Flow equation for the scalar model in the large N expansion and its applications

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in collaboration with

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base on

S. Aoki, J. Balog, T. Onogi, P. Weisz,
``Flow equation for the scalar model in the large N expansion and its applications",
arXiv:1701.00046[hep-th].

related works

S. Aoki, J. Balog, T. Onogi, P. Weisz,
``Flow equation for the large N scalar model and induced geometries",
PTEP 2016(2016) 8, 083B04 (arXiv:1605.02413[hep-th]).

S. Aoki, K. Kikuchi, T. Onogi,
``Geometries from field theories"
PTEP 2015(2015)10, 101B01 (arXiv:1505.00131[hep-th]).

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Introduction

Holography

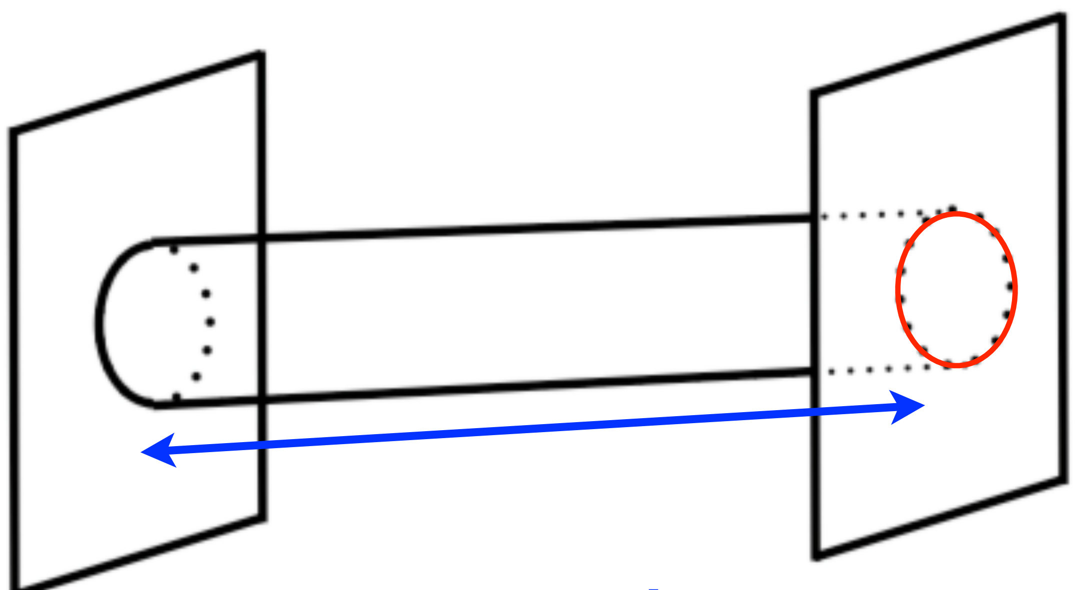
AdS/CFT correspondence ➡ **Gravity/Gauge**

Maldacena 1997

huge numbers of evidences but no proof

open string/closed string duality ?

D-brane



closed string = gravity

D-brane

open string

=

gauge theory

Different viewpoint

We propose a general method

Quantum Field Theory ⇒ Geometry

cf. Geometry of classical gauge theories

covariant derivative $D_\mu = \partial + igA_\mu$ connection

field strength (curvature) $F_{\mu\nu} \propto [D_\mu, D_\nu]$

Our proposal

S. Aoki, K. Kikuchi, T. Onogi,

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Proposal

d-dim. field -> (d+1)-dim. field

$$\varphi^a(x) \Rightarrow \sigma^a(t,x)$$

d dimensions　　d+1 dimensions

(d+1)-dim. field -> (d+1)-dim. induced metric

$$\hat{g}_{\mu\nu}(z) := h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z) \qquad z = (t,x)$$

