Holography, Quantum Entanglement and Higher Spin Gravity
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# Relative Entropy and Conformal Interfaces

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Based on arXiv1702.*****

Collaboration with
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# Relative Entropy

[cf: Ugajin's talk]

$\sigma$ , $\rho$ :density matrices

**Relative Entropy**

$$S(\rho||\sigma) \equiv \mathrm{Tr}\rho\log\rho - \mathrm{Tr}\rho\log\sigma$$

- Distinguishability or "Distance" between quantum states
- $S(\rho||\sigma) = 0 \quad \leftrightarrow \quad \rho = \sigma$
- Related to Holography

[Blanco-Casini-Hung-Myers, 13]

[Jafferis-Lewcowycz-Maldacena-Suh, 15]

[Dong-Harlow-Wall, 15]

- used to the entropic proof of g-theorem

[Casini-Landea-Torroba, 16]

# Renyi Version of Relative Entropy

We choose

$$S^{(n)}(\rho||\sigma) = \frac{1}{1-n}\left[\log\frac{\mathrm{Tr}(\rho\sigma^{n-1})}{\mathrm{Tr}\rho(\mathrm{Tr}\sigma)^{n-1}} - \log\frac{\mathrm{Tr}(\rho^n)}{(\mathrm{Tr}\rho)^n}\right]$$

For normalized $\rho$ and $\sigma$ ,

$$S^{(n)}(\rho||\sigma) = \frac{1}{1-n}\left[\log\mathrm{Tr}(\rho\sigma^{n-1}) - \log\mathrm{Tr}(\rho^n)\right]$$

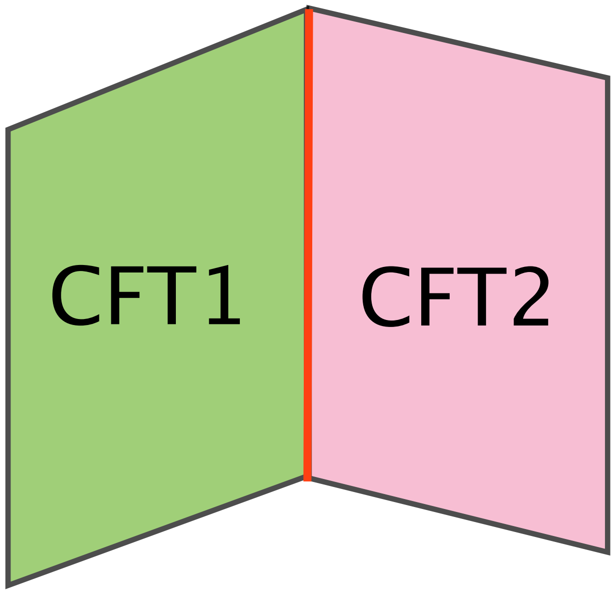
# Conformal Interface

Conformal invariance

⇒ continuity cond.

$$L_n^{(1)} - \tilde{L}_{-n}^{(1)} = L_n^{(2)} - \tilde{L}_{-n}^{(2)}$$

at the interface

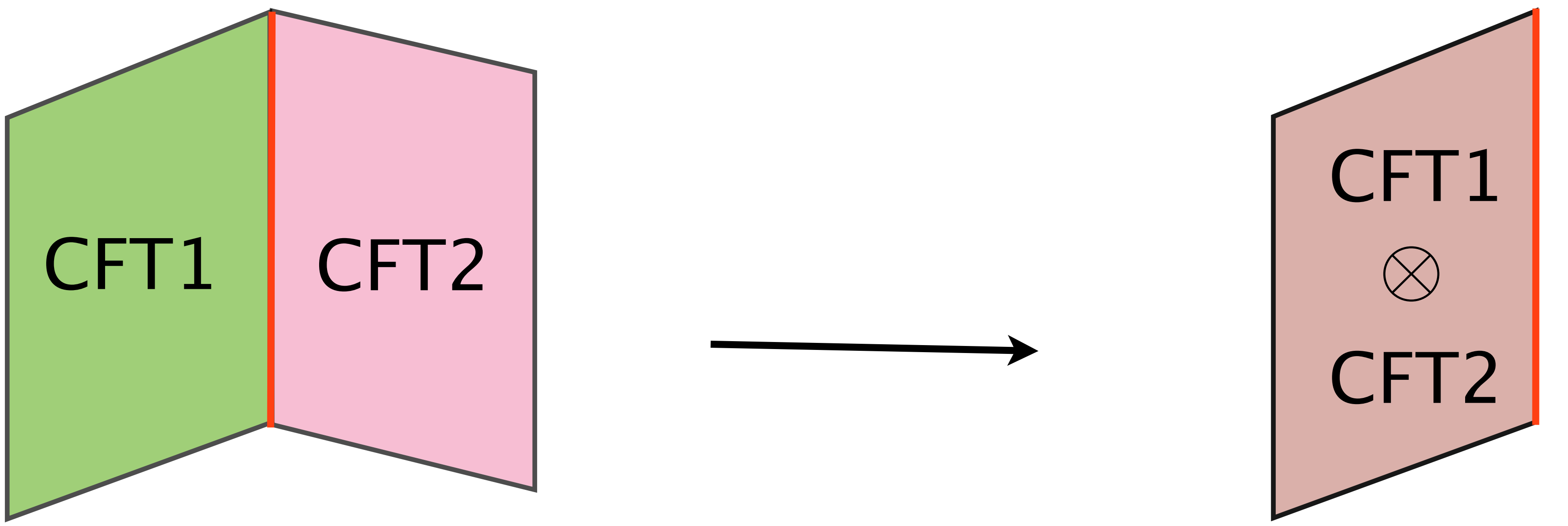



Interface

# Folding trick

[Affleck-Oshikawa, 96]

[Bachas- de Boer-Dijkgraaf-Ooguri, 02]




Interface

boundary
in CFT1⊗CFT2

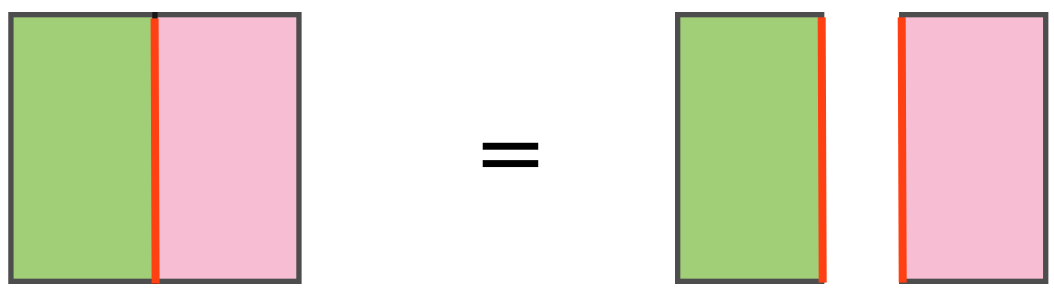
Condition for "Interface state" in CFT1 $\otimes$ CFT2:

$$L_n^{(1)} + L_n^{(2)} - \tilde{L}_{-n}^{(1)} - \tilde{L}_{-n}^{(2)} |B\rangle = 0$$

