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# First-class constraints and the BV formalism


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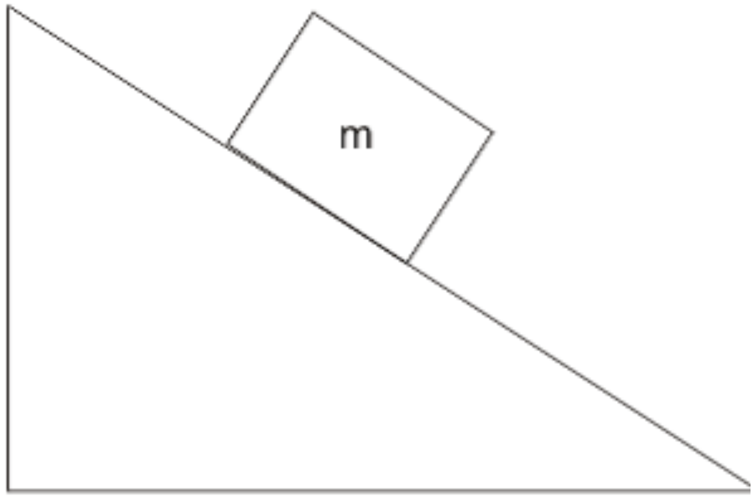
# Constrained systems are everywhere

- Redundant variables




A block on a ramp

- gauge theory
- gravity

Batalin-Vilkovisky (BV) formalism is an efficient way to treat these constrained systems.

# Motivating analogy

One way to compute holographic Weyl anomalies employ the flow equation (a.k.a. Hamiltonian constraint) $\{S, S\}$, which is

- a 2nd order functional differential eq.,
- a result of the (Hamiltonian) constraint.

Question. Can antibracket $(S, S)$ reproduce flow eq.?

$$\{S, S\} \leftrightarrow (S, S)$$

# Outline

1. Introduction
2. BV formalism revisited
3. 1st-class constraints from BV
4. Summary

# BV formalism revisited 1/4

recipe

1. For all fields $\Phi^n = (\phi, A, c, \dots)$ introduce antifields $K_n = (K_\phi, K_A, K_c, \dots)$.
2. For arbitrary functionals $F[\Phi, K], G[\Phi, K]$, define antibracket

$$(F, G) := \int d^d x \left\{ \frac{\delta^R F}{\delta \Phi^n(x)} \frac{\delta^L G}{\delta K_n(x)} - \frac{\delta^R F}{\delta K_n(x)} \frac{\delta^L G}{\delta \Phi^n(x)} \right\}.$$

3. Define extended action $S[\Phi, K]$ as a solution of the master eq. $(S, S) = 0$:

$$S = S_c + S_K = S_c - \int d^d x \left( \boldsymbol{\delta}_{\mathrm{BRS}} \Phi^n \right) K_n.$$

# BV formalism revisited 2/4

some observations

- Extended action is given by classical action plus a linear combination of antifields $S = S_c + (K\text{-linear})$
- A linear combination of antifields generates gauge trans. $(S_K, \Phi^n) = \boldsymbol{\delta}_{\mathsf{BRS}} \Phi^n$.
- Nilpotent transformation $(S, (S, \forall F)) = 0 \longrightarrow$ cohomology
- Antifield $=$ canonical momenta

$$(\Phi^m(x), K_n(y)) = \delta^m_n \delta^{(d)}(x-y), \quad (K_m, K_n) = 0$$

# BV formalism revisited 3/4

These observations would lead to the following identifications;

$$
\begin{aligned}
q &\leftrightarrow \Phi; \text{generalized coordinates} \\
\pi_q &\leftrightarrow K; \text{canonical momenta} \\
\{\cdot,\cdot\}_P &\leftrightarrow (\cdot,\cdot); \text{brackets} \\
H &\leftrightarrow S; \text{time evolution generator} \\
\varphi &\leftrightarrow K; \text{1st-class constraints}.
\end{aligned}
$$

# BV formalism revisited 4/4

1st-class constraints of systems can be systematically obtained by a simple prescription

$$(S, K_n)\Big|_{K=0} \sim 0.$$

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# 1st-class constraints from BV 1/4

ex) scalar QED

$$S_c[A,\phi]=\int d^dx\Big\{\frac{1}{4}F_{\mu\nu}F^{\mu\nu}+V(\phi)+\frac{1}{2}L_{IJ}(\phi)D^\mu\phi^I D_\mu\phi^J\Big\}$$

Gauss's law is correctly reproduced (in Lagrangian variables):

$$(S,K_A^0)=\partial_\mu F^{0\mu}-L_{IJ}(\phi)(ie\phi^I)D^0\phi^J\sim 0.$$

- Perfect agreement in terms of Hamiltonian variables.

# 1st-class constraints from BV 2/4

ex2) scalar coupled to gravity

$$S_c[g,\phi] = \int d^d x \sqrt{g} \Big\{ V(\phi) - R_{(d)} + \frac{1}{2} L_{IJ}(\phi) g^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J \Big\}$$

With the help of ADM decomposition

$$g_{\mu\nu} = \begin{pmatrix} N^2 + \lambda^k \lambda_k & \lambda_j \\ \lambda_i & h_{ij} \end{pmatrix},$$

the prescription again reproduces 1st-class constraints:

$$(S, K_N)\Big|_{K=0} \sim 0, \quad \text{(Hamiltonian const.)}$$

$$(S, K_\lambda^j)\Big|_{K=0} \sim 0. \quad \text{(momentum const.)}$$

# 1st-class constraints from BV 3/4

ex3) (classical) “bulk” action

$$S_c[g, \phi, A]$$

$$= \int d^d x \sqrt{g} \left\{ V(\phi) - R_{(d)} + \frac{1}{2} L_{IJ}(\phi) g^{\mu\nu} D_\mu \phi^I D_\nu \phi^J + \frac{1}{4} B(\phi) F^a_{\mu\nu} F^{a\mu\nu} \right\}$$