(d+1)-dim. induced metric-> geometry

$$G_{\mu\nu}(z) := \langle G_{\mu\nu}(\hat{g}_{\mu\nu}(z)) \rangle$$

quantum average of
Einstein tensor

$\Rightarrow$ ``geometry” of
d+1 dimensional space

d-dim. field -> (d+1)-dim. field

(Gradient) Flow equation

large N index

action for original d-dim. theory

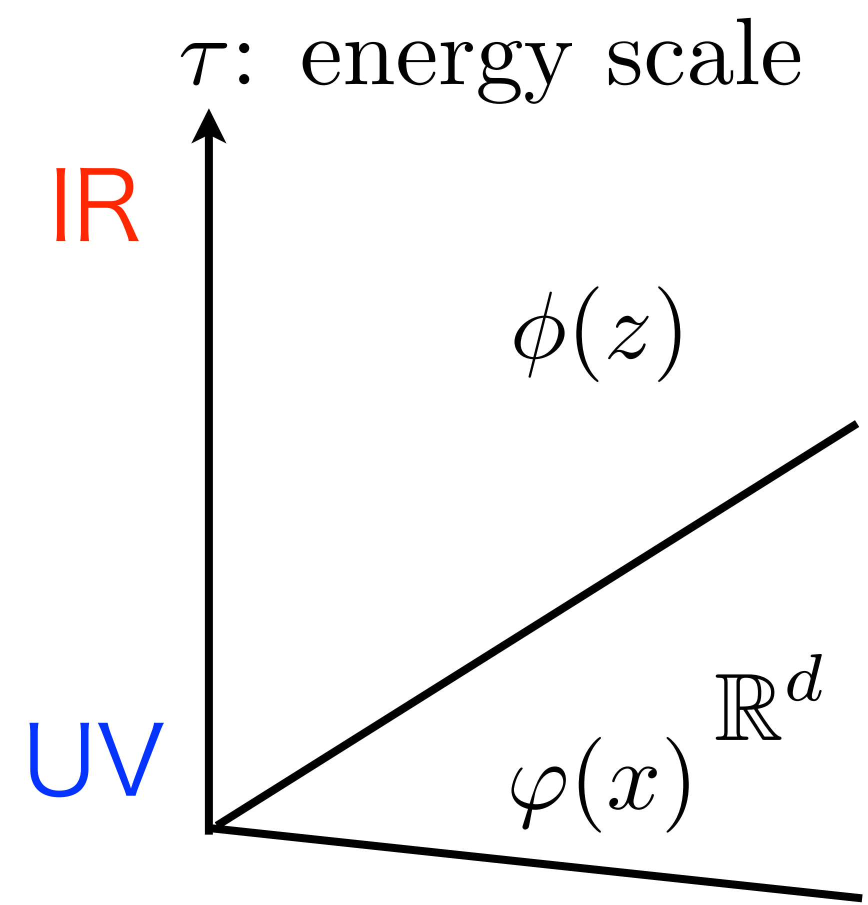
$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S_f(\varphi)}{\delta\varphi^a(x)}\right|_{\varphi^a(x)\to\phi^a(t,x)}$$

(d+1)-dim. field

d-dim. field

$$\phi^a(0,x) = \varphi^a(x)$$ initial condition

$$z = (\tau = \sqrt{t}, x) \in (\mathbb{R}^+, \mathbb{R}^d)$$



Remark

$\varphi(x)$ is the field in the path integral (NOT the operator).

What is the (gradient) flow equation ?

Free theory $$\frac{\partial}{\partial t}\phi^a(t,x) = \left(\Box - m^2\right)\phi^a(t,x)$$

$$\Rightarrow \quad \phi^a(t,x) = \frac{e^{-m^2 t}}{(4\pi t)^{d/2}}\int d^d y\, e^{-(x-y)^2/t}\varphi^a(y)$$