<u>Special cases</u>

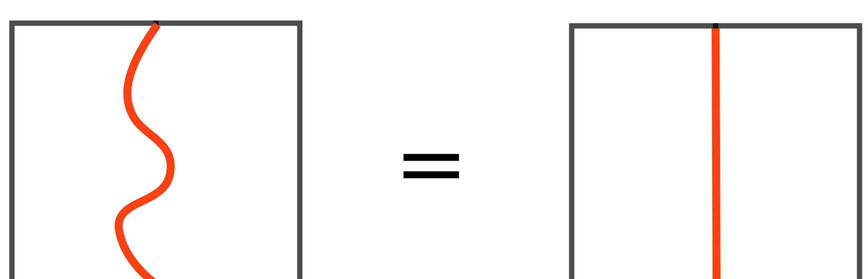
1) If $L_n^{(1)} - \tilde{L}_{-n}^{(1)} |B\rangle = 0$ and $L_n^{(2)} - \tilde{L}_{-n}^{(2)} |B\rangle = 0$

$\Rightarrow$ $|B\rangle = |B_1\rangle \otimes |B_2\rangle$ (perfectly reflective)




2) If $L_n^{(1)} - \tilde{L}_{-n}^{(2)} |B\rangle = 0$ and $L_n^{(2)} - \tilde{L}_{-n}^{(1)} |B\rangle = 0$

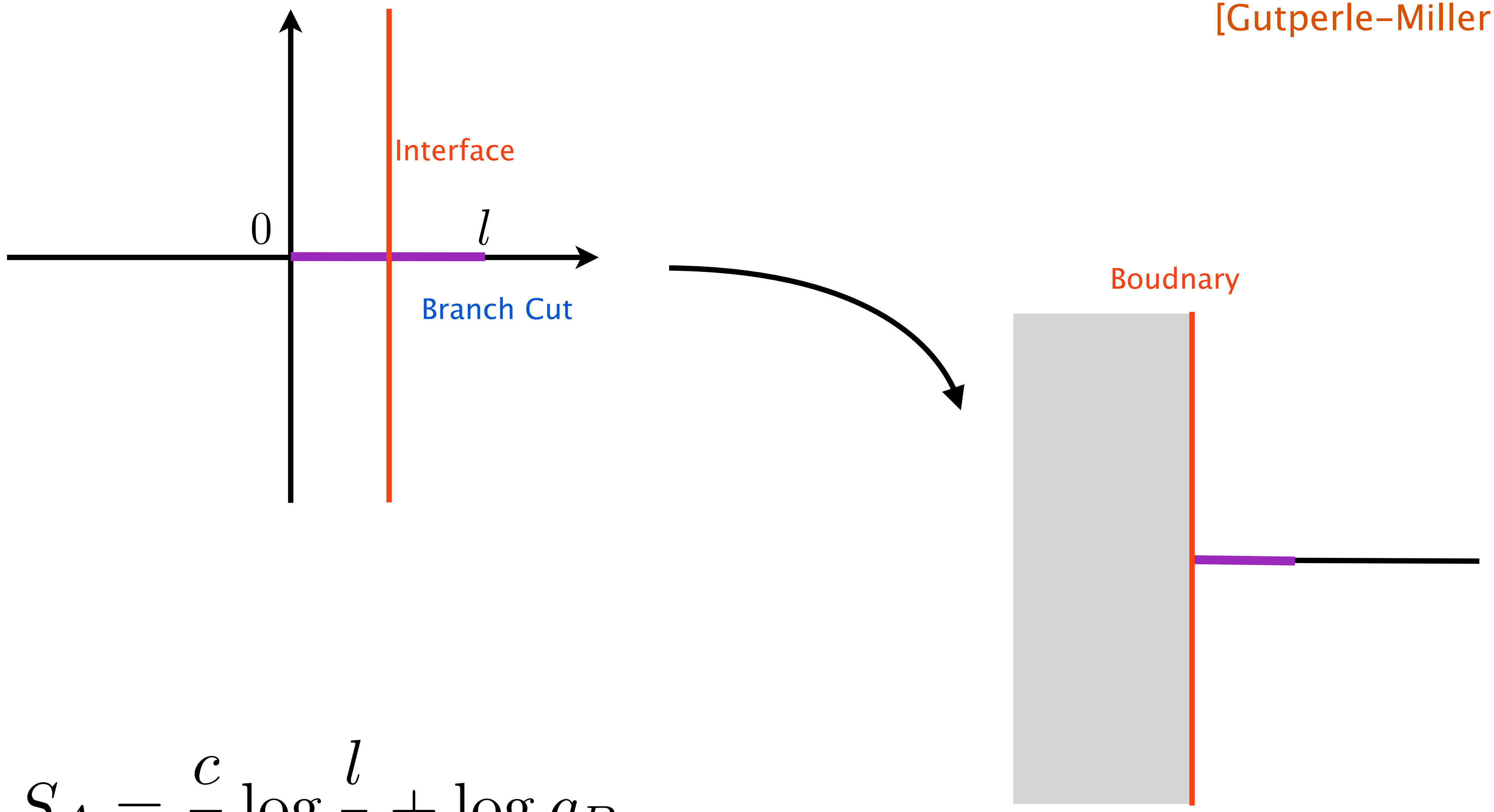
$\Rightarrow$ Interface is topological (perfectly transmissive)



# Interface on the center

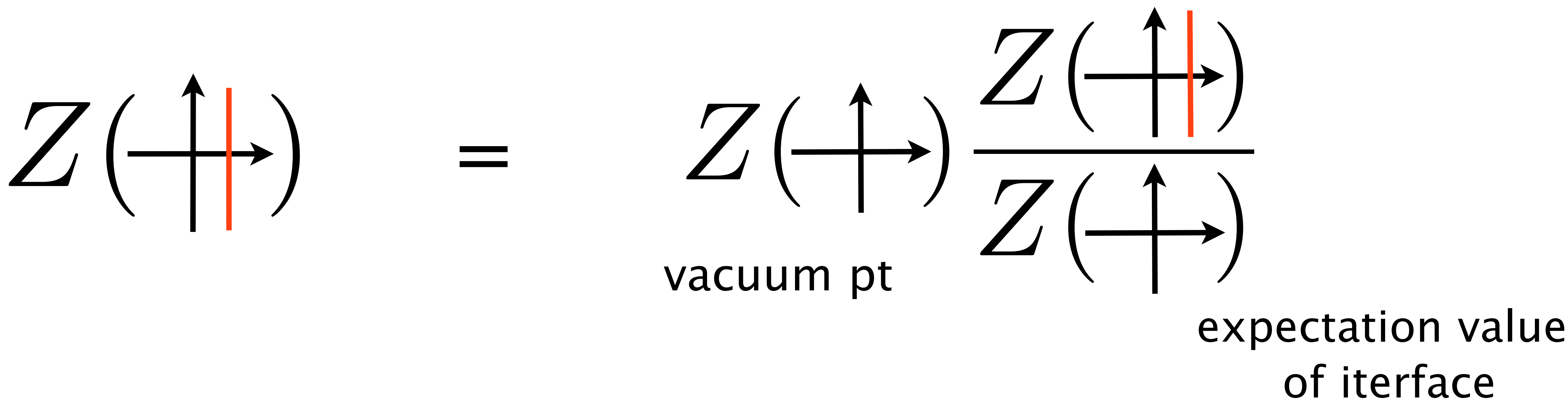
[Brehm-Brunner-Jaud and Schidt-Colinet, 16]

