

Time Evolution of Entanglement Entropy in Orbifold CFTs

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1701.03110[hep-th]

[OTOc in Orbifold CFTs *to appear*
→ Pawel's Talk]

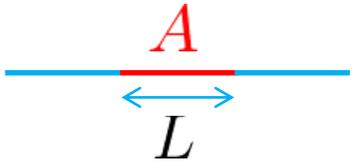
w/ Pawel Caputa (Nordita → YITP)

Yuya Kusuki (YITP)

Tadashi Takayanagi (YITP)

Small subsystem :

$(L \rightarrow 0)$



The 1st law of EE



Universal property of EE

$$T_{ent} \cdot \Delta S_A = \Delta E_A$$

[Bhattacharya-Nozaki-Takayanagi-Ugajin 12]

[Blanco-Casini-Hung-Myers 13]...



EE for excited states

$$\Delta S_A = S_A - S_{A,0}$$

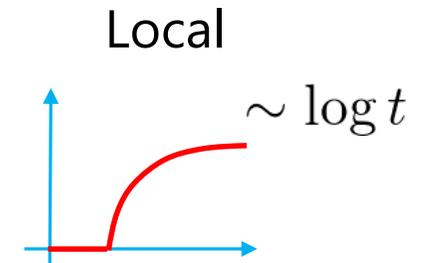
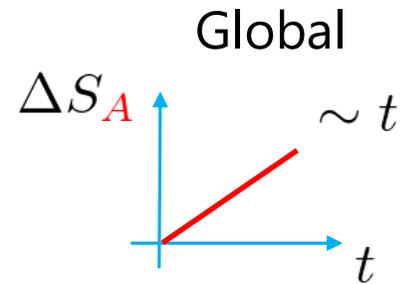
Large subsystem :

$(L \rightarrow \infty)$



Quantum quench

since [Calabrese – Cardy 05]

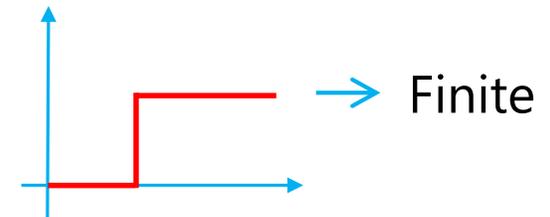


✓ Excited states by local operators

$$e^{-itH} e^{-\epsilon H} \mathcal{O}(x) |0\rangle$$

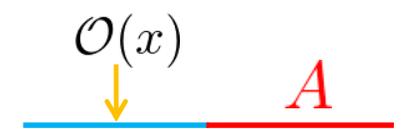
since [Nozaki-Numasawa-Takayanagi 14]

Free, Rational



Excited states by local operators

$$e^{-itH} e^{-\epsilon H} \mathcal{O}(x) |0\rangle$$



Free Scalar [Nozaki-Numasawa-Takayanagi 14, Nozaki 14]

Rational CFTs [He-Numasawa-Takayanagi-KW 14]

Large-N free YM, Large-c [Caputa-Nozaki-Takayanagi 14]

Large-c Heavy op [Asplund-Bernamonti-Galli-Hartman 14]

Finite T [Caputa-Simon-Stikonas-Takayanagi 14]

Descendent ops [Caputa - Veliz-Osorio, Chen-Guo-He-Wu 15]

Free Fermion [Nozaki-Numasawa-Matsuura 15]

Charged Renyi [Caputa-Nozaki-Numasawa 15]

Maxwell [Nozaki-Watamura 16]

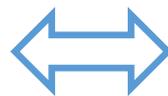
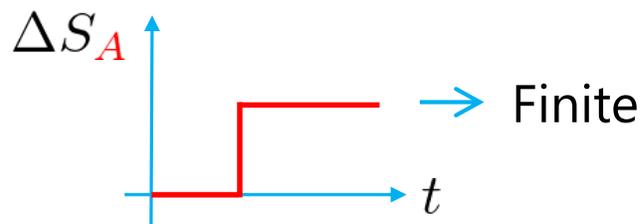
Many ops in RCFTs [Numasawa 16]

Aharonov-Bohm Effect [Shiba 17]

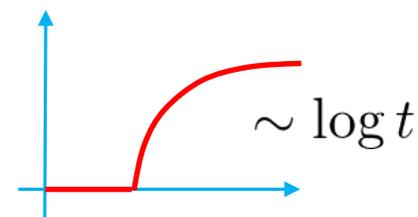
etc ...