Heat kernel

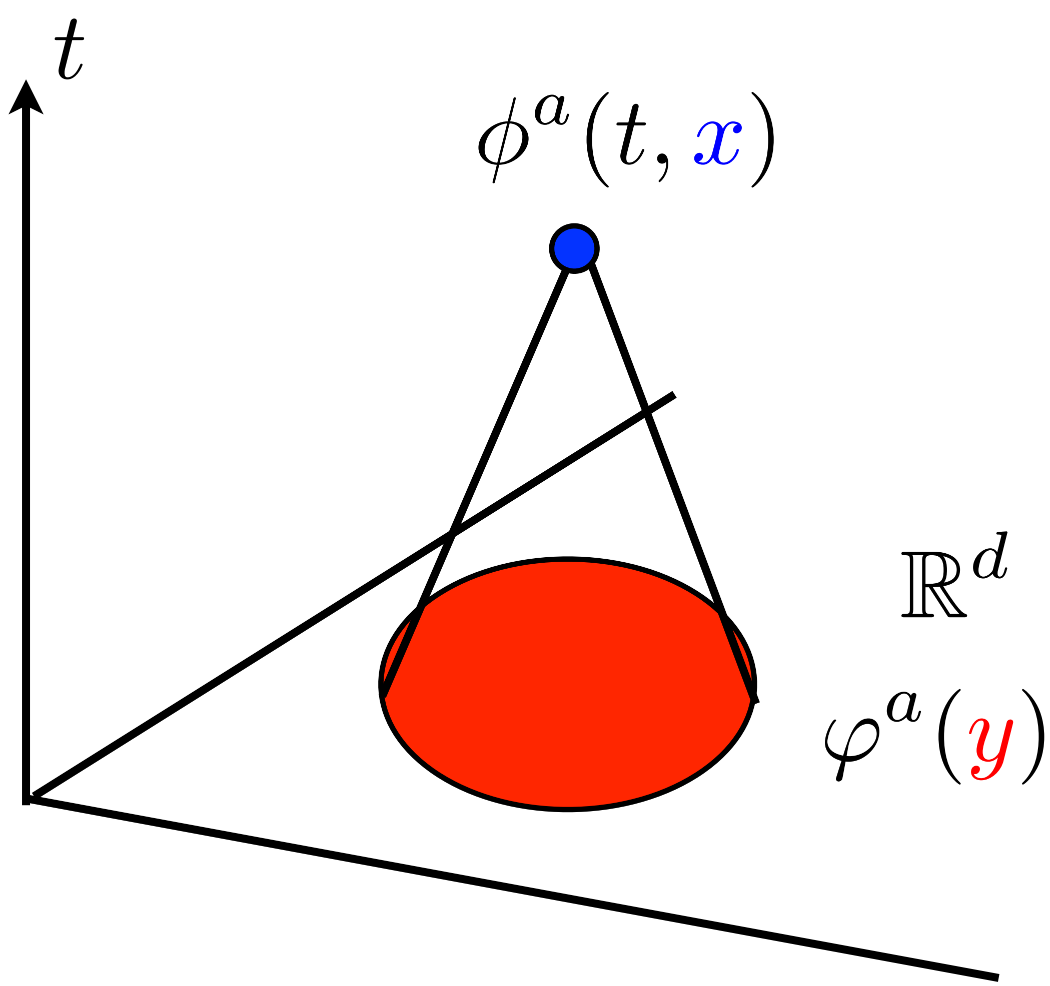
Lattice QCD

introduced to smooth out UV fluctuations of gauge fields

Narayanan-Neuberger 2006, Luescher 2010

flow gauge field is UV finite

Luescher-Weisz 2011



cf. Ricci flow $$\frac{d}{dt}g_{ij} = -2R_{ij}$$

used to prove Poincare conjecture by Perelman

Normalized flow field

$$\sigma^a(z) := \frac{\phi^a(z)}{\sqrt{\langle \phi^2(z) \rangle}}$$

Non-Linear Sigma Model (NLSM) normalization

$$\langle \sigma^2(z) \rangle = 1$$

Quantum average

d-dimension

$$\langle \mathcal{O}(\varphi) \rangle := \langle \mathcal{O}(\varphi) \rangle_S = \frac{1}{Z} \int \mathcal{D}\varphi \, \mathcal{O}(\varphi) e^{-S(\varphi)}, \qquad Z := \int \mathcal{D}\varphi \, e^{-S(\varphi)}$$

Flow equation

integrate out UV modes

Normalization

renormalization of field

define ``Renormalization Group" transformation

Remarks One may take different normalization conditions instead of NLSM.

$S \neq S_f$ is allowed. If $S = S_f$, we call it **"gradient flow"**.