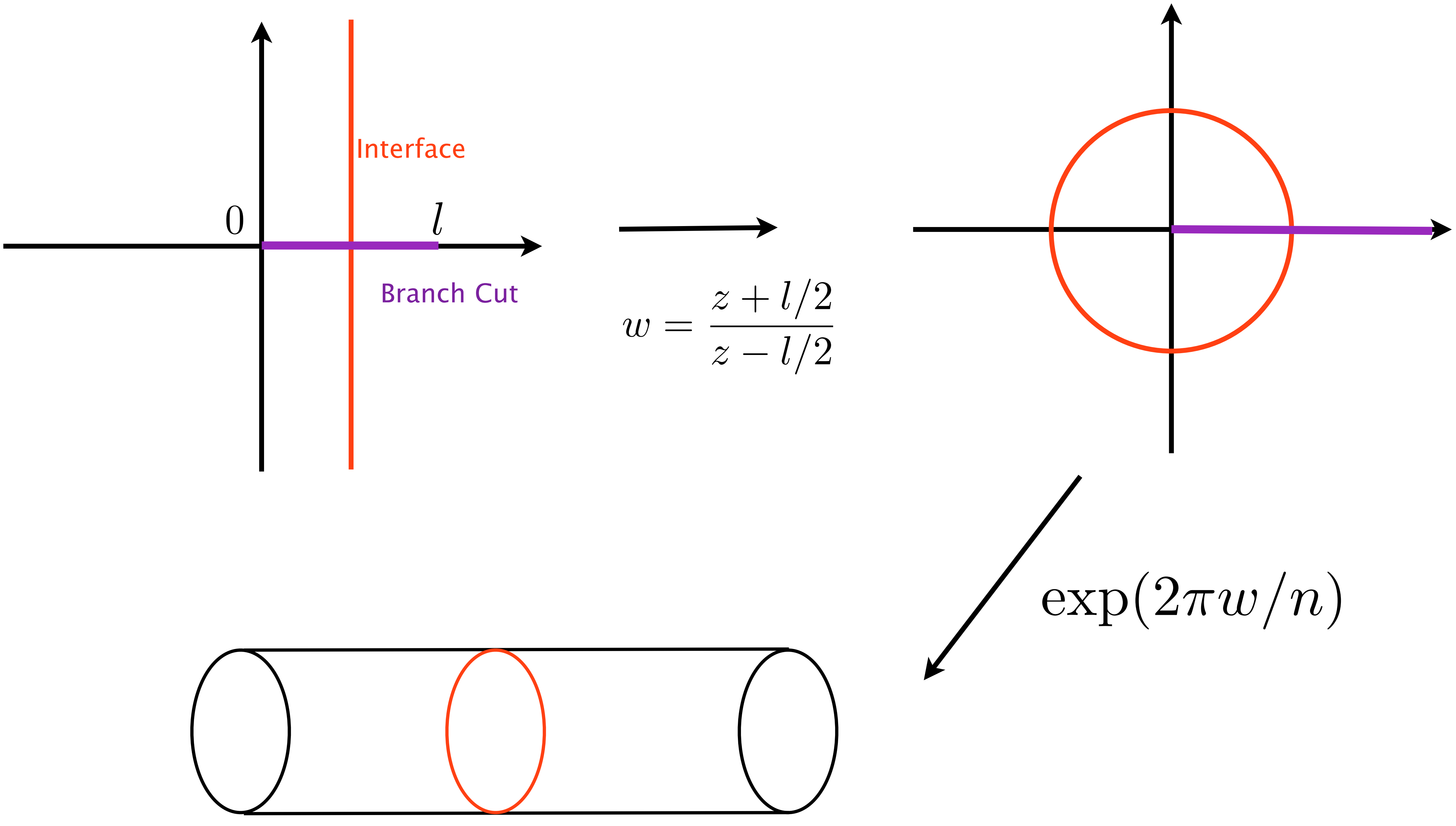
[Gutperle-Miller, 16]



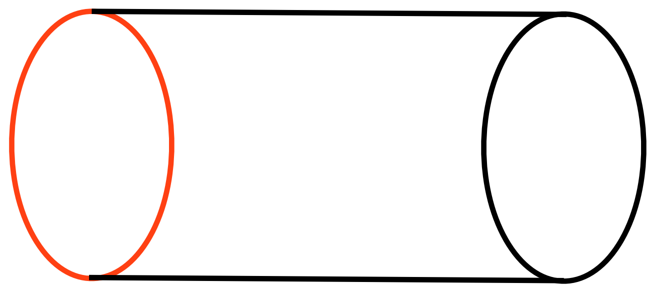

$$S_A = \frac{c}{3} \log \frac{l}{\epsilon} + \log g_B$$

[Calabrese-Cardy, 04]


