Relative entropy in CFT

Tonomori Ugajin (KITP)
Works in collaboration with Gabor Sarosi,
[1603.03057]+[1611.02959]
And myself [1611.03163] + in progress

Holography, Quantum Entanglement and Higher Spin Gravity, Kyoto

The Motivation

- Microstates of a black hole $|V\rangle \quad |W\rangle$

- They are “almost” same -> The foundation of the statistical mechanical interpretation of BH entropy.

- The difference between them is also important for information loss .

▪ In addition to this, usually we employ a coarse grained view of the system,

$$\rho_W = tr_H |W\rangle\langle W| \qquad \rho_V = tr_H |V\rangle\langle V|$$

-> How can we quantify the difference between ρ_V and ρ_W ?

Relative entropy

$$S(\rho||\sigma) = \mathrm{tr}\rho\log\rho - \mathrm{tr}\rho\log\sigma$$

It measures the distance between ρ and σ

Asymmetric under the exchange $\rho \leftrightarrow \sigma$

It can be written by the modular Hamiltonian $K_\sigma = -\log\sigma$

$$S(\rho||\sigma) = \Delta\langle K_\sigma\rangle - \Delta S \qquad \Delta\langle K_\sigma\rangle = \mathrm{tr}\rho K_\sigma - \mathrm{tr}\sigma K_\sigma$$

$$\Delta S = S(\rho) - S(\sigma)$$

Relative entropy

$$S(\rho||\sigma) = \mathrm{tr}\rho\log\rho - \mathrm{tr}\rho\log\sigma$$

Nice properties

1. Positive definite; $S(\rho||\sigma) \geq 0$
2. A generalization of Free energy ; $S(\rho||\sigma) = \langle\rho K_\sigma\rangle - S(\rho)$
3. Monotonicity $S_A(\rho||\sigma) > S_B(\rho||\sigma) \qquad A \supset B$

Positivity of relative entropy

1. Rigorous formulation of Bekenstein bound in QFTs $\Delta K \geq \Delta S$ [Casini]

2. Proof of
   - Generalized second law [Wall].....
   - Quantum Busso bound [Busso Casini Fisher Maldacena]
   - Averaged null energy condition [Faulkner et al]

Calculations of Relative entropy

A Calculation of relative entropy itself remains to be a difficult task.

Holographic computations: Available only for $S(\rho_V||\rho_0)$

RT formula + the vacuum modular Hamiltonian ex [Blanco Casini Hung Myers]

CFT computations: 2d free scalar[Lashkari]

Calculations of Relative entropy

A Calculation of relative entropy itself remains to be a difficult task.

Holographic computations: Available only for $S(\rho_V||\rho_0)$

RT formula + the vacuum modular Hamiltonian ex [Blanco Casini Hung Myers]..

CFT computations: 2d Free scalar[Lashkari]

Problem: Compute $S(\rho_V||\rho_W)$ for generic two excited states $|V\rangle$ $|W\rangle$ in an arbitrary CFT.

Calculations of Relative entropy

A Calculation of relative entropy itself remains to be a difficult task.