The prescription $(S, K_n)|_{K=0} \sim 0$ yields

$$(S, K_A^{a0})\Big|_{K=0} \sim 0, \quad \text{(Gauss's law)}$$

$$(S, K_N)\Big|_{K=0} \sim 0, \quad \text{(Hamiltonian const.)}$$

$$(S, K_\lambda^j)\Big|_{K=0} \sim 0. \quad \text{(momentum const.)}$$

# 1st-class constraints from BV 4/4

e.g. Hamiltonian constraint (flow equation)

$$
\begin{aligned}
&(S, K_N) \\
&= \sqrt{h}\Bigg\{ V(\phi) - R_{(d-1)} + \frac{1}{2} L_{IJ}(\phi) h^{ij} D_i \phi^I D_j \phi^J + \frac{1}{4} B(\phi) h^{ik} h^{jl} F^a_{ij} F^a_{kl} \\
&\qquad + \frac{1}{h}\left( \frac{1}{d-2}\pi^2 - \pi^{ij}\pi_{ij} \right) - \frac{1}{2h}(L^{-1}(\phi))^{IJ}\pi_I \pi_J - \frac{1}{2hB(\phi)} h_{ij}\pi^{ai}\pi^{aj} \Bigg\} \\
&\qquad + (K \ \text{terms})
\end{aligned}
$$

BV correctly reproduces the known forms of the flow eq.

[de Boer-Verlinde-Verlinde '99, Fukuma-Matsuura-Sakai '00, KK-Sakai '15]

# Summary

- The motivating analogy $\{S, S\} \leftrightarrow (S, S)$ works.
- 1st-class constraints are systematically obtained by
$$(S, K_n)\Big|_{K=0} \sim 0.$$
- Another analogy $\varphi \leftrightarrow K$ would enable us to treat (1st-class) constraints consistently just in Lagrangian variables.
- future direction: 2nd-class constraints, higher spin, etc.

# Appendix

# Some details of the BV formalism

Antifields are assigned (quantum) numbers as follows so that $S_K = -\int(\boldsymbol{\delta}_{\mathsf{BRS}}\Phi^n)K_n$ is bosonic and has ghost # zero:

| antifield | $\epsilon[\cdot]$ mod 2 | ghost # |
|---|---|---|
| $K_n$ | $\epsilon[\Phi^n]+1$ | $-\mathsf{gh}[\Phi^n]-1$ |

.

For arbitrary functionals $F[\Phi, K], G[\Phi, K]$ and $H[\Phi, K]$,

- $(F, G) = (-)^{(\epsilon[F]+1)(\epsilon[G]+1)}(G, F)$,
- $(-)^{\epsilon[F]\epsilon[H]+\epsilon[G]}(F, (G, H)) + (\text{cyclic terms}) = 0$.

# BV cohomology and gauge fixing

Since $(S, \cdot)$ is nilpotent on the space of functionals $F[\Phi, K] \in \mathcal{F}$, we can define cohomology as usual:

$$
\begin{aligned}
\mathcal{C} &:= \{X \in \mathcal{F} | (S, X) = 0\}, \\
\mathcal{E} &:= \{X \in \mathcal{F} | \exists Y \in \mathcal{F} \; s.t. \; X = (S, Y)\}, \\
\mathcal{H} &:= \mathcal{C}/\mathcal{E}.
\end{aligned}
$$

Then gauge fixing is realized by adding an exact term to an old extended action

$$
S \mapsto S' := S + (S, \Psi[\Phi, K]).
$$

# ADM decomposition in BV formalism

A definition

$$g_{\mu\nu} = \begin{pmatrix} N^2 + \lambda^k \lambda_k & \lambda_j \\ \lambda_i & h_{ij} \end{pmatrix}$$

and a requirement (or definition)

$$(\boldsymbol{\delta}_{\mathsf{BRS}} g_{\mu\nu}) K_g^{\mu\nu} = (\boldsymbol{\delta}_{\mathsf{BRS}} N) K_N + (\boldsymbol{\delta}_{\mathsf{BRS}} \lambda_j) K_\lambda^j + (\boldsymbol{\delta}_{\mathsf{BRS}} h_{ij}) K_h^{ij}$$

give new antifields in terms of old ones:

$$K_N := 2N K_g^{00}, \quad K_\lambda^j := 2\lambda^j K_g^{00} + 2K_g^{j0}, \quad K_h^{ij} := K_g^{ij} - \lambda^i \lambda^j K_g^{00}.$$

# Explicit forms of the other (1st-class) constraints

$$(S, K_A^{a0}) = D_\mu\Big[N\sqrt{h}B(\phi)F^{a0\mu}\Big] - \frac{\sqrt{h}}{N}L_{IJ}(\phi)\Big(D_0\phi^I - \lambda^j D_j\phi^I\Big)(iT^a\phi)^J$$

$$+ (K \text{ terms}),$$

$$(S, K_\lambda^i) = \sqrt{h}\Bigg\{\frac{2}{N}h^{ij}(R_{0j} - \lambda^k R_{kj}) - \frac{1}{N}L_{IJ}(\phi)h^{ij}(D_0\phi^I - \lambda^k D_k\phi^I)D_j\phi^J$$

$$- \frac{B(\phi)}{N}h^{ij}h^{kl}F_{jl}^a(F_{0k}^a - \lambda^p F_{pk}^a)\Bigg\} + (K \text{ terms}).$$