vacuum pt
expectation value
of iterface

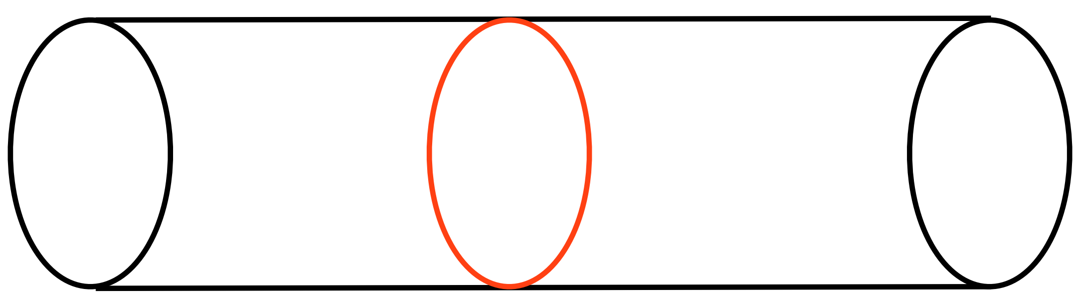

Interface
0
$l$
Branch Cut
$w = \frac{z + l/2}{z - l/2}$
$\exp(2\pi w/n)$

after folding



$$= \langle 0 | B \rangle = g_B$$

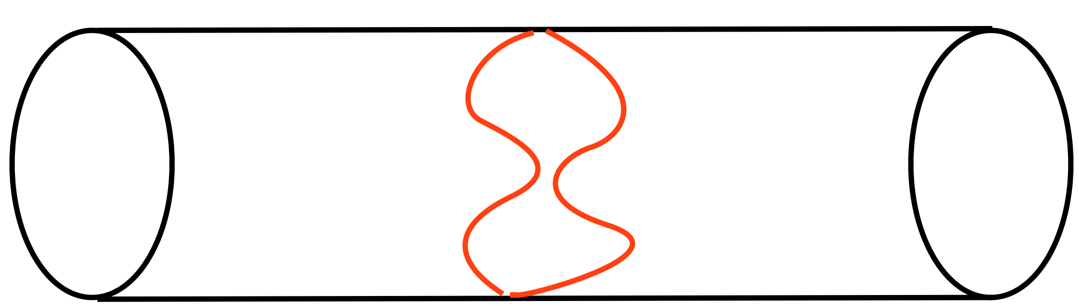
or, by unfolding



$$= \langle I \rangle$$

$I$ : interface operator

※ If the interface is not centered,

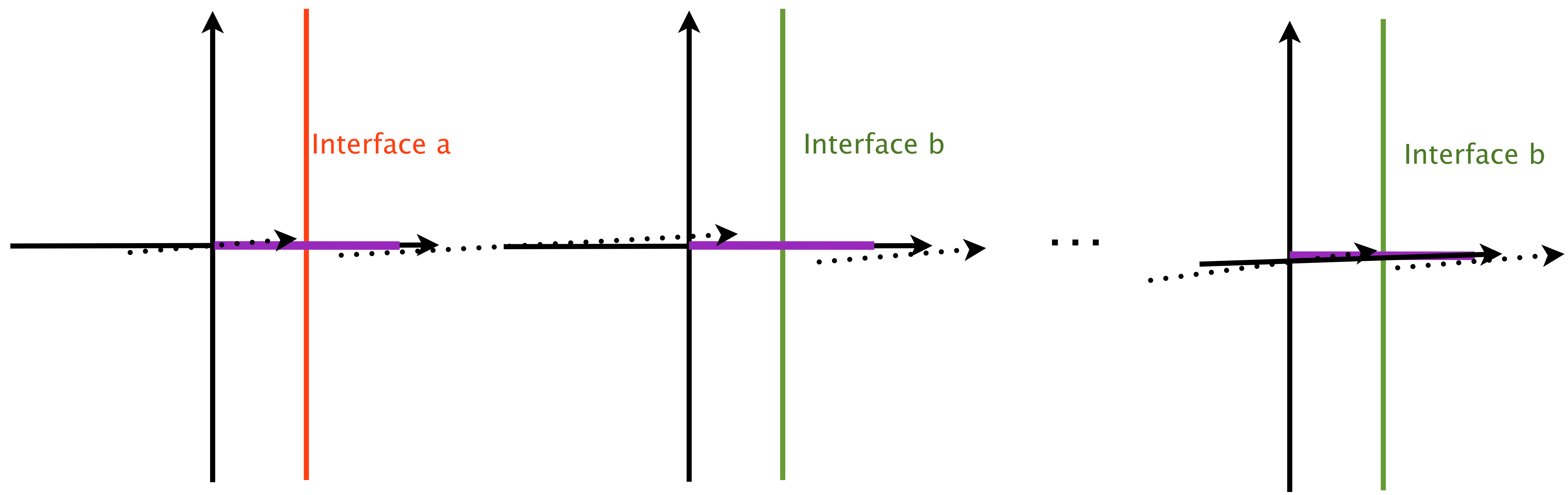


# Relative Entropy of Interfaces

[TN-Ryu-Ugajin-Wen, Work in Progress]

$$\mathrm{Tr}(\rho_a \rho_b^{n-1})$$

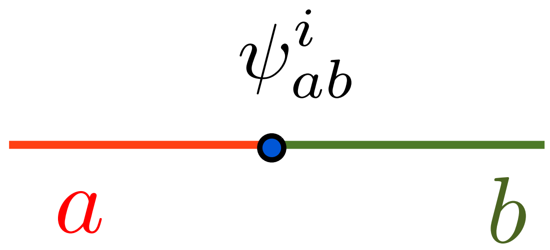
=




→We need to connect two interfaces

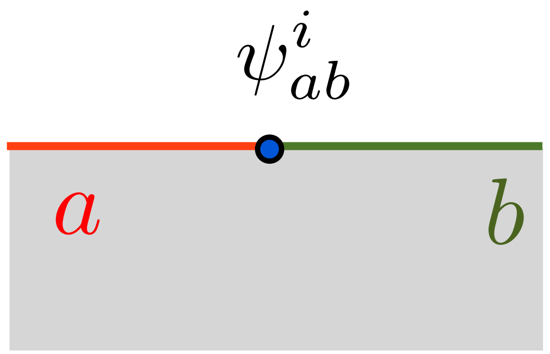
=Change of interface at some points

introduce an "Interface Changing Operator"