(d+1)-dim. field -> (d+1)-dim. metric-> geometry

$$\sigma^a(z) : \mathbb{R}^+ \times \mathbb{R}^d \longrightarrow \mathbb{R}^N$$

h: constant with mass dimension -2

$$\hat{g}_{\mu\nu}(z) := h \sum_{a=1}^{N} \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z)$$

$h = R^2$

Induced metric on a $d+1$ dim. manifold $\mathbb{R}^+ \times \mathbb{R}^d$ from a manifold in $\mathbb{R}^N$, defined by $\sigma^a(z)$ with $\langle \sigma^2(z) \rangle = 1$

any correlation functions can be calculated using

functional integral in d-dimensions

$$\langle \hat{g}_{\mu\nu}(z) \rangle := \langle \hat{g}_{\mu\nu}(z) \rangle_S, \Rightarrow \text{geometry}$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \hat{g}_{\mu_2\nu_2}(z_2) \rangle_S,$$

$$\langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle := \langle \hat{g}_{\mu_1\nu_1}(z_1) \cdots \hat{g}_{\mu_n\nu_n}(z_n) \rangle_S,$$

$\Rightarrow$ quantum corrections

key properties

1 $\hat{g}_{\mu\nu}(z) \propto \partial_\mu \sigma^a(z) \partial_\nu \sigma^a(z)$ may give finite results for $\tau \neq 0$

Flow: a heat kernel type smearing $\quad \tau \to 0$ is UV while $\tau \to \infty$ is IR

Finiteness as QFT is NOT guaranteed in general but true in the large N limit.

cf. d dimensional induced metric $g_{\mu\nu}(x) \sim \partial_\mu \varphi(x) \partial_\nu \varphi(x)$ is badly divergent

2 metric becomes classical in the large N limit

$$\langle \hat{g}_{\mu\nu}(z_1) \hat{g}_{\alpha\beta}(z_2) \rangle = \langle \hat{g}_{\mu\nu}(z_1) \rangle \langle \hat{g}_{\alpha\beta}(z_2) \rangle + O\left(\frac{1}{N}\right)$$

large N factorization

$$\Rightarrow \quad \langle G_{\mu\nu}(\hat{g}_{\mu\nu}) \rangle = G_{\mu\nu}(\langle \hat{g}_{\mu\nu} \rangle) + O\left(\frac{1}{N}\right)$$

classical geometry after quantum averages d-dim. quantum field theory

large N limit

(d+1)-dim. classical metric

Geometry in d+1 dimensions

Model and large N expansion

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O(N) scalar Model

φ^4 model

$$S(\mu^2, u) = N \int \mathrm{d}^d x \left[\frac{1}{2} \partial^k \varphi(x) \cdot \partial_k \varphi(x) + \frac{\mu^2}{2} \varphi^2(x) + \frac{u}{4!} \left(\varphi^2(x) \right)^2 \right]$$

$u = 0$: free, $u \to \infty$: NLSM

$\varphi^2(x) \equiv \varphi(x) \cdot \varphi(x) = \sum_{a=1}^N \varphi^a(x) \varphi^a(x)$

large N limit

$$\langle \varphi^a(x) \varphi^b(y) \rangle = \delta^{ab} \frac{1}{N} \int \mathrm{d}p \, \frac{\mathrm{e}^{ip(x-y)}}{p^2 + m^2}$$

mass renormalization

$$\mu^2 = m^2 - \frac{u}{6} Z(m)$$

$$Z(m) = \int Dp \frac{1}{p^2 + m^2} \geq 0, \qquad Dp \equiv \frac{d^d p}{(2\pi)^d}$$

$$Z(m) \to \infty \ (d > 1)$$

Flow equation and SDE

Flow equation

$$\frac{\partial}{\partial t}\phi^a(t,x) = -\left.\frac{\delta S(\mu_f^2, u_f)}{\delta\varphi^a(x)}\right|_{\varphi\to\phi} = \left(\Box - \mu_f^2\right)\phi^a(t,x) - \frac{u_f}{6}\phi^a(t,x), \quad \phi^a(0,x) = \varphi^a(x)$$

$$\Downarrow$$

Schwinger-Dyson Equation (SDE)

$$\langle D_z^f \phi^a(z)^\forall\mathcal{O}\rangle = -\frac{u_f}{6}\langle\phi^a(z)\phi^2(z)^\forall\mathcal{O}\rangle, \qquad D_z^f \equiv \frac{\partial}{\partial t} - (\Box - \mu_f^2)$$

Solve SDE order by order in the 1/N expansion.