Holographic computations: Available only for $S(\rho_V||\rho_0)$

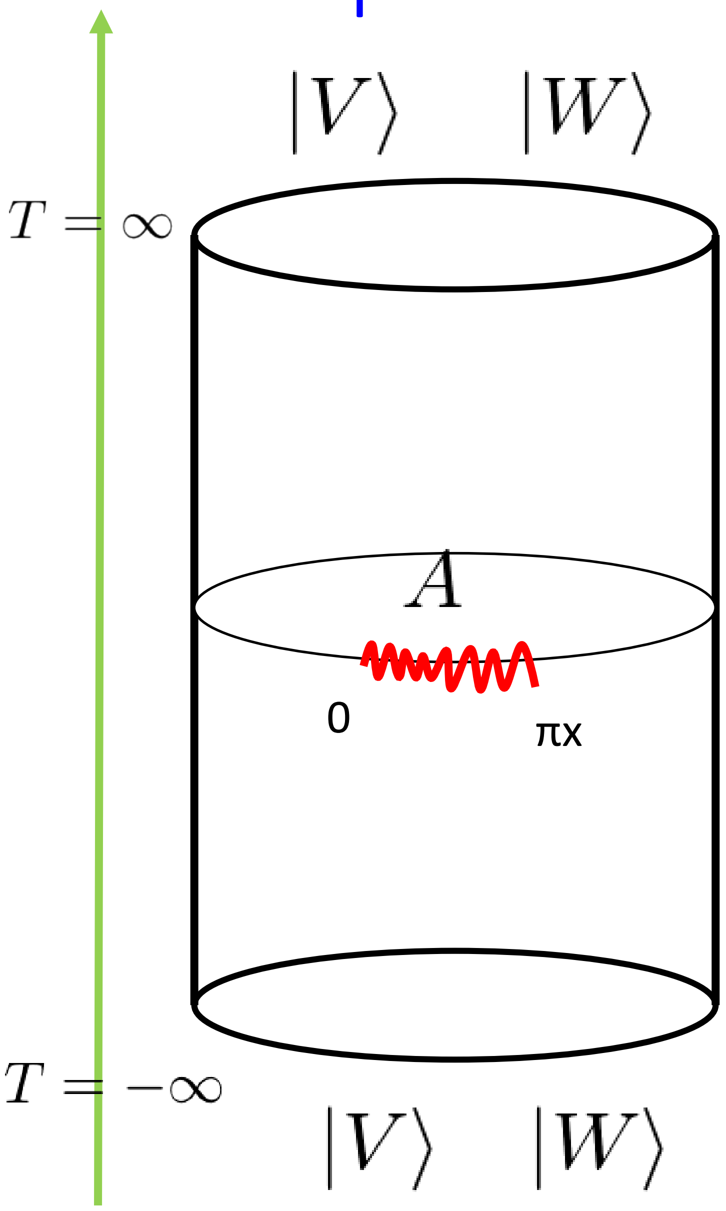
RT formula + the vacuum modular Hamiltonian ex [Blanco Casini Hung Myers]..

CFT computations: 2d Free scalar[Lashkari]

Problem: Compute $S(\rho_V||\rho_W)$ for generic two excited states $|V\rangle$ $|W\rangle$ in an arbitrary CFT.

Main claim: We found a general formula for it, in the small subsystem size limit.

Set up



- A CFTd on a Cylinder $\mathbb{R}\times S^{d-1}$
- A subsystem $[0,\theta_0]\times S^{d-2}$
- Excited states $|V\rangle$ $|W\rangle$ at $T=\pm\infty$
- Reduced density matrices $\rho_V = \mathrm{tr}_{A_c}|V\rangle\langle V|$

$$\rho_W = \mathrm{tr}_{A_c}|W\rangle\langle W|$$

Main result

In the small interval limit θ->0, the leading behavior of relative entropy is given by

$$S(\rho_V||\rho_W) = \frac{\Gamma(\frac{3}{2})\Gamma(\Delta+1)}{\Gamma(\Delta+\frac{3}{2})}\sum_\alpha (C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW})^2(\pi x)^{2\Delta}$$

$$\theta_0 = \pi x$$

$\{\mathcal{O}_\alpha\}$ Set of lightest primary operators with $C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW} \neq 0$

$$C_{\mathcal{O}_\alpha VV} = \langle V|\mathcal{O}_\alpha|V\rangle \qquad \Delta = \dim \mathcal{O}_\alpha$$

Main result

In the small interval limit x->0, the leading behavior of relative entropy is given by

$$S(\rho_V||\rho_W) = \frac{\Gamma(\frac{3}{2})\Gamma(\Delta+1)}{\Gamma(\Delta+\frac{3}{2})}\sum_\alpha (C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW})^2(\pi x)^{2\Delta}$$

$$\theta_0 = \pi x$$

$\{\mathcal{O}_\alpha\}$ Set of lightest primary operators with $C_{\mathcal{O}_\alpha VV} - C_{\mathcal{O}_\alpha WW} \neq 0$

$$C_{\mathcal{O}_\alpha VV} = \langle V|\mathcal{O}_\alpha|V\rangle \qquad \Delta = \dim \mathcal{O}_\alpha$$

At the leading order, the relative entropy is captured by the difference of the one point function of these excited states V and W.