with dim. $\Delta_i$

After the folding,

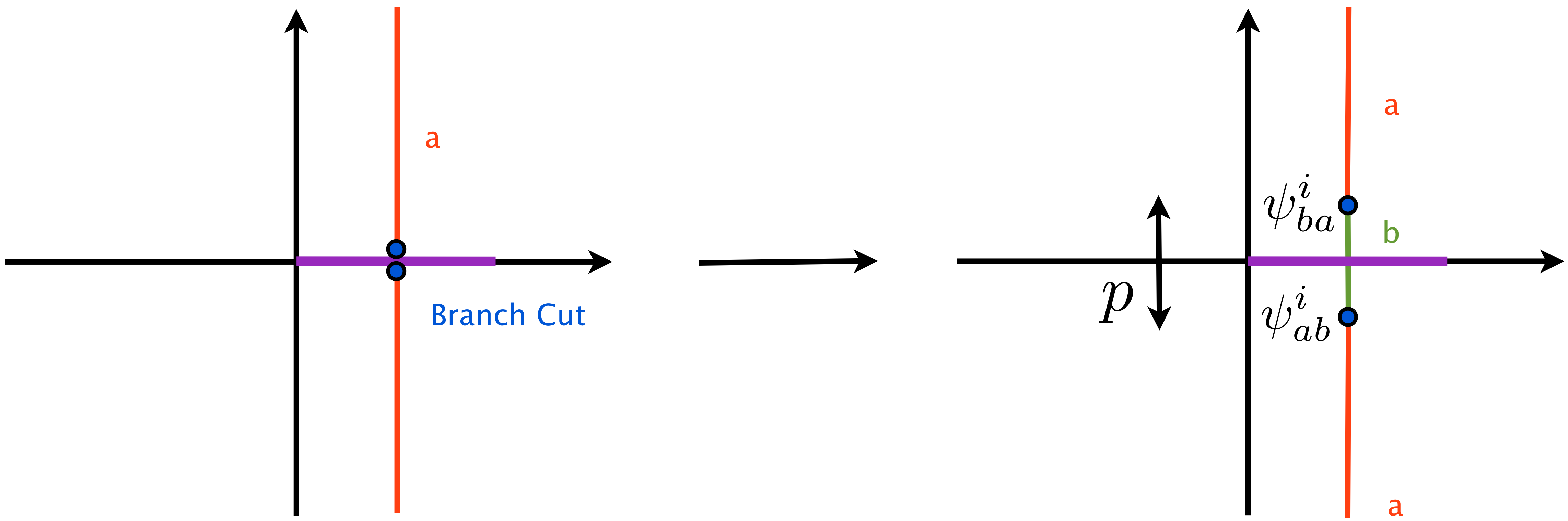



with dim. $\Delta_i$

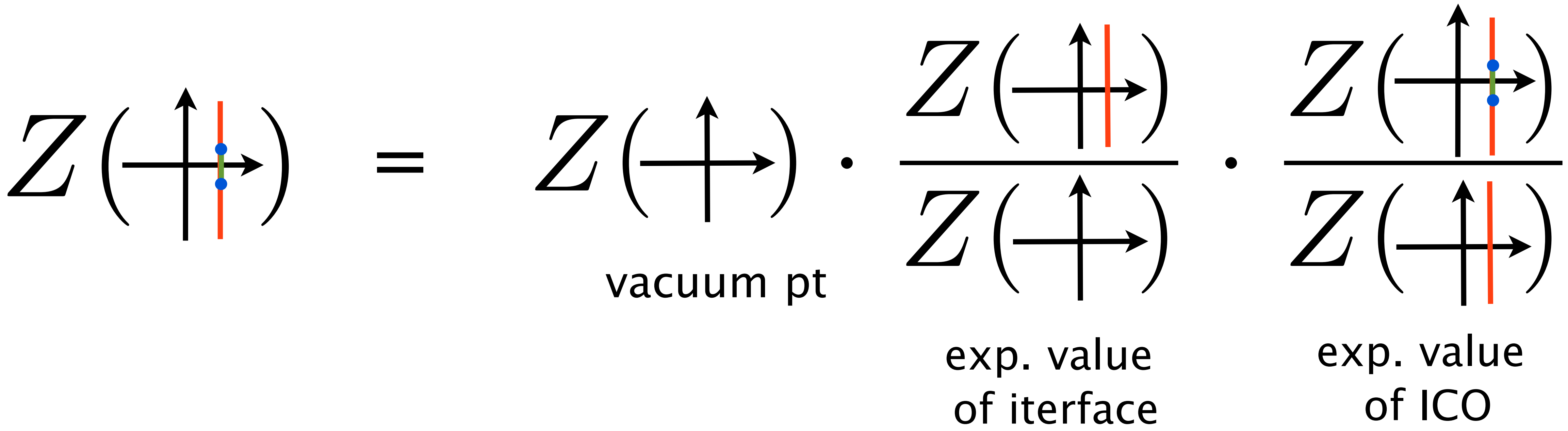
$\Rightarrow$ $\psi^i_{ab}$ : Boundary Changing Operator

To avoid a divergence at $n \to 1$, we introduce a cutoff to $\rho_a$

$$S^{(n)}(\rho_a||\rho_b) = \frac{1}{1-n}\left[\log \frac{\mathrm{Tr}(\rho_a \rho_b^{n-1})}{\mathrm{Tr}\rho_a (\mathrm{Tr}\rho_b)^{n-1}} - \log \frac{\mathrm{Tr}(\rho_a^n)}{(\mathrm{Tr}\rho_a)^n}\right]$$

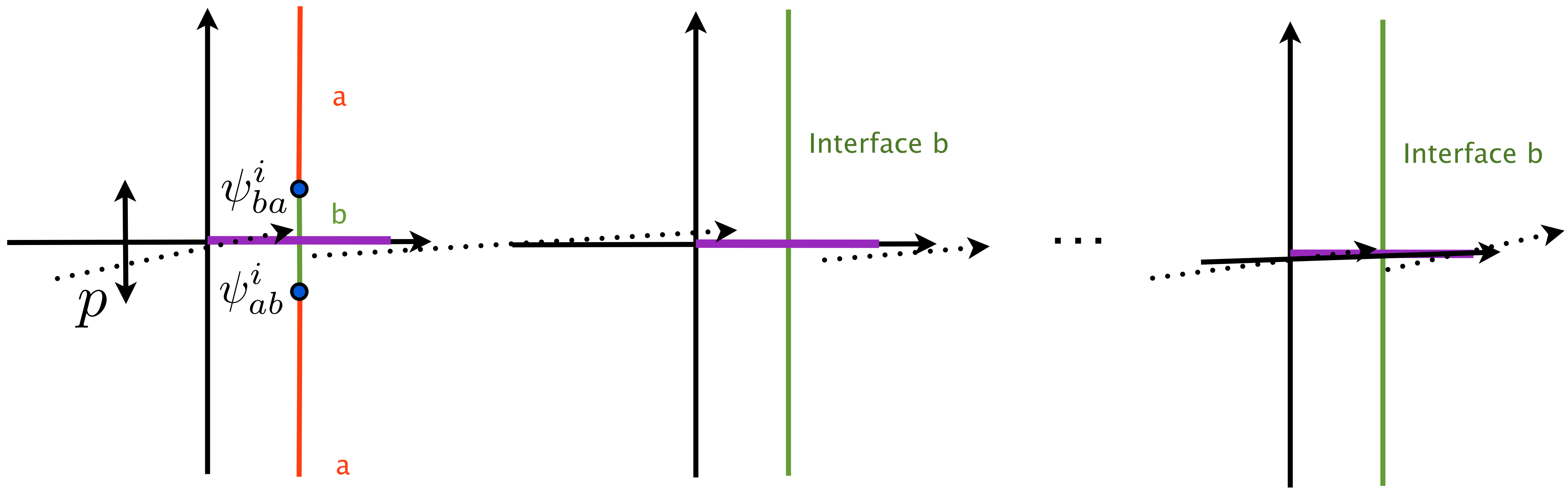



$$|0\rangle\langle 0| \to e^{-\frac{p}{2}H}\psi^i_{ab}|0\rangle\langle 0|\psi^i_{ba}e^{-\frac{p}{2}H}$$



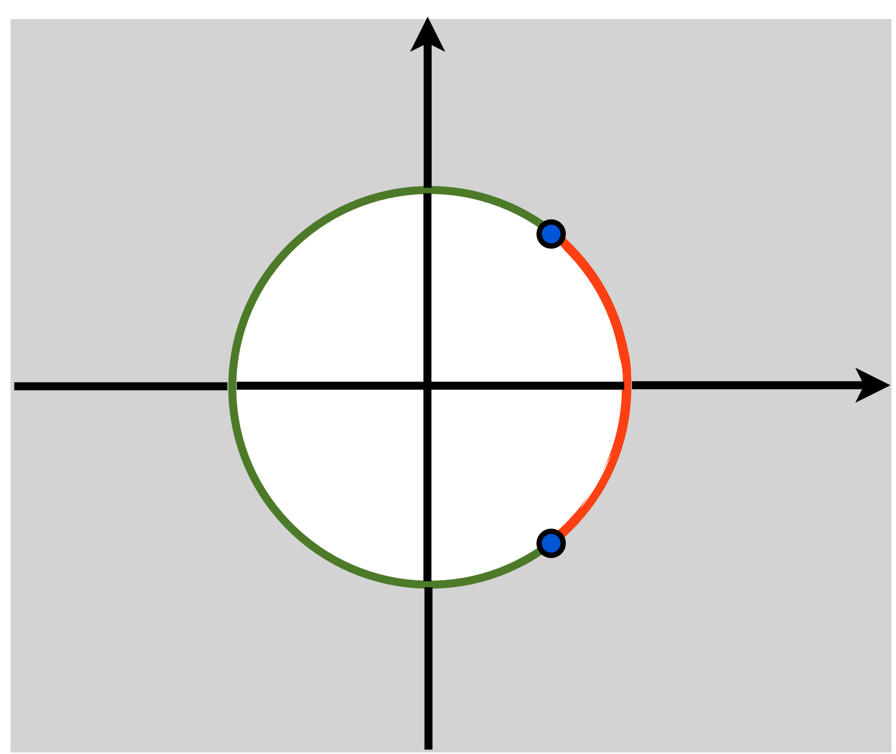

$$= Z_n^{ground} \cdot \langle I_a \rangle \cdot \langle \psi^i_{ab} \psi^i_{ba} \rangle$$

