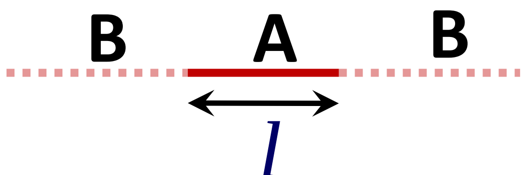
Note: In time-dependent black holes, the definition of BH entropy is not unique.

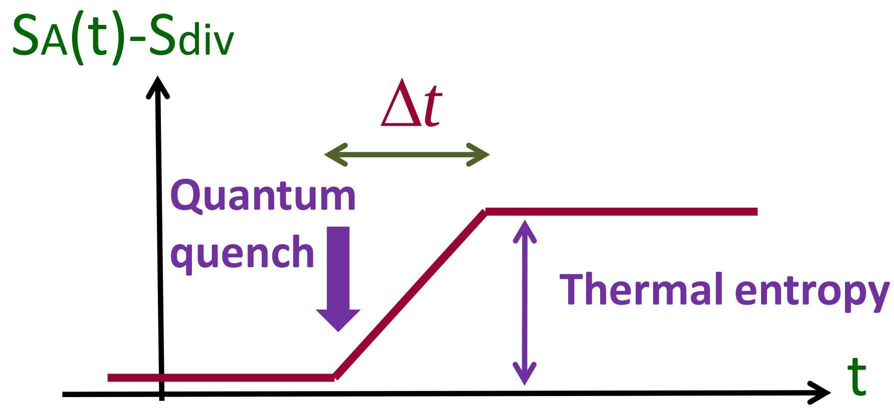
⇒ We need to specify how coarse-grain the system. HEE offers us one convenient example of this.

Time Evolutions of HEE under Quantum Quenches

In 1+1 dim. CFTs, we expect a linear growth of EE after a quantum quench.

[Calabrese-Cardy 05]





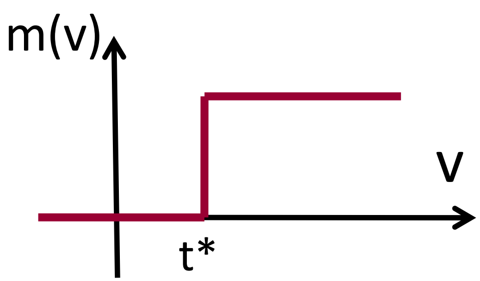
Causality $\rightarrow$ $\Delta t \approx \dfrac{l}{2}$

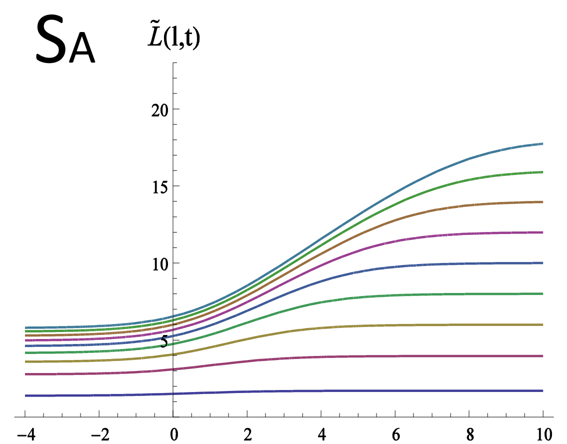
HEE reproduced the same result.

[Arrastia-Aparicio-Lopez 10]

Vaidya BH

$$ds^2 = -(r^2 - m(v))dv^2 + 2drdv + r^2 dx^2$$

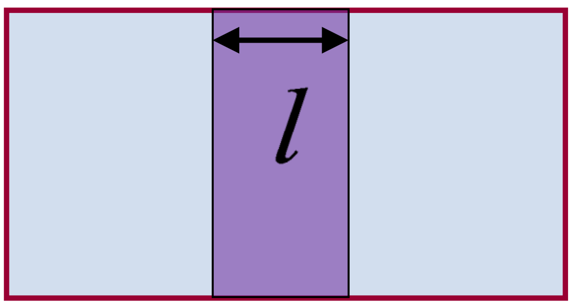


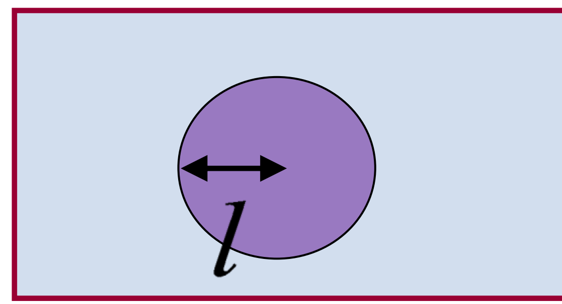


The time evolution of HEE in higher dim. have been conducted recently. [Albash-Johnson 10, Balasubramanian-Bernamonti-de Boer-Copland-Craps- Keski-Vakkuri-Müller-Schäfer-Shigemori-Staessens 10, 11, ...]

$\Rightarrow$ In higher dim., Δt depends on the shape of A.

HEE predicts: A = strip → $\Delta t > \dfrac{l}{2}$, A = round disk → $\Delta t \approx \dfrac{l}{2}$





(7-2) An Solvable Example in 2D CFT: Free Dirac Fermion

As an explicit example in CFT side, we would like to study quantum quench in the 2D free Dirac fermion. In this case, we can calculate the time evolution of EE with the finite size effect.

AdS/CFT: free CFT $\Longleftrightarrow$ quantum gravity
with a lot of quantum corrections !