2 extreme classes :

Free, Rational
(integrable)



Holographic
(maximally chaotic)



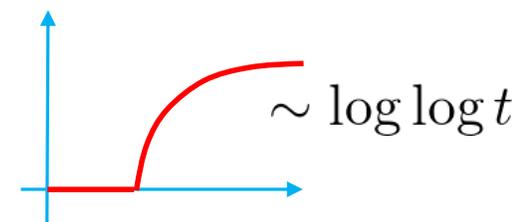
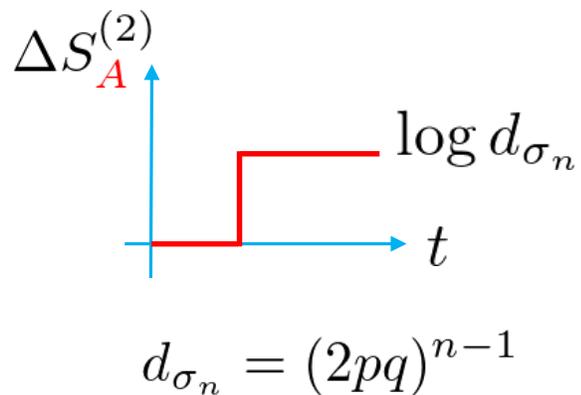
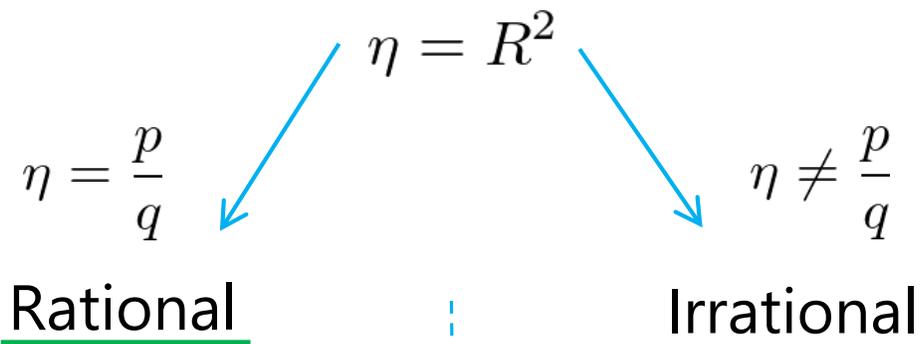
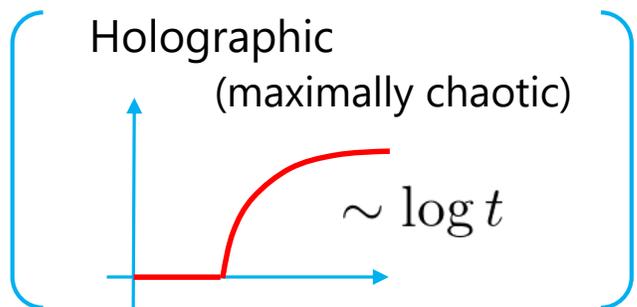
Middle Class ??

Time Evolution of Entanglement Entropy in Orbifold CFTs

[Caputa-Kusuki-Takayanagi-KW 17]

2d Complex Compact Free Scalar $c = 2$ \rightarrow $(T^2)^n / \mathbb{Z}_n$ Orbifold CFTs $c_n = nc$

The orbifold twist op



Middle Class !!