# Evaluation of $\mathrm{Tr}(\rho_a \rho_b^{n-1})$



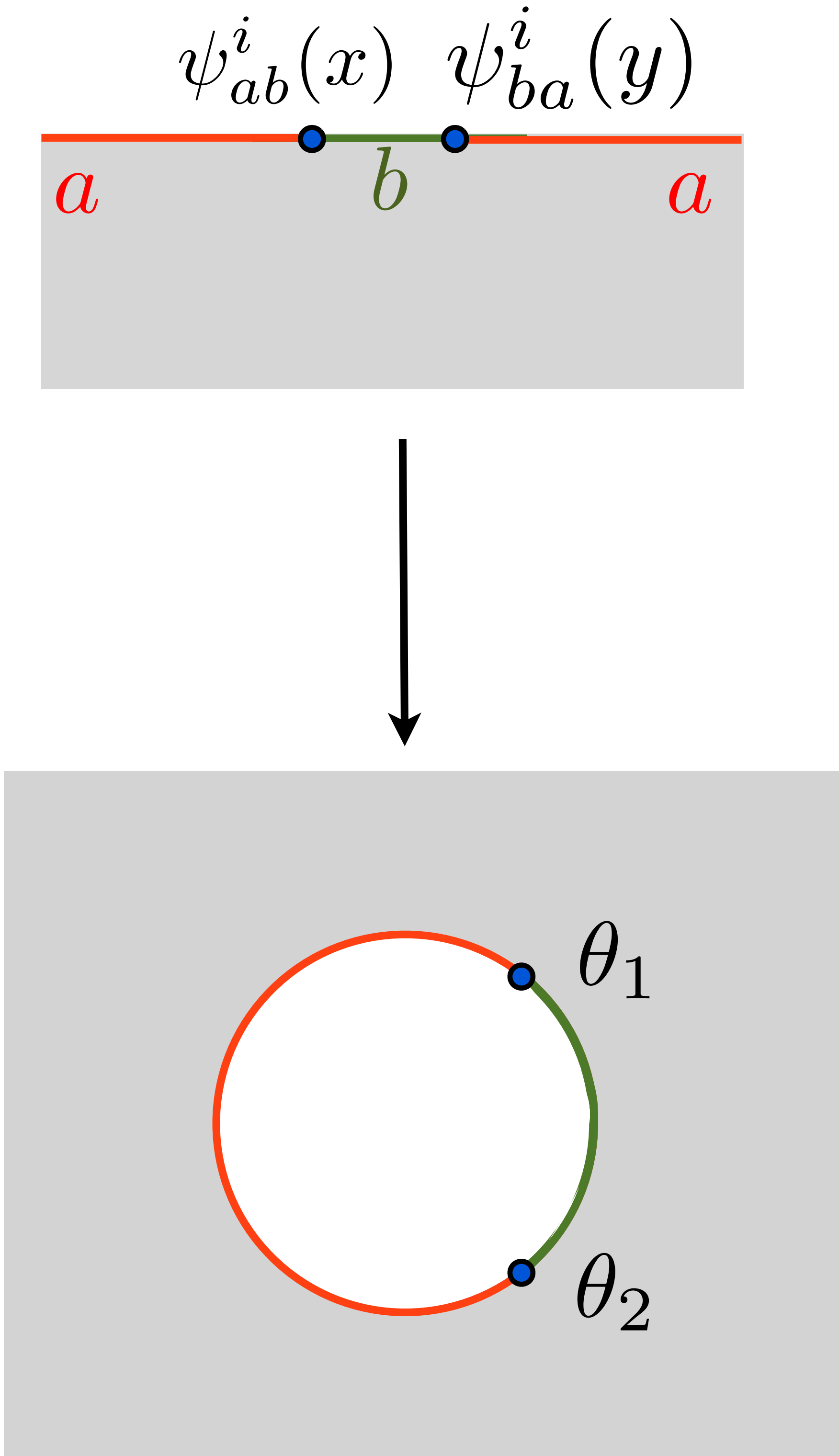

$$z = \left( \frac{w + l/2}{w - l/2} \right)^{\frac{1}{n}}$$

+ folding



two point func
of Boundary ops

# Correlation function of BCO



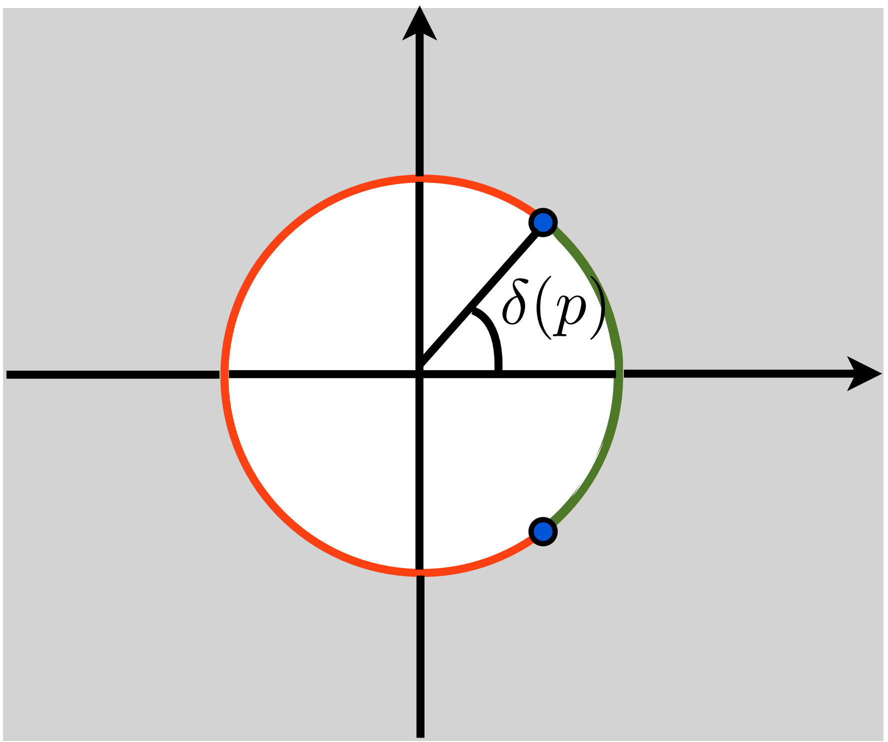

$$\langle \psi^i_{ab}(x)\psi^i_{ba}(y)\rangle_{UHP}$$

$$= \frac{A^i_{ab}}{|x-y|^{\Delta_i}}$$

$$\langle \psi^i_{ab}(x)\psi^i_{ba}(y)\rangle_{D}$$

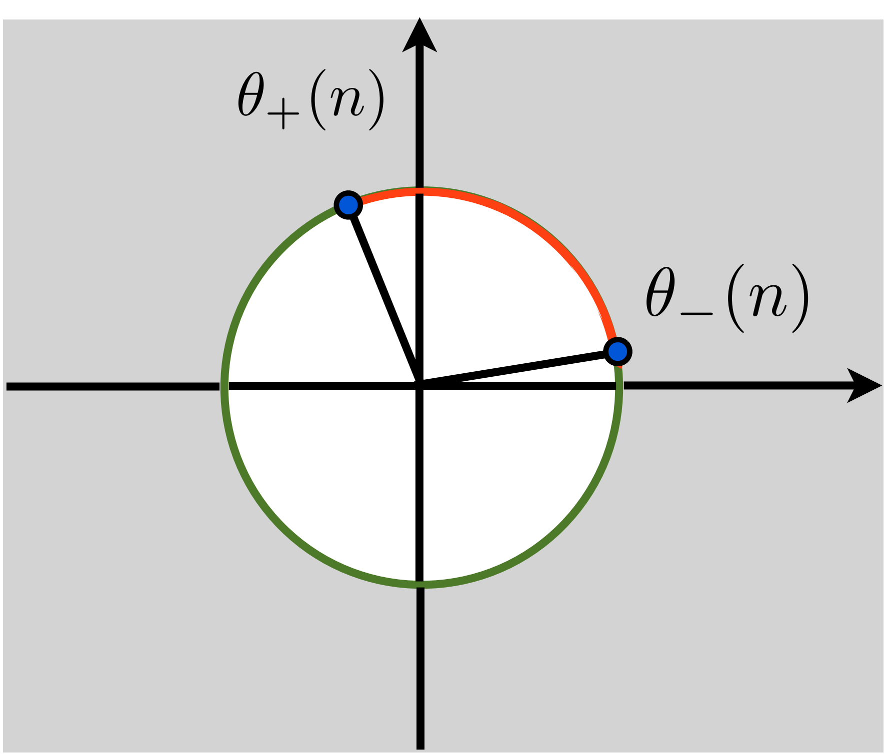
$$= \frac{A^i_{ab}}{|\sin(\theta_1-\theta_2)|^{\Delta_i}}$$

1 sheet ( $\mathrm{Tr}\rho_a$ )




$$\tan\frac{\delta(p)}{2} = \frac{p}{l}$$

n sheet ( $\mathrm{Tr}(\rho_a\rho_b^{n-1})$ )




$$\theta_+(n) = \frac{2\pi}{n} - \frac{\delta(p)}{n}$$

$$\theta_-(n) = \frac{\delta(p)}{n}$$

Using these ingredients,

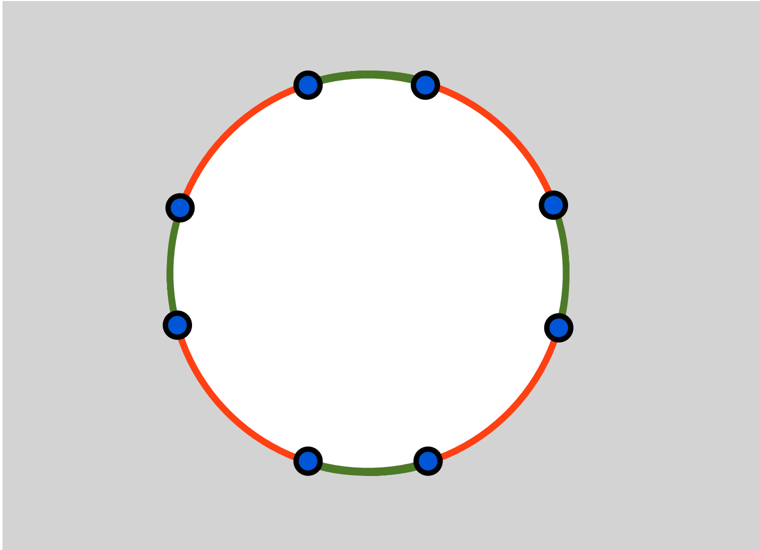
$$\log \mathrm{Tr} \rho_a \rho_b^{n-1} = \log Z_n^{ground} + \log \langle I_a \rangle + \log \langle \psi_{ab}^i \psi_{ba}^i \rangle$$

$$= \log Z_n^{ground} + \log g_a + \log \left( \frac{4l}{n(p^2 + l^2)} \right)^{2\Delta_i} \frac{A_{ab}^i}{|\sin(\frac{2\pi}{n} - \frac{\delta(p)}{n})|^{2\Delta_i}}$$