Assuming that the initial wave function $|\Psi_0\rangle$ flows into a boundary fixed point as argued in [Calabrese-Cardy 05], we can identify

$$|\Psi_0\rangle = e^{-\varepsilon H}|B\rangle \quad ,$$

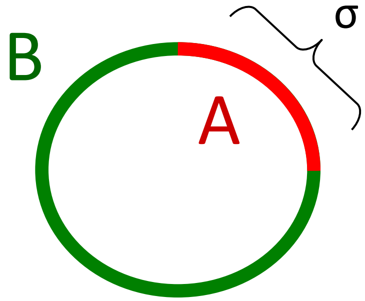
where $|B\rangle$ is the boundary state. The constant ε is a regularization paramter and measures the strength of the quantum quench:

$$\Delta m \sim \varepsilon^{-1}$$

The final result of entanglement entropy is given by

$$S_A(t,\sigma)=\frac{1}{3}\log\frac{2\varepsilon}{\pi a}+\frac{1}{6}\log\frac{\left|\theta_1\left(\frac{i\sigma}{4\varepsilon}\middle|\frac{\pi i}{2\varepsilon}\right)\right|^2\cdot\left|\theta_1\left(\frac{\varepsilon+it}{2\varepsilon}\middle|\frac{\pi i}{2\varepsilon}\right)\right|^2}{\eta\left(\frac{\pi i}{2\varepsilon}\right)^6\cdot\left|\theta_1\left(\frac{2\varepsilon+2it+i\sigma}{4\varepsilon}\middle|\frac{\pi i}{2\varepsilon}\right)\right|\cdot\left|\theta_1\left(\frac{2\varepsilon+2it-i\sigma}{4\varepsilon}\middle|\frac{\pi i}{2\varepsilon}\right)\right|},$$

where $a=\mathrm{UV\ cut\ off}$ and $0\leq\sigma<2\pi$.



This satisfies

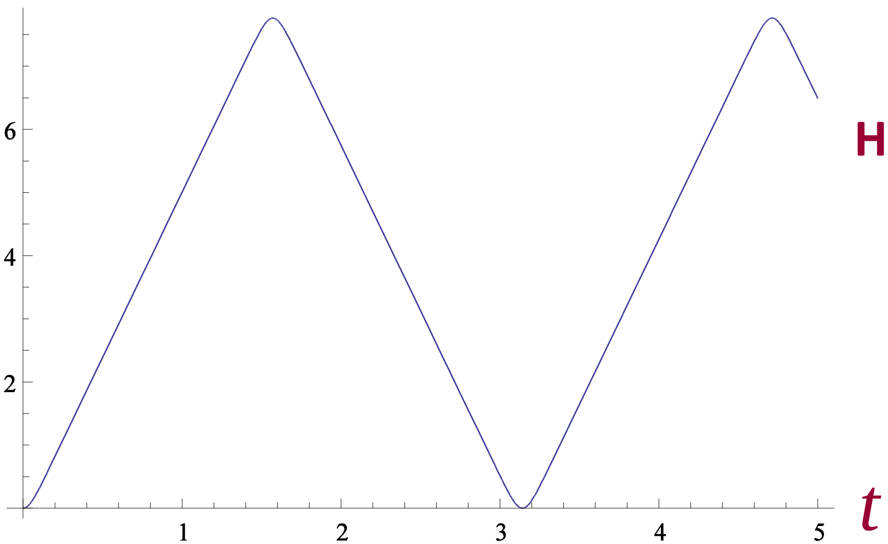
$$S_A(t,\sigma)=S_A(t,2\pi-\sigma)\equiv S_B(t,\sigma). \quad \Rightarrow \quad \text{Pure State}$$

$$S_A(t+\pi,\sigma)=S_A(t,\sigma) \quad \Rightarrow \text{Recurrence special to the free field theory}$$

(much shorter th an the Poincare recurrence)

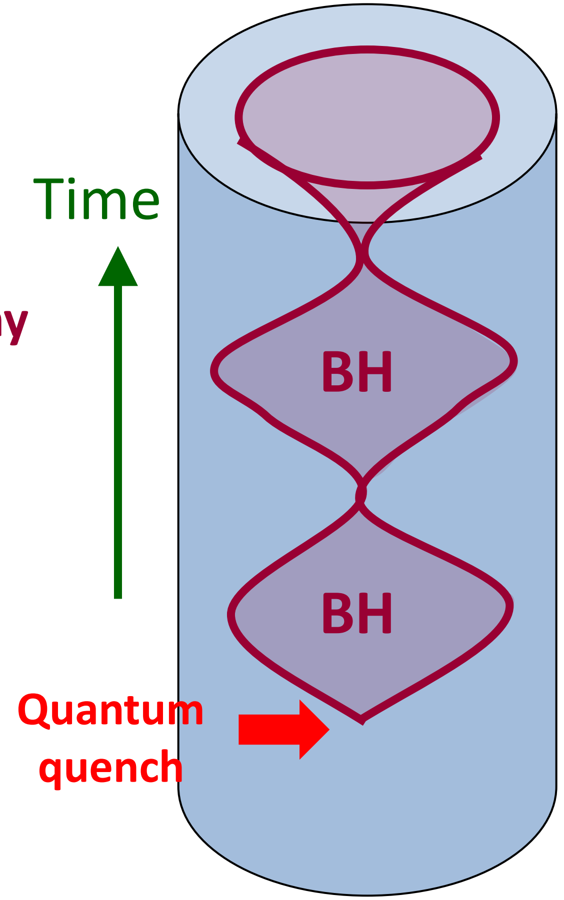
Time evolution of entanglement entropy

$S_A(t,\pi)_{\varepsilon=0.2} - S_{div}$



Quantum quench in free CFT

Holography



BH formation and evaporation in extremely quantum gravity

⑧ Fermi Surfaces and HEE

[Ogawa-Ugajin-TT, 11]

(8-1) Logarithmic Violation of Area Law

In d dim. lattice models that the area law of EE is violated logarithmically in free fermion theories. [Wolf 05, Gioev-Klich 05]

$$S_A \sim L^{d-1} \log L, \qquad (L = size\ of\ A).$$

Comments:

(i) This property can be understood from the logarithmic EE in 2D CFT, which approximates the radial excitations of fermi surface.