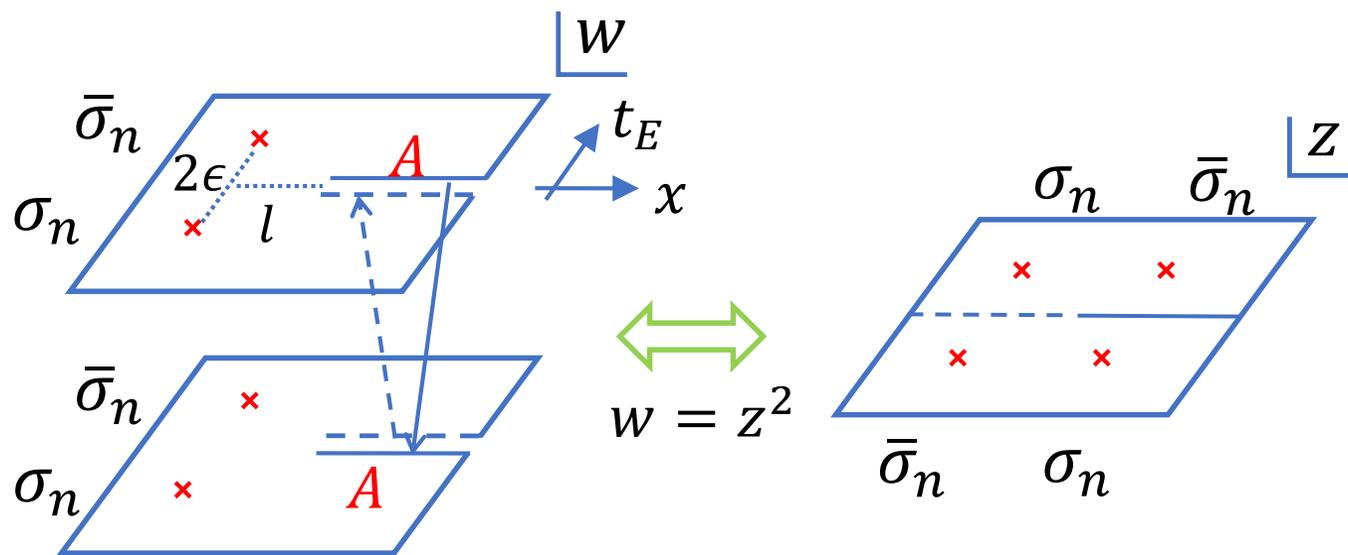
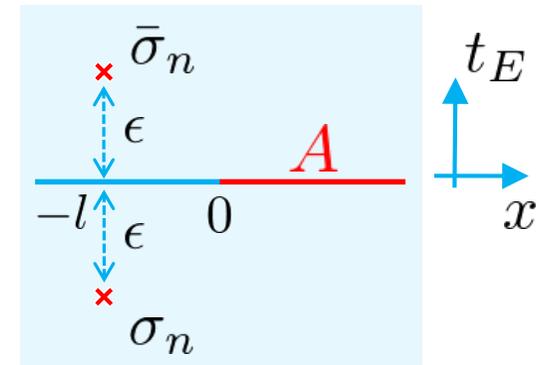
Some More Details ...

- Setup
- Results
- Remarks

What We Compute

$$\rho(t) = e^{-(\epsilon+it)H} \sigma_n(-l) |0\rangle \langle 0| \bar{\sigma}_n(-l) e^{-(\epsilon+it)H}$$

$$\Rightarrow \Delta S_A^{(2)} = -\log \frac{\langle \sigma_n \bar{\sigma}_n \sigma_n \bar{\sigma}_n \rangle}{\langle \sigma_n \bar{\sigma}_n \rangle \langle \sigma_n \bar{\sigma}_n \rangle} = -\log F_n(z, \bar{z})$$



→ This is almost same setup as

EE for 2 disjoint intervals

[Calabrese-Cardy-Tonii 09]

but w/ indep. cpx. z, \bar{z}

Ref: [Calabrese-Cardy-Tonii 12]

[Cosser-Tagliacozzo-Tonii 13]

We can apply the well-known result : [\[Calabrese-Cardy-Tonii 09\]](#) [\[Calabrese-Cardy-Tonii 12\]](#) [\[Coser-Tagliacozzo-Tonii 13\]](#)

$$F_n(z, \bar{z}) = \frac{2^{n-1} \eta^{n-1}}{\prod_{k=1}^{n-1} I_{k/n}(z, \bar{z})} \cdot \Theta(0|\eta\Gamma)^2 \quad \text{Normalized s.t. } F_n(0, 0) = 1$$