$$\log \frac{\mathrm{Tr}(\rho_a \rho_b^{n-1})}{\mathrm{Tr}\rho_a (\mathrm{Tr}\rho_b)^{n-1}}$$

$$= \log \frac{Z_n^{ground}}{(Z_1^{ground})^n} + (1-n) \log g_b - \frac{2\Delta_i}{n-1} \log \frac{n|\sin(\frac{2\pi}{n} - \frac{\delta(p)}{n})|}{\sin 2\delta(p)}$$

$$\left( \quad S^{(n)}(\rho||\sigma) = \frac{1}{1-n} \left[ \log \frac{\mathrm{Tr}(\rho \sigma^{n-1})}{\mathrm{Tr}\rho (\mathrm{Tr}\sigma)^{n-1}} - \log \frac{\mathrm{Tr}(\rho^n)}{(\mathrm{Tr}\rho)^n} \right] \quad \right)$$



$$\log \frac{\mathrm{Tr}\rho_a^n}{(\mathrm{Tr}\rho_a)^n}$$

$$= \log \frac{Z_n^{ground}}{(Z_1^{ground})^n} + \log \frac{\langle I_a \rangle}{\langle I_a \rangle^n} + \log \frac{\langle \psi_{ab}^i \psi_{ba}^i \cdots \psi_{ab}^i \psi_{ba}^i \rangle}{\langle \psi_{ab}^i \psi_{ba}^i \rangle^n}$$

cancel at $p \to 0$

$$= \log \frac{Z_n^{ground}}{(Z_1^{ground})^n} + (1-n) \log g_a$$

does not depend on ICO

Relative Renyi Entropy is

$$S^{(n)}(\rho_a||\rho_b) = \log g_b - \log g_a + \frac{2\Delta_i}{n-1}\log\frac{n|\sin(\frac{2\pi}{n} - \frac{\delta(p)}{n})|}{\sin 2\delta(p)}$$