(ii) It is natural to expect that this property is true for non-Fermi liquids. [Swingle 09,10, Zhang-Grover-Vishwanath 11 etc.]

Note in this lattice calculation assumes

$$\varepsilon^{-1} \quad \text{(UV cut off)} \ \sim \ k_F \ .$$

Instead, in our holographic context which corresponds to a continuous limit, we are interested in the case $\varepsilon^{-1} >> k_F$.

In this case, we expect

$$S_A = (div.) + \eta \cdot (L \cdot k_F)^d \log Lk_F + \cdots .$$

Below we would like to see if we can realize this behavior in HEE.

We assume that all physical quantities can be calculable in the classical gravity limit (i.e. $\exists O(N^2)$ ***Fermi surfaces).***

(8-2) Holographic Construction

The metric ansatz: $$ds^2 = \frac{R_{AdS}^2}{z^2}\left(-f(z)dt^2 + g(z)dz^2 + dx^2 + dy^2\right).$$

$$\text{Asymp. AdS} \Rightarrow f(0) = g(0) = 1.$$

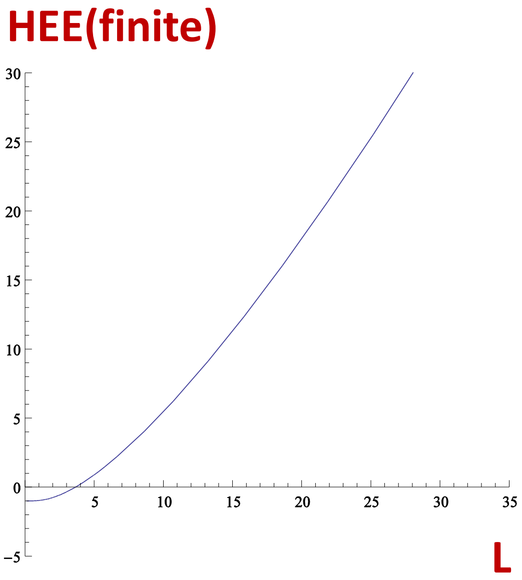
(Below we work d=2 i.e. AdS4/CFT3 setup.)

The logarithmical behavior of EE occurs iff

$$g(z) \to \left(\frac{z}{z_F}\right)^2 \qquad (z \to \infty).$$

Note: f(z) does not affect the HEE.

z_F^{-1} is dual to the fermi energy.



(8-3) Null Energy Condition

To have a sensible holographic dual, a necessary condition is known as the null energy condition:

$$T_{\mu\nu}N^{\mu}N^{\nu} \geq 0 \qquad \text{for any null vector } N^{\mu}.$$

In the IR region, the null energy condition argues

$$g(z) \propto z^{2}, \qquad f(z) \propto z^{-2m} \quad \Rightarrow \quad m \geq 2.$$

The specific heat behaves like

$$C \propto T^{\alpha} \quad \text{with} \quad \alpha \leq \frac{2}{3}.$$

Notice that this excludes standard Landau fermi liquids.

[Ogawa-Ugajin-TT, see also Huijse-Sachdev-Swingle 11, Shaghoulian 11]

In summary, we find that ***classical gravity duals only allow non-fermi liquids***.

Comments:

(i) Our definition of classical gravity duals is so restrictive that it does not include either the emergent AdS2 geometry [Faulkner-Liu-McGreevy-Vegh 09, Cubrovic-Zaanen-Schalm 09] nor the electron stars (or Lifshitz) [Hartnoll-Polchinski-Silverstein-Tong 09, Hartnoll-Tavanfar 10] .

(ii) More generally, the background with $g(z) \propto z^{2n}$ leads to

$$S_A^{finite} \propto L^{\frac{2n}{n+1}} \Rightarrow \text{In general, the area law is violated !}$$

(iii) We can embed this background in an effective gravity theory:

$$S_{EMS} = \frac{1}{16G_N}\int dx^{d+2}\sqrt{-g}\left[R - 2\Lambda - W(\phi)F_{\mu\nu}F^{\mu\nu} - \partial_\mu\phi\partial^\mu\phi - V(\phi)\right].$$

if W and V behave in the large ϕ limit as follows

$$V(\phi) + 2\Lambda \approx -\frac{(p^2 + 12p + 32)}{4R_{AdS}^2}\cdot e^{-\sqrt{\frac{2}{p-2}}\phi},$$

$$W(\phi) \approx \frac{8A^2}{z_F^2\, p(8+p)R^2}\, e^{3\sqrt{\frac{2}{(p-2)}}\phi},$$

$$\Rightarrow f(z) \propto z^{-p}, \quad g(z) \propto z^2, \qquad (p > 2).$$

Later, it has been pointed out that, such a background is understood as the violation of hyperscaling