NLO solution

2-pt function

$$\langle \phi^{a_1}(z_1)\phi^{a_2}(z_2)\rangle = \frac{\delta_{a_1a_2}}{N}\frac{Z(m_f)}{\zeta(t_1)\zeta(t_2)}\int Dp\,\frac{e^{-p^2(t_1+t_2)}e^{ip(x_1-x_2)}}{p^2+m^2}\left[1+\frac{G_1(t_1,t_2|p)}{N}\right]$$

4-pt function

$$\langle \phi^{a_1}(z_1)\phi^{a_2}(z_2)\phi^{a_3}(z_3)\phi^{a_4}(z_4)\rangle = \frac{1}{N^3}\left[\delta_{a_1a_2}\delta_{a_3a_4}K_0(z_1z_2;z_3z_4) + \ 2 \text{ permutations}\right]$$

$$K_0(z_1z_2;z_3z_4) = \int dP_4\,\hat{\delta}\hat{g}(t_1t_2;t_3t_4|p_1p_2;p_3p_4), \quad dP_4 \equiv \prod_{j=1}^4 Dp_j\sqrt{\frac{Z(m_f)}{\zeta(t_j)}\,\frac{e^{ip_jx_j}e^{-p_j^2t_j}}{p_j^2+m^2}}$$

$$\hat{\delta} \equiv (2\pi)^d\delta(p_1+p_2+p_3+p_4)$$

$G_1(t_1,t_2|p)$ and $\hat{g}(t_1t_2;t_3t_4|p_1p_2;p_3p_4)$ are very complicated.

Results in the large N limit

S. Aoki, J. Balog, T. Onogi, P. Weisz,
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PTEP 2016(2016) 8, 083B04 (arXiv:1605.02413[hep-th]).

Induced metric

VEV of the metric

$$g_{\mu\nu}(z) := \langle \hat{g}_{\mu\nu}(z) \rangle = \begin{pmatrix} g_{\tau\tau}(\tau) & 0 \\ 0 & g_{ij}(\tau) \end{pmatrix}$$

$$g_{\tau\tau}(\tau) = \frac{h\tau^2}{16} \frac{\mathrm{d}^2 \log \zeta_0(t)}{\mathrm{d}t^2}, \qquad g_{ij}(\tau) = -\delta_{ij} \frac{h}{2d} \frac{\mathrm{d} \log \zeta_0(t)}{\mathrm{d}t}.$$

$$\zeta_0(t) = \frac{m^{d-2} \mathrm{e}^{2m^2 t}}{(4\pi)^{d/2}} \Gamma(1 - d/2, 2m^2 t).$$

incomplete gamma function

Massless limit $m^2 \to 0$

Massive UV limit $m\tau \ll 1$

$$g_{\tau\tau}(\tau) = h\frac{d-2}{2}\frac{1}{\tau^2}, \quad g_{ij} = h\frac{d-2}{d}\frac{1}{\tau^2}$$

$d \geq 3$

Euclidean AdS

Einstein tensor

$$G_{\mu\nu}(\tau) \simeq -\Lambda_{\mathrm{UV}} g_{\mu\nu}(\tau) \quad \Lambda_{\mathrm{UV}} = -\frac{d(d-1)}{h(d-2)}$$