The result

- When the lightest operator is the stress tensor, then,

$$S(\rho_V||\rho_W) = \frac{1}{4C_T}\frac{d}{d-1}\frac{\Gamma(\frac{1}{2})\Gamma(d+1)}{\Gamma(d+\frac{3}{2})}(\varepsilon_V - \varepsilon_W)^2(\theta_0)^{2d}$$

CT: the coefficient of the two point function of the stress tensor. $\langle TT\rangle \sim \frac{C_T}{x^{2d}}$

$\varepsilon_V = \frac{\Delta_V}{\mathrm{Vol}(S^{d-1})}$ energy density of the excited state V.

Sketch of the derivation

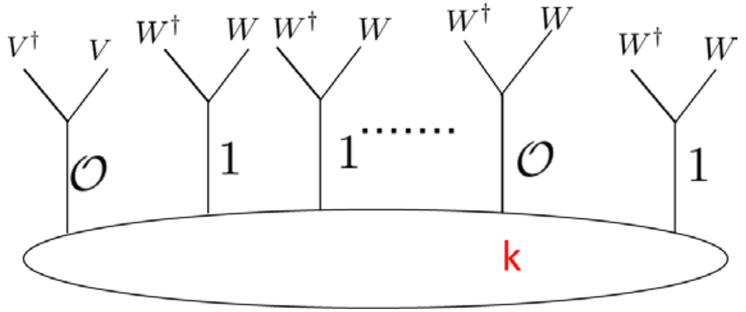
- Step1: Introduce the replica trick[Lashkari]

$$S(\rho_V||\rho_W) = \lim_{n\to 1}\frac{1}{n-1}\left(\mathrm{tr}\rho_V^n - \mathrm{tr}\rho_V\,\rho_W^{n-1}\right)$$

- Step2 : Write each term by a correlation function on $\Sigma_n = S^1 \times H^{d-1}$:

$$\mathrm{tr}\rho_V\,\rho_W^{n-1} = \frac{\langle V^\dagger(\infty_0)V(0_0)\prod_{k=1}^{n-1}W^\dagger(\infty_k)W(0_k)\rangle_{\Sigma_n}}{\langle V(\infty)V(0)\rangle_{\Sigma_1}\langle W(\infty)W(0)\rangle_{\Sigma_1}^{n-1}}$$

- Step 3: expand these correlation functions in the x->0 limit, using OPEs



[Alcaraz,Berganza,Sierra], [Carabrese Cardy Tonni], [Headrick]+ a lot of papers

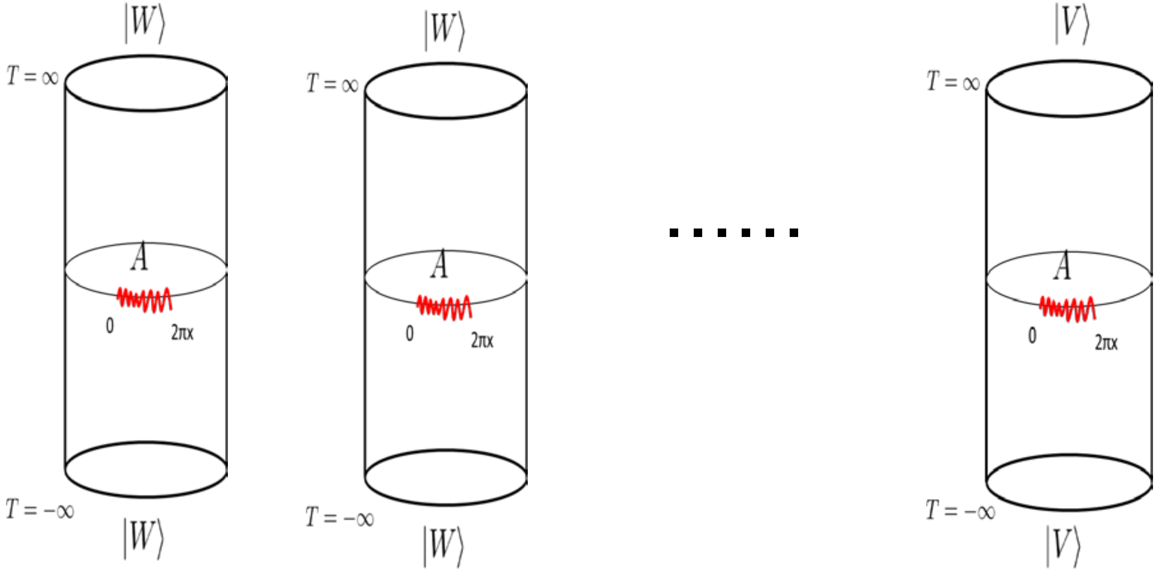
Replica Trick [Lashkari]