$\Rightarrow$ A generalization of Lifshitz spacetime

[Huijse-Sachdev-Swingle 11, Dong-Harrison-Kachru-Torroba-Wang 12]

$$ds^2_{(d+2)} = r^{-(d-\theta)}\left(-r^{-2(z-1)}dt^2 + dr^2 + \sum_{i=1}^{d} dx_i^2\right) \quad .$$

$$\Rightarrow C \propto S \propto T^{(d-\theta)/z}.$$

$d-1<\theta<d: \quad S_A \sim \mathrm{L}^{\alpha}, \quad d\text{-}1<\alpha<d \quad \rightarrow$ Violation of Area law

$\theta = d\text{-}1 \quad : \quad S_A \sim (L)^{d\text{-}1}\log L$ Fermi surface

$0<\theta<d\text{-}1 \quad : \quad S_A \sim \mathrm{L}^{\alpha}, \quad 0<\alpha<d-1$

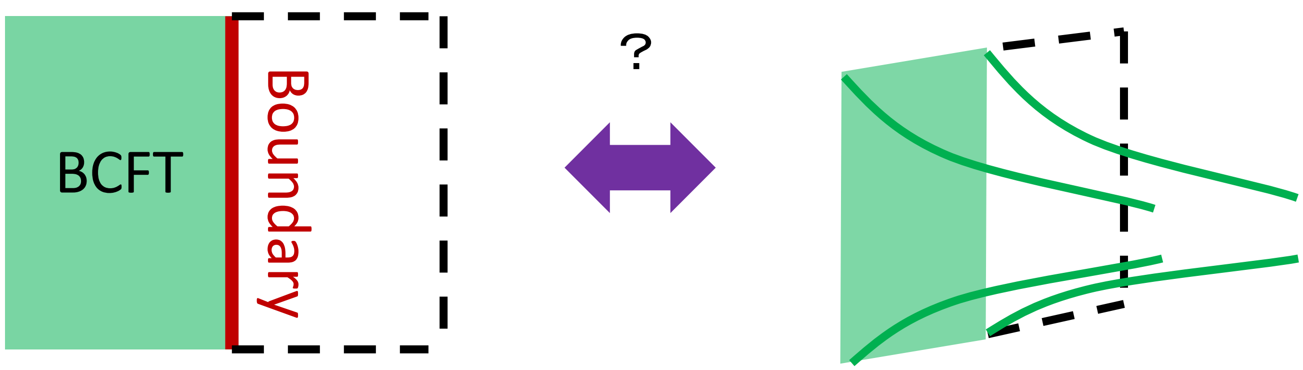
⑨ HEE and BCFT

(9-1) AdS/BCFT

What is a holographic dual of CFT on a manifold with Boundary (BCFT) ?

$$\mathrm{CFT}_d:\ SO(d,2) \quad \Leftrightarrow \quad AdS_{d+1}$$

$$\mathrm{BCFT}_d:\ SO(d-1,2) \quad \Leftrightarrow \quad AdS_d$$



[Earlier studies: Karch-Randall 00 (BCFT,DCFT),…
Bak-Gutperle-Hirano 03, Clark-Freedman-Karch-Schnabl 04 (Janus CFT)
Sugra Sol. D'Hoker-Estes-Gutperle 07,
Aharony-Berdichevsky-Berkooz-Shamir 11]

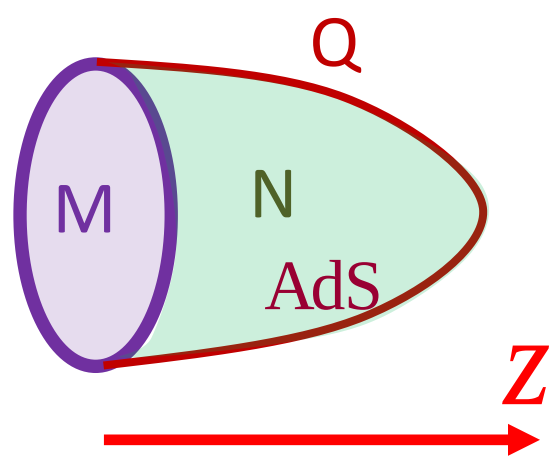
AdS/BCFT Proposal [Fujita-Tonni-TT 11]

In addition to the standard AdS boundary M,
we include an extra boundary Q, such that ∂Q=∂M.

$$I_E = -\frac{1}{16\pi G_N}\int_N \sqrt{g}(R-2\Lambda-L_{matter}) - \frac{1}{8\pi G_N}\int_Q \sqrt{h}(K-L^Q_{matter}).$$

EOM at boundary leads to
the Neumann b.c. on Q :