Massive IR limit $m\tau \gg 1$

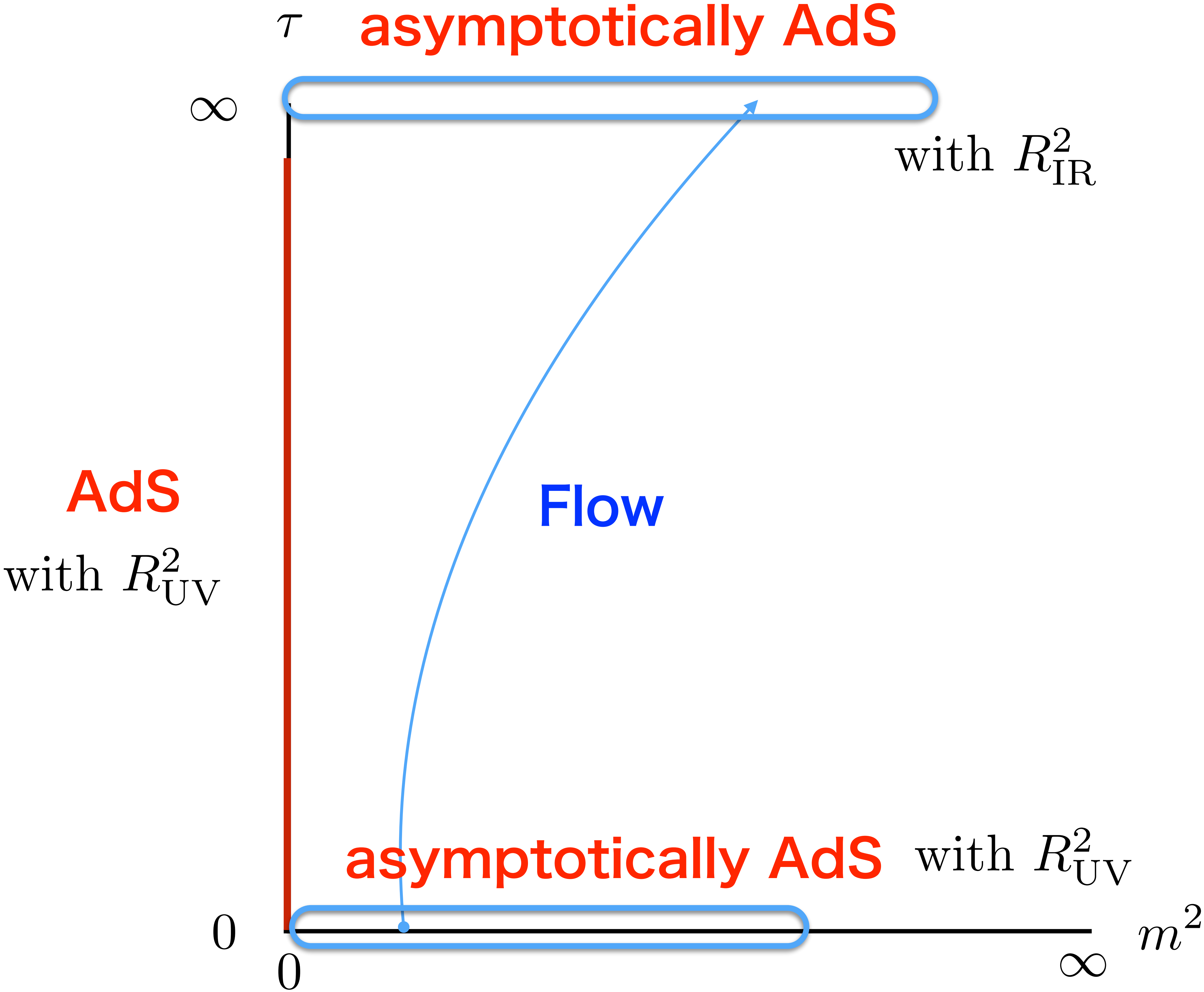
$$g_{\tau\tau}(\tau) = \frac{hd}{2\tau^2}, \quad g_{ij}(\tau) = \frac{h\delta_{ij}}{\tau^2}$$

Euclidean AdS

$$G_{\mu\nu}(\tau) \simeq -\Lambda_{\mathrm{IR}} g_{\mu\nu}(\tau) \quad \Lambda_{\mathrm{IR}} = -\frac{d-1}{h}$$

AdS radius
$(m^2 \neq 0)$

$$R_{\mathrm{UV}}^2 = -\frac{h(d-2)}{d(d-1)} = \frac{d-2}{d} R_{\mathrm{IR}}^2 < R_{\mathrm{IR}}^2$$



τ
asymptotically AdS
∞
with R^2_{IR}
AdS
with R^2_{UV}
Flow
asymptotically AdS
with R^2_{UV}
0
m^2
0
∞

NLO corrections to massless theory at d=3

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arXiv:1701.00046[hep-th].