$$S(\rho_V||\rho_W) = \lim_{n\to 1}\frac{1}{n-1}\left(\mathrm{tr}\rho_V^n - \mathrm{tr}\rho_V\,\rho_W^{n-1}\right)$$

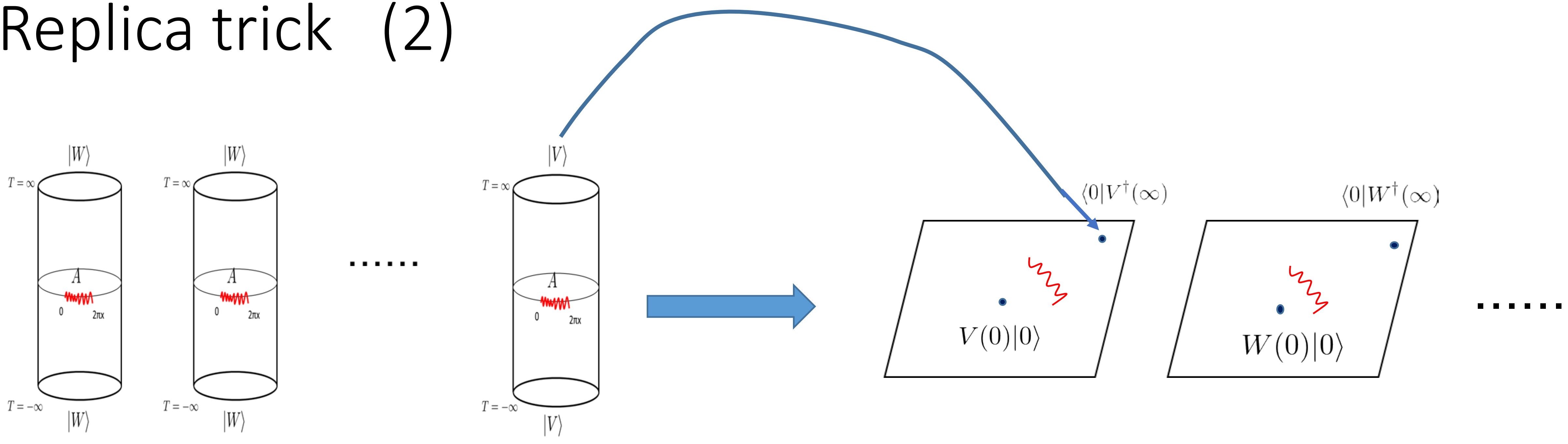
$$= \mathrm{tr}\ \rho_V \log\rho_V - \mathrm{tr}\ \rho_V \log\rho_W$$

Each term in the RHS can be computed by a Path integral on the n sheet cover of the cylinder with the bdy condition specifying the excited states .

$$\mathrm{tr}\rho_V\,\rho_W^{n-1} =$$



Replica trick (2)