$$K_{ab} - Kh_{ab} = 8\pi G_N T^Q_{ab} \quad .$$



Conformal inv. ⇒ $T^Q_{ab} = -Th_{ab}.$

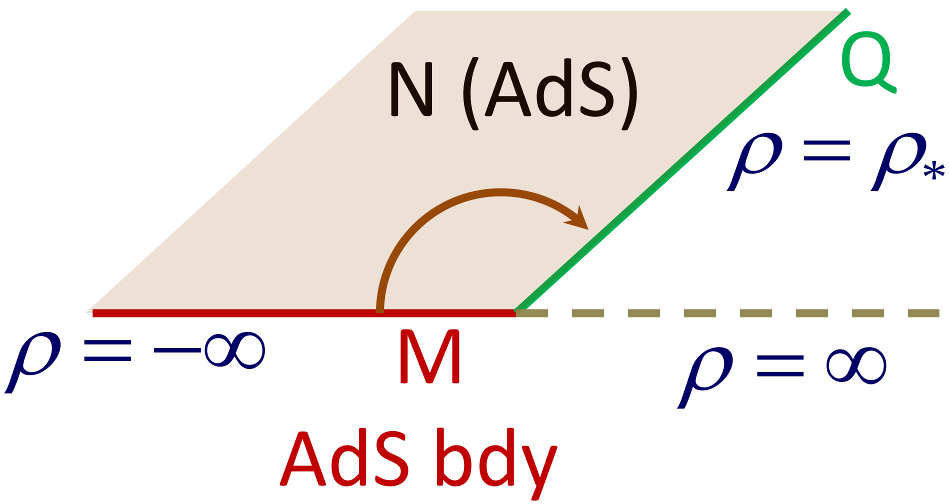
(9-2) Simplest Example

Consider the AdS slice metric:

$$ds^2_{AdS(d+1)} = d\rho^2 + \cosh^2(\rho/R)ds^2_{AdS(d)} \quad .$$

Restricting the values of ρ to $-\infty < \rho < \rho_*$ solves the boundary condition with

$$T = \frac{d-1}{R}\tanh\frac{\rho_*}{R} \quad .$$



(9-3) Holographic Boundary Entropy

The boundary entropy [Affleck-Ludwig 91]

Sbdy measures the degrees of freedom at the boundary.

The g-theorem:

Sbdy monotonically decreases under the RG flow in CFT.

[proved by Friedan -Konechny 04]

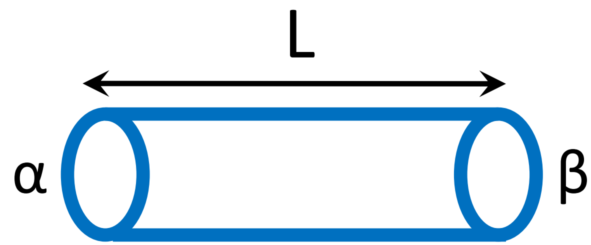
Definition 1 (Disk Amplitude)

It is simply defined from the disk amplitude

$$S_{bdy(\alpha)} = \log g_\alpha \,, \qquad g_\alpha \equiv \langle 0 | B_\alpha \rangle .$$

Definition 2 (Cylinder Amplitude)

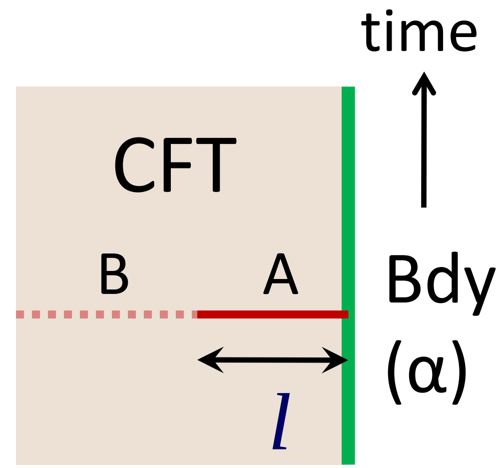
$$Z^{cylinder}_{(\alpha,\beta)} = \langle B_\alpha | e^{-HL} | B_\beta \rangle \underset{L\to\infty}{\approx} \underbrace{g_\alpha g_\beta}_{\text{Boundary Part}} \underbrace{e^{-E_0 L}}_{\text{Bulk Part}} .$$



Definition 3 (Entanglement Entropy)

$$S_A = -\mathrm{Tr}\left[\rho_A \log \rho_A\right],$$

$$\rho_A = \mathrm{Tr}_B \rho_{tot} \quad .$$



In 2D BCFT, the EE generally behaves like

$$S_A = \underbrace{\frac{c}{6}\log\frac{l}{\varepsilon}}_{\text{Bulk Part}} + \underbrace{\log g_\alpha}_{\text{Boundary Part}} \quad .$$

[Calabrese-Cardy 04]

In our setup, HEE can be found as follows

$$S_A = \frac{\text{Length}}{4G_N} = \frac{1}{4G_N}\int_{-\infty}^{\rho_*} d\rho = \frac{c}{6}\log\frac{l}{\varepsilon} + \underline{\frac{\rho_*}{4G_N}} \quad .$$