$\left[\epsilon \rightarrow 0 \text{ for } 0 < t < l \right]$

$$\Gamma : 2(n-1) \times 2(n-1) \text{ sym. matrix w/ } \text{Im } \Gamma \geq 0 \quad \eta = R^2$$

$$I_{k/n}(z, \bar{z}) = {}_2F_1\left(\frac{k}{n}, 1 - \frac{k}{n}, 1; 1 - z\right) \cdot {}_2F_1\left(\frac{k}{n}, 1 - \frac{k}{n}, 1; \bar{z}\right) + (z \leftrightarrow \bar{z})$$

$$\Theta(0|\eta\Gamma) = \sum_{m \in \mathbb{Z}^{2(n-1)}} e^{i\pi m^T \cdot \Gamma \cdot m} \quad \text{: the Riemann-Siegel theta function}$$

Our main work

⇒ Evaluate $F_n(z, \bar{z})$ in the late time limit $(z, \bar{z}) \rightarrow (1, 0) \left[\epsilon \rightarrow 0 \text{ for } t > l \right]$

Rough Sketch of the Evaluation

$$\delta = \frac{\pi}{\log\left(\frac{4t^2}{\epsilon^2}\right)}$$

In the late time limit $\delta \rightarrow 0$ ($\epsilon \rightarrow 0$ for $t > l$)

For simplicity, assume $t \gg l$

$$\left[\begin{array}{l} z \simeq 1 - \frac{\epsilon^2}{4t^2} = 1 - e^{-\frac{\pi}{\delta}} \\ \bar{z} \simeq \frac{\epsilon^2}{4t^2} = e^{-\frac{\pi}{\delta}} \end{array} \right] \longrightarrow \left[\begin{array}{l} I_{k/n} \simeq \frac{\sin^2\left(\frac{\pi k}{n}\right)}{\delta^2} \\ \Gamma \simeq \begin{pmatrix} i\delta \cdot (\Omega_0)_{r,s} & -\frac{1}{2}(\Lambda_0)_{r,s} \\ -\frac{1}{2}(\Lambda_0)_{r,s} & i\delta \cdot (\Omega_0)_{r,s} \end{pmatrix} \end{array} \right]$$

$$\Rightarrow F_n(z, \bar{z}) = \frac{2^{n-1} \eta^{n-1}}{\prod_{k=1}^{n-1} I_{k/n}(z, \bar{z})} \cdot \Theta(0|\eta\Gamma)^2$$

$$\simeq 2^{n-1} \eta^{n-1} \cdot \left(\frac{2^{n-1} \delta^{n-1}}{n} \cdot \sum_{l, m \in \mathbb{Z}^{n-1}} e^{-\pi \delta \eta m^T \cdot \Omega_0 \cdot m - i \pi \eta l^T \cdot \Lambda_0 \cdot m - \pi \delta \eta l^T \cdot \Omega_0 \cdot l} \right)^2$$

the Poisson resummation

$$\frac{1}{\sqrt{\det(\eta \delta \Omega_0)}} \cdot \sum_{l, \tilde{m} \in \mathbb{Z}^{n-1}} e^{-\frac{\pi}{\delta \eta} (\tilde{m}^T - \frac{\eta}{2} l^T \cdot \Lambda_0^T) \cdot \Omega_0^{-1} \cdot (\tilde{m} - \frac{\eta}{2} \Lambda_0 \cdot l) - \pi \delta \eta l^T \cdot \Omega_0 \cdot l}$$

→ Evaluate the dominant saddle contribution

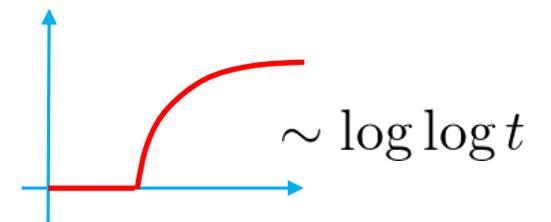
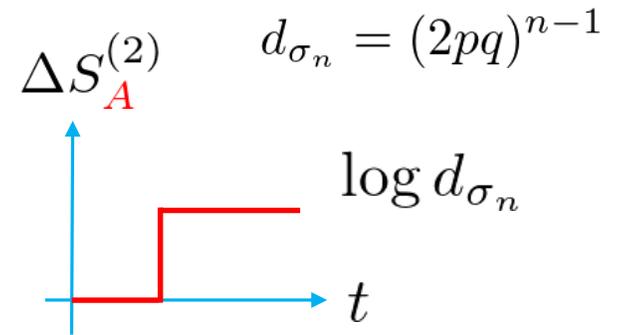
Results