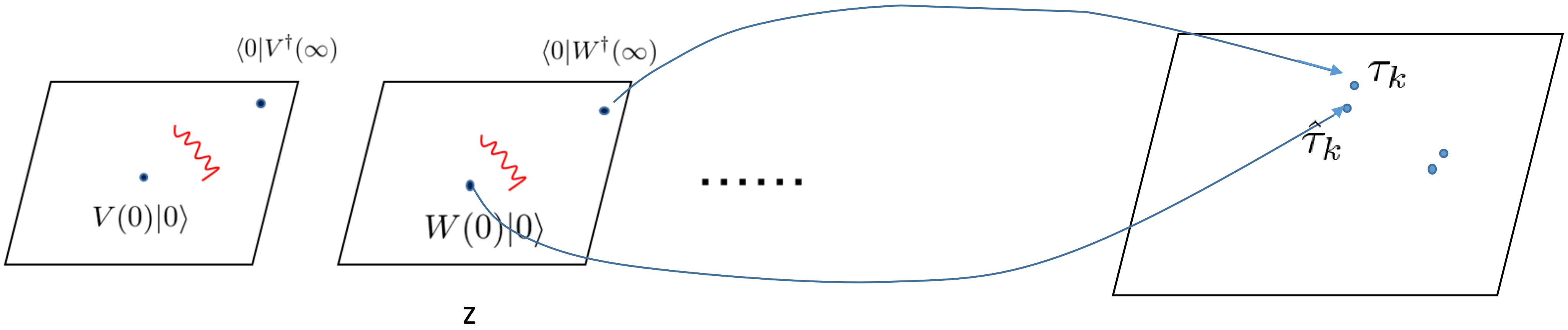


- Each cylinder is mapped to a plane by the exponential map. The excited states are mapped to 0 , and ∞.

  - State operator correspondence: $|V\rangle = V(0)|0\rangle \qquad \langle V| = \langle 0|V^\dagger(\infty)$

We can express each term by a 2n point function on the conical space.

$$\mathrm{tr}\rho_V \rho_W^{n-1} = \frac{\langle V^\dagger(\infty_0)V(0_0)\prod_{k=1}^{n-1} W^\dagger(\infty_k)W(0_k)\rangle_{\Sigma_n}}{\langle V(\infty)V(0)\rangle_{\Sigma_1}\langle W(\infty)W(0)\rangle_{\Sigma_1}^{n-1}}$$



The n sheeted plane is further mapped to $S^1 \times H^{d-1}$ $ds^2_{S^1\times H^{d-1}} = d\tau^2 + du^2 + \sinh^2 u\, d\Omega^2_{d-2}$

$$\tau \sim \tau + 2\pi n$$

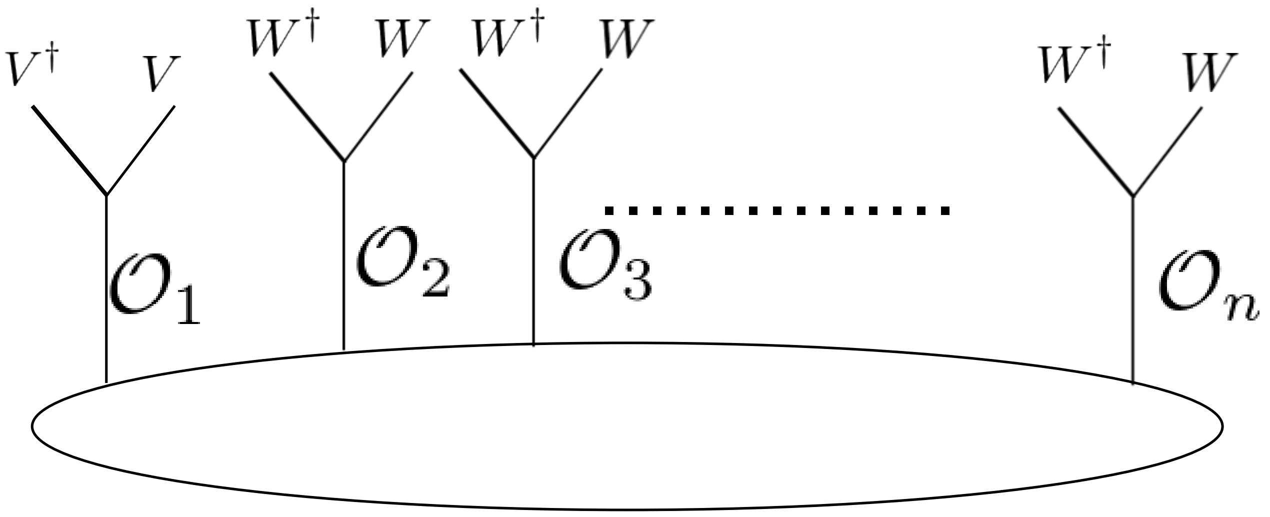
$$\infty_k \longrightarrow \tau_k = (2k+1)\pi + \theta_0$$