Boundary Entropy

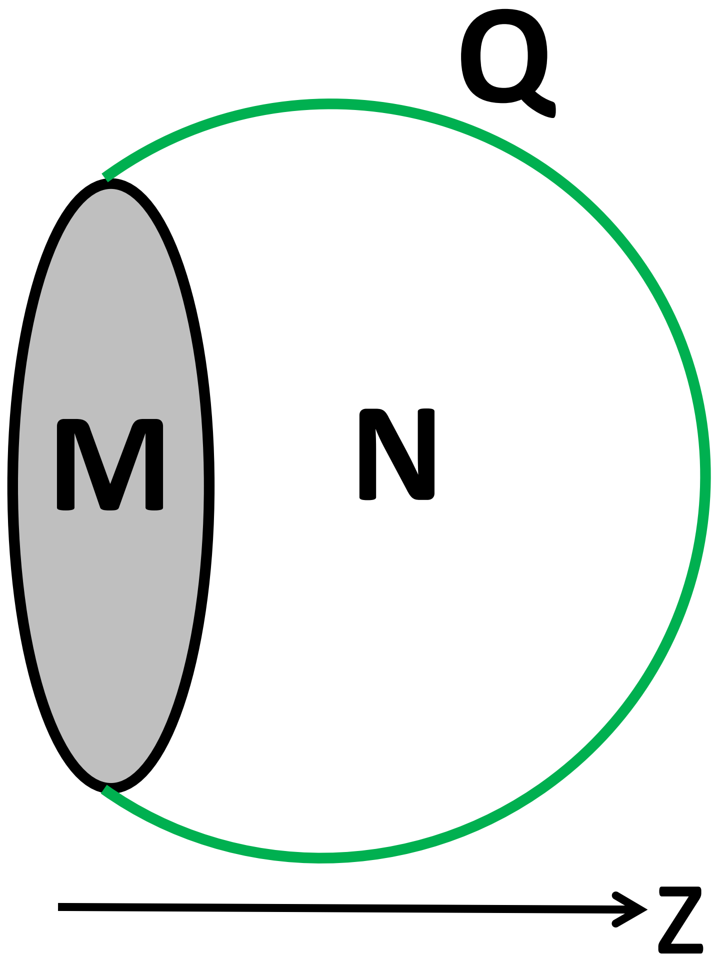
[Earlier calculations: Azeyanagi-Karch-Thompson-TT 07 (Non-SUSY Janus),
Chiodaroli-Gutperle-Hung, 10 (SUSY Janus)]

Also $S_{bdy} = \rho_* / 4G_N$ can be confirmed in other two definitions.

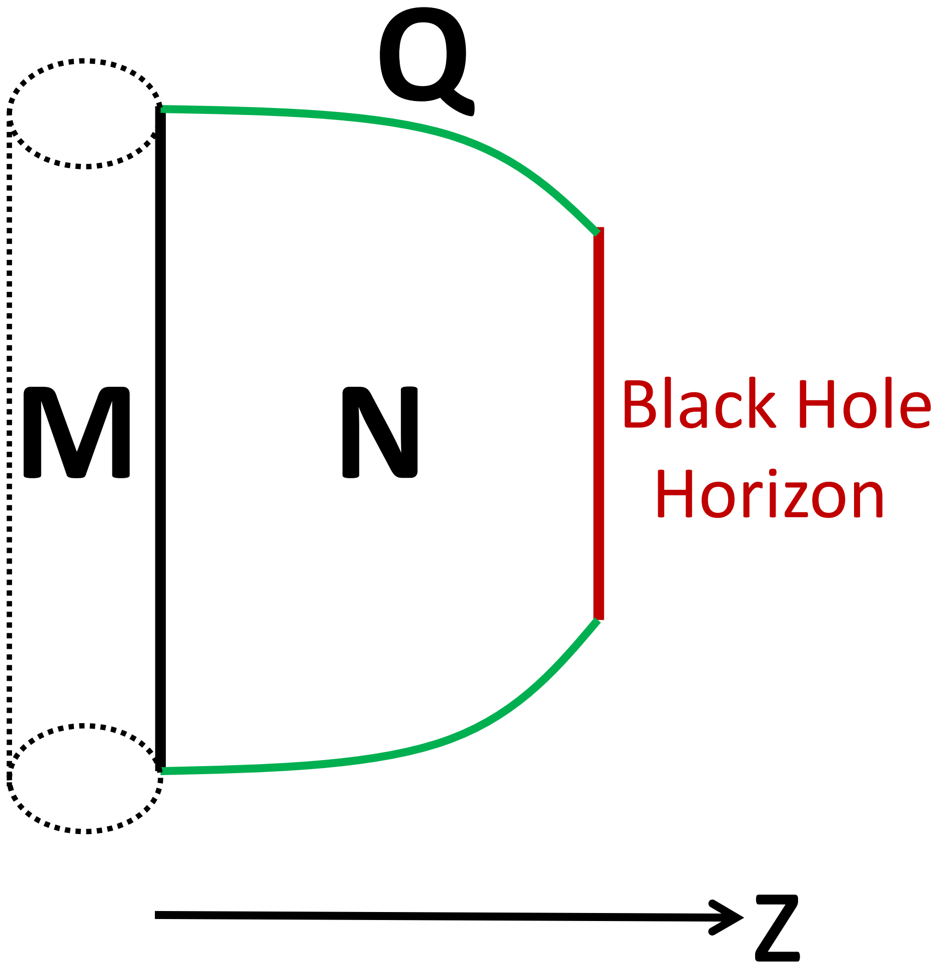
$$I_{Disk} = \frac{R}{4G_N}\left(\frac{r^2}{2\varepsilon^2} + \frac{r\sinh(\rho_*/R)}{\varepsilon} + \log\frac{\varepsilon}{r} - \underline{\frac{\rho_*}{R}} - \frac{1}{2}\right).$$

$$I_{Cylinder} = \frac{\pi}{3} c \cdot l \cdot T_{BH} + \underline{\frac{\rho_*}{2G_N}}.$$

Holographic Dual of Disk



Holographic Dual of Cylinder



Hawking-Page Transition for BCFT on an interval

$$I_E = -\frac{\pi}{24}\cdot\frac{c}{L\cdot T_{BCFT}}, \quad \text{(Low temp.)}$$

$$I_E = -\frac{\pi}{6}cLT_{BCFT} - \underbrace{\frac{\rho_*}{2G_N}}_{-2S_{bdy}}. \quad \text{(High temp .)}$$