Running coupling from flowed fields

(dimensionless) running coupling

$$g(t) = -3\hat{g}(t,t;t,t|\{p\}_{\rm sym})t^{2-d/2}$$

4-pt function

$$(p_1+p_2)^2 = (p_3+p_4)^2 = 1/t$$

d=3, massless $u_f = 0$

$$g(t) = \frac{u\sqrt{t}}{1+u\sqrt{t}/48} \simeq \begin{cases} \to 0, & t \to 0 \\ \to 48, & t \to \infty \end{cases}$$

Asymptotic free UV fixed point

Wilson-Fisher IR fixed point

c.f. $u_f \neq 0$

$$g(t) = G_1 + G_2 \frac{u\sqrt{t}/48}{1+u\sqrt{t}/48}, \quad G_1 \simeq 21, G_2 \simeq 2$$

$\to G_1$ UV

$\to G_1 + G_2$ IR

NLO corrections to induced metric

d=3, massless

$$g_{\tau\tau}(\tau) = \frac{R_0^2}{2\tau^2}\left[1 + \frac{\tau}{2N}\int DQ h_{\text{total}}(Q^2)\frac{\bar{u}(Q^2)(1+3\bar{u}(Q^2)\tau/2)}{(1+\bar{u}(Q^2)\tau/2)^3}\right]$$

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1 + \frac{\tau}{N}\int DQ h_{\text{total}}(Q^2)\frac{\bar{u}(Q^2)}{(1+\bar{u}(Q^2)\tau/2)^2}\right]$$

$$\bar{u}(Q^2) = \frac{u}{48\sqrt{Q^2}}$$

Q:dimensionless

UV limit $\tau \to 0$

$$g_{\tau\tau}(\tau) \simeq \frac{R_0^2}{2\tau^2}\left[1 + \frac{\tau}{2N}\int DQ h_{\text{total}}(Q^2)\bar{u}(Q^2)\right]$$

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1 + \frac{\tau}{N}\int DQ h_{\text{total}}(Q^2)\bar{u}(Q^2)\right]$$

NLO is less singular than LO

$\Rightarrow$ $$R_{\text{AdS}}^{\text{LO}} = R_{\text{AdS}}^{\text{NLO}}$$

IR limit $\tau \to \infty$

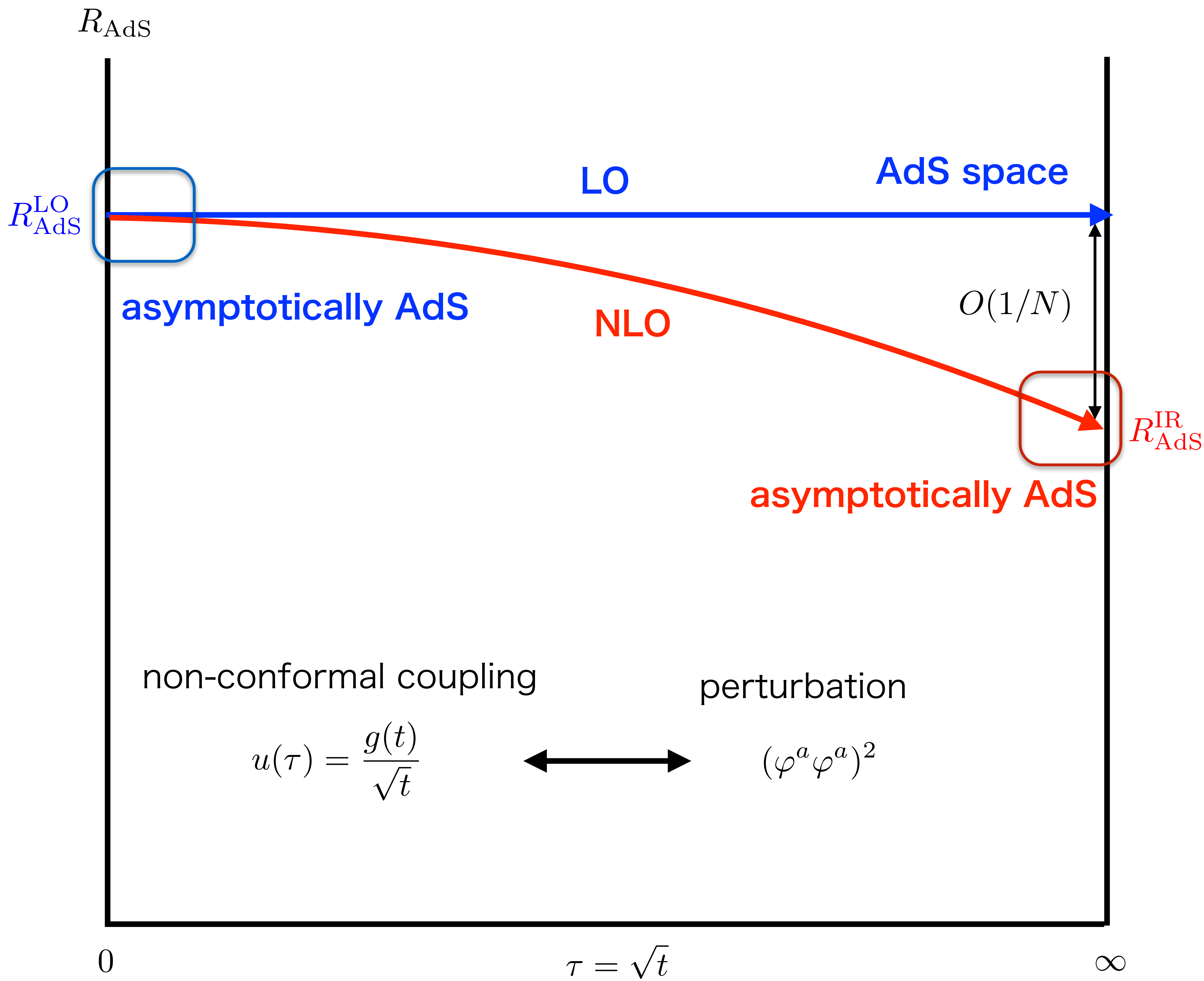
$$g_{\tau\tau}(\tau) = \frac{R_0^2}{2\tau^2}\left[1 + \frac{r}{N}\right]$$