Relative Entropy is

$$S(\rho_a||\rho_b) = \log g_b - \log g_a + \frac{\pi l}{p}\Delta_i$$

$$S(\rho_a||\rho_b) = \log g_b - \log g_a + \frac{\pi l}{p}\Delta_i$$

$$S(\rho_b||\rho_a) = \log g_a - \log g_b + \frac{\pi l}{p}\Delta_i$$

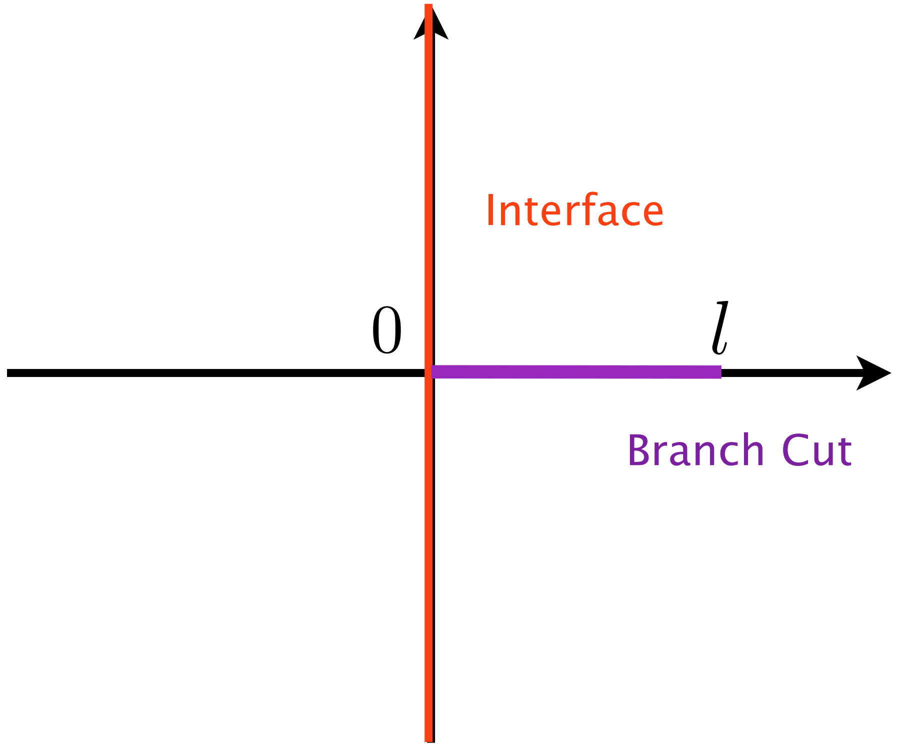
Both of them are positive

⇒ If $g_a \neq g_b$ ,then $\Delta_i \neq 0$

We cannot connect two interface without introducing ICO with $\Delta_i \neq 0$ (even if interfaces are topological)

## Another placement

We can also consider an interface on the entangling surface




- Entanglement Entropy

(1)Topological case

[Brehm-Brunner-Jaud and Schidt-Colinet, 16]
[Gutperle-Miller, 16]

$$S_A = \frac{c}{3}\log\frac{l}{\epsilon} - 2\sum_i |S_{ai}|^2 \log\frac{S_{ai}}{S_{0i}}$$

(2)Conformal interface in Free boson

[Sakai-Satoh, 08]

$$S_A = \frac{\sigma(|s|)}{3}\log\frac{l}{\epsilon} - \log k_1 k_2$$

$s$ : control permeability

$\sigma(s)$ : monotonic, $\sigma(0) = 0$ , $\sigma(1) = 1$

- Relative Entropy

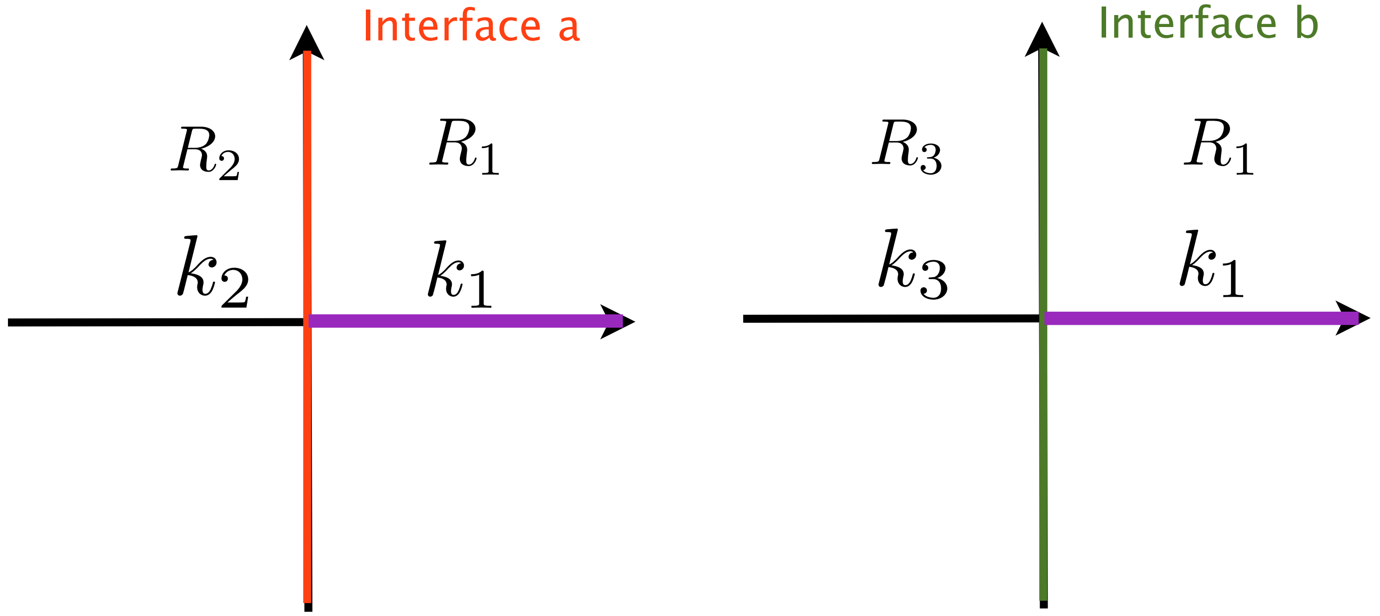
[TN-Ryu-Ugajin-Wen, Work in Progress]

(1)Topological case

$$S(\rho_a||\rho_b) = \sum_i |S_{ai}|^2 \log \frac{|S_{ai}|^2}{|S_{bi}|^2}$$

(2)Conformal Interface in free boson

For same transmission coeff $k_2 R_2 = k_3 R_3$




$$S^{(n)}(\rho_a||\rho_b) = \log \frac{k_2}{k_3} - \frac{1}{1-n} \log \frac{k_1}{\gcd(k_1, k_2)}$$

general case, the computation is difficult except for n=2

# Conclusion /Future Works

- We computed relative entropy between interfaces located on the center.

- Positivity of relative entropy leads that the dimension of Interface changing operator should not be 0

- If we computed relative entropy when we put interface on the entangling surface.

- Can we compute for non-centered case?
- Holographic dual ?