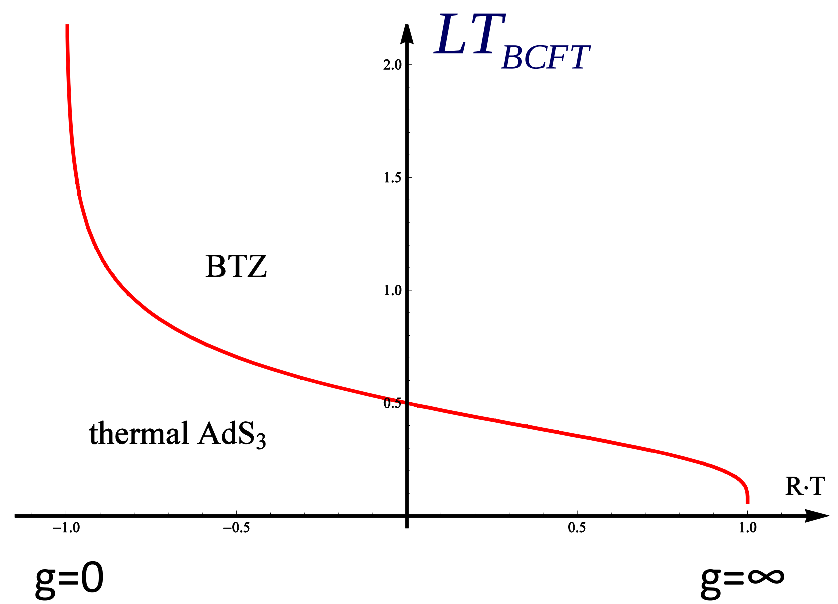
$$\Rightarrow \quad S_{thermal} = \frac{\pi}{3}cLT_{BCFT} + 2S_{bdy}$$

The phase transition occurs

when $I_E(\text{Low}) = I_E(\text{High})$

i.e.

$$T_{BCFT} = -\frac{1}{\pi L}\operatorname{arctanh}(RT) + \frac{1}{L}\sqrt{\frac{1}{4}+\frac{1}{\pi^2}\operatorname{arctanh}^2(RT)}\ .$$



(9-4) Holographic g-Theorem

Consider the surface Q defined by $x = x(z)$ in the Poincare metric

$$ds^2 = R^2\left(\frac{dz^2 - dt^2 + dx^2 + (d\vec{w})^2}{z^2}\right).$$

We impose the null energy condition for the boundary matter

i.e. $T^Q_{ab} N^a N^b \geq 0$ for any null vector N^a.

[cf. Hol. C-theorem: Freedman-Gubser-Pilch-Warner 1999, Myers-Sinha 2010]

For the null vector, $N^t = 1,\ \ N^z = 1/\sqrt{1+(x')^2}\ ,\ N^x = x'/\sqrt{1+(x')^2}\ ,$

we find the constraint

$$(K_{ab} - Kh_{ab})N^a N^b = -\frac{R \cdot x''}{z(1+(x')^2)^{3/2}} \geq 0.$$

Thus we simply get $x''(z) \leq 0$ from the null energy condition.

Define the holographic g-function:

$$\log g(z) = \frac{R^{d-1}}{4G_N} \cdot \mathrm{Arcsinh}\left(\frac{x(z)}{z}\right) = \frac{R^{d-2}}{4G_N} \cdot \rho_*(z).$$

Then we find $\dfrac{\partial \log g(z)}{\partial z} = \dfrac{x'(z)z - x(z)}{\sqrt{z^2 + x(z)^2}} \leq 0$,

because $(x'z - x)' = x''z \leq 0$.

For d=2, at fixed points $\log g(z)$ agrees with the boundary entropy.

For any d, $\rho_*(z)$ is a monotonically decreasing function w.r.t. z.

➡ This is our holographic g-theorem !

Example: AdS4/BCFT3

In this case, we obtain

$$I_E = \frac{R^2}{2G_N}\left[\frac{\pi}{2} + \arctan\left(\sinh\frac{\rho_*}{R}\right) - \frac{1}{24}\sinh\frac{3\rho_*}{R}\right.$$
$$\left. \underbrace{-\left(\sinh\frac{\rho_*}{R}\right)\log r_B}_{} + \left(\log\cosh\frac{\rho_*}{R} - \frac{33}{24} - \log 2\right)\sinh\frac{\rho_*}{R}\right].$$

Conformal anomaly
in odd dim. CFT ?

➡ This should come from the 2 dim. boundary !

Boundary central charge

As the usual central charge in 2 dim. CFT, we can define a boundary central charge in BCFT3 as follows:

$$r_B \frac{\partial \log Z_{Ball}}{\partial r_B} = -\frac{1}{2\pi}\left\langle \int_\Sigma dx^2 \sqrt{g_b}\, T^{\mu}_{\mu} \right\rangle = \frac{c_{bdy}}{6}\chi(\Sigma).$$

In our holographic calculation, we obtain

$$c_{bdy} = \frac{3R^2}{2G_N}\sinh\frac{\rho_*}{R}.$$

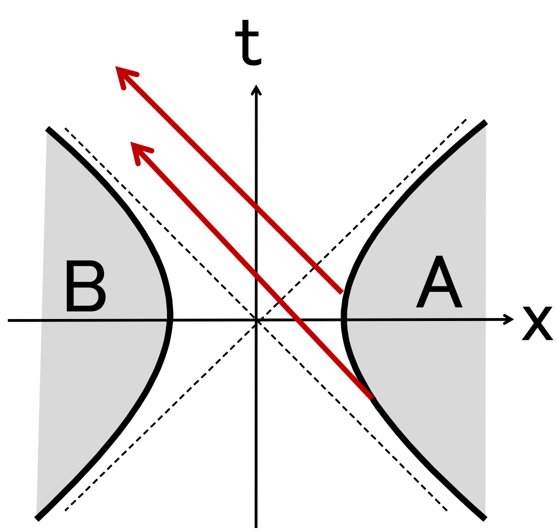
Our holographic g-theorem leads to a c-theorem for c_{bdy} .