In the late time limit $(z, \bar{z}) \rightarrow (1, 0)$

$$\left(\begin{array}{l} \epsilon \rightarrow 0 \text{ for } t \gg l \\ \text{or } \delta \rightarrow 0 \quad \delta = \frac{\pi}{\log\left(\frac{4t^2}{\epsilon^2}\right)} \end{array} \right)$$

$$F_n(1, 0) \simeq \left[\begin{array}{ll} \frac{1}{(2pq)^{n-1}} & \text{for rational } \eta = \frac{p}{q} \\ \frac{2^{n-1}}{n} \cdot \delta^{n-1} & \text{for irrational } \eta \neq \frac{p}{q} \end{array} \right.$$

$$\Rightarrow \Delta S_A^{(2)} \simeq \left[\begin{array}{ll} (n-1) \cdot \log(2pq) & \text{for rational } \eta = \frac{p}{q} \\ (n-1) \cdot \log\left(\frac{\log(4t^2/\epsilon^2)}{2\pi}\right) + \log n & \text{for irrational } \eta \neq \frac{p}{q} \end{array} \right.$$



\Rightarrow **New Scaling Class !!**

Remarks

For Rational CFTs :

[He-Numasawa-Takayanagi-KW 14]

The finite saturation value \longrightarrow Log[Quantum dim. of the operator]

$$\Delta S_A^{(m)} = \log d_{\sigma_n} \quad \text{for any } m \quad d_{\sigma_n} = \frac{S_{0\sigma_n}}{S_{00}} = (2pq)^{n-1}$$

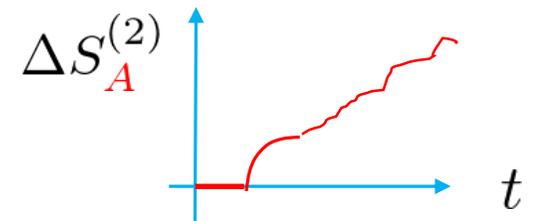
$$\left(\text{For } S_n\text{-orbifold twist op } d_{\sigma_n} = (n-1)! \cdot (2pq)^{n-1} \right)$$

For Irrational CFTs :

Strictly speaking,

$F_n(z, \bar{z})$ oscillates in time or has many plateau-like structures

It seems to be because the continued fraction expansions of irrational numbers generate many rational numbers ...



$$\text{E.g. } \sqrt{2} = 1 + \frac{1}{2 + \frac{1}{2 + \dots}}$$

$$\longrightarrow \left\{ 1, \frac{3}{2}, \frac{7}{5}, \frac{17}{12}, \dots \right\}$$

Summary

1701.03110[hep-th]

[Caputa-Kusuki-Takayanagi-KW 17]

2d Complex Compact Free Scalar

→ $(T^2)^n / \mathbb{Z}_n$ Orbifold CFTs

The orbifold twist op



$$\eta = R^2$$

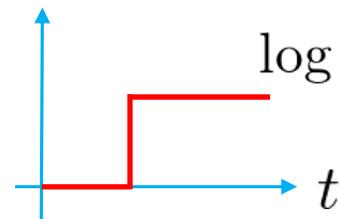
$$\eta = \frac{p}{q}$$

Rational

$$\eta \neq \frac{p}{q}$$

Irrational

$$\Delta S_A^{(2)}$$



$$\log d_{\sigma_n}$$

$$d_{\sigma_n} = (2pq)^{n-1}$$

$$\left[= (n-1)! \cdot (2pq)^{n-1} \right. \\ \left. \text{for } S_n \right]$$

$$\sim \log \log t$$

Holographic
(maximally chaotic)

A graph with $\Delta S_A^{(2)}$ on the vertical axis and t on the horizontal axis. The curve is zero until a certain time, then rises and levels off. The label $\sim \log t$ is next to it.

⇒ Middle Class !!

Future Works

For irrational cases :

- Higher Renyi entropies
- Symmetric Orbifold CFTs
- Large- c (Large- n) limit

Other quantities :

- OTOC → Pawel's Talk
- Relative Entropy
- Modular Hamiltonian