$$g_{ij}(\tau) = \delta_{ij}\frac{R_0^2}{3\tau^2}\left[1 + \frac{r}{N}\right]$$

$$r = -0.41869(1)$$

$\Rightarrow$ $$R_{\text{AdS}}^{\text{IR}} = R_{\text{AdS}}^{\text{LO}}\left(1 + \frac{r}{2N}\right) < R_{\text{AdS}}^{\text{UV}}$$

u_f independent as long as $u_f \neq 0$



$R_{\rm AdS}$
$R_{\rm AdS}^{\rm LO}$
LO
AdS space
asymptotically AdS
NLO
$O(1/N)$
$R_{\rm AdS}^{\rm IR}$
asymptotically AdS
non-conformal coupling
perturbation
$u(\tau) = \frac{g(t)}{\sqrt{t}}$
$(\varphi^a \varphi^a)^2$
0
$\tau = \sqrt{t}$
∞

Discussions

prediction from F-theorem

Free energy on S³ (conformal coupling, zeta-function reg.)

$$NF_S = N\left(\frac{\log 2}{8} - \frac{3\zeta(3)}{16\pi^2}\right), \quad F_S \simeq 0.0638$$

N massless free scalars

$$F_{WF}(N) = NF_S\left(1 + \frac{r_0}{N}\right), \quad r_0 \simeq -0.2386$$

O(N) scalars $u \to \infty$

(Wilson-Fisher FP ?)

$F_{WF}(N) < NF_S$

IR UV

Holographic dual $F = \dfrac{\pi R^2}{2G_4}$

$$R^2_{\mathrm{IR}} \simeq R^2_{\mathrm{UV}}\left(1 - \frac{0.2386}{N}\right)$$

c.f. our result on R³

$$R^2_{\mathrm{IR}} \simeq R^2_{\mathrm{UV}}\left(1 - \frac{0.41869}{N}\right)$$

It is interesting to repeat our calculation on S³.

Future directions

- some other quantities in 3-d theory
  - expectations from higher spin theories ?
  - your suggestions are very welcome
- 2-pt function for the metric at NLO ?

$$\langle g_{\mu_1\nu_1}(z_1) g_{\mu_2\nu_2}(z_2)\rangle_c = O\left(\frac{1}{N}\right)$$

- finite Temperature -> black hole ?