➡ Our conjecture: this is true for all BCFT3.

(9-5) Time-dependent solution

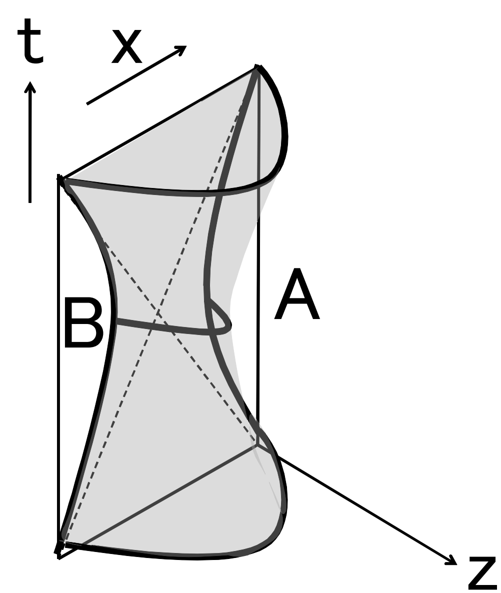
The analytical continuation to the Lorenzian signature $\tau = it$ leads to the following time-dependent solution

$$\mathrm{Q}: \quad -t^2 + x^2 + \left(z - r_D \sinh \frac{\rho_*}{R} \right)^2 = \left(r_D \cosh \frac{\rho_*}{R} \right)^2 .$$



Two BCFTs

A and B are causally disconnected !



Gravity dual

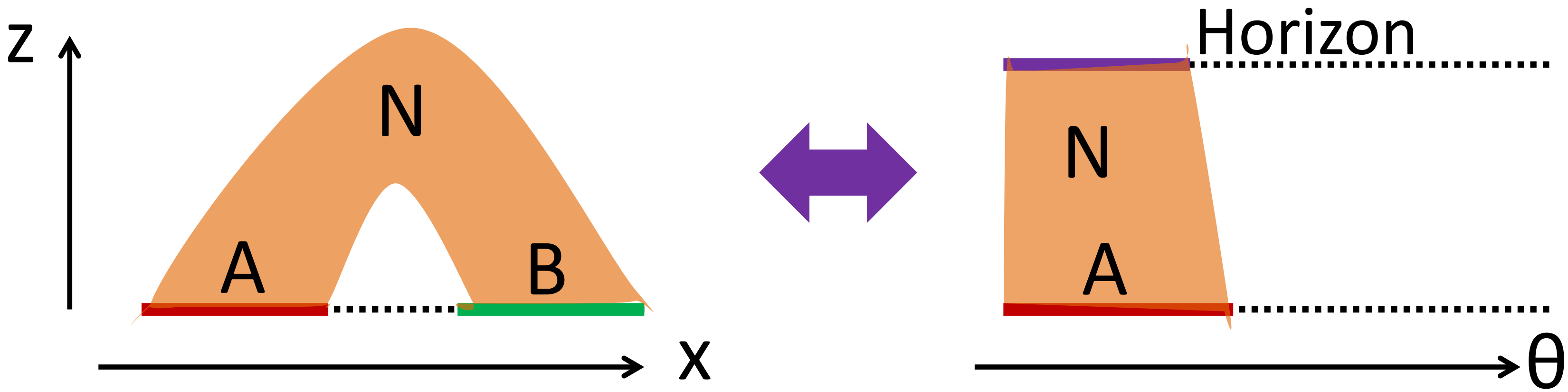
In the BCFT side, these two BCFTs are entangled with each other.
The entanglement entropy between them is calculated as

$$S_A = \frac{R}{4G_N}\int_{r_D e^{\rho_*/R}}^{z_{IR}} \frac{dz}{z}.$$

This is equal to the entropy $S_{BH} = \frac{R}{4G_N}\int_{\log(r_D e^{\rho_*/R})}^{\log z_{IR}} d\theta = S_A$.

of the BTZ Black hole: $ds^2 = -R^2\left(\frac{r^2}{r_+^2}-1\right)d\tau^2 + R^2\frac{dr^2}{r^2-r_+^2} + \frac{R^2}{r_+^2}r^2 d\theta^2.$

They are indeed related by a coordinate transformation.



⑩ Conclusions

- The entanglement entropy (EE) is a useful bridge between gravity (string theory) and cond-mat physics.

Gravity	⟺	Entanglement	⟺	Cond-mat. systems
$g_{\mu\nu}$		$S_A \approx \mathrm{Area}$		$\lvert \Psi \rangle$

- EE can characterize various phases of ground states (CFT, mass gap, fermi surfaces, topological etc.) .

  In odd dim. CFT, it provides an analogue of central charge.

- Especially in higher dimensions, the HEE offers us a powerful way to calculate EE for strongly coupled systems.

- EE is helpful for understanding s of various (quantum) gravity phenomena such as black hole formations, singularities etc.

<u>Future Problems</u>

- Proof of HEE ?
- Complete Higher derivative corrections to HEE ?
- 1/N corrections to HEE ?
- More on HEE and Fermi Liquids ?
- HEE for non-AdS spacetimes ?
- What is an analogue of the Einstein eq. for HEE ?
- A New Formulation of QG in terms of Quantum Entanglement