U=0

$$0_k \longrightarrow \hat{\tau}_k = (2k+1)\pi - \theta_0$$

On the $S^1 \times H^{d-1}$, two operators lived in the same sheet in the n-sheeted plane get close to each other $\tau_k \to \hat{\tau}_k$ in the small interval limit $\theta_0 \to 0$.

⇒ We can expand the 2n point functions Gn in this OPE channel to obtain the leading behavior.

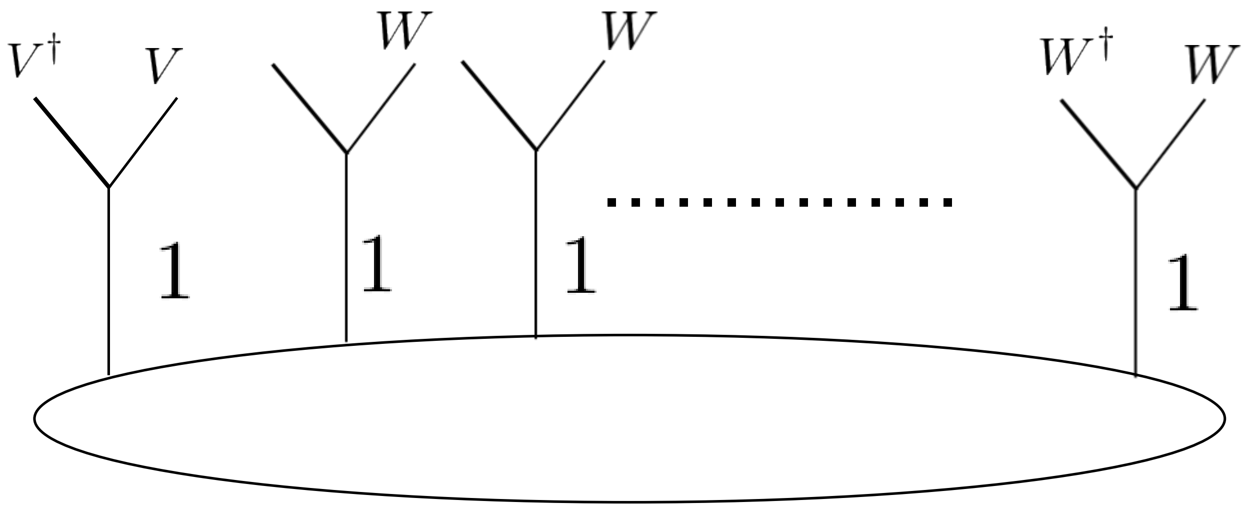
G2n= $\displaystyle\sum_{\mathcal{O}_1,\cdots\mathcal{O}_n}$ 

・By using the OPEs n times, the 2n point functions can be expressed as a sum over all exchanges of internal operators $\mathcal{O}_1, \cdots \mathcal{O}_n$.

・ Each term with the fixed internal operators scales like $x^{\sum_i \Delta_{\mathcal{O}_i}}$: This sum is regarded as an expansion of G2n with respect to the subsystem size x

・In the small subsystem size limit x->0, the effects of light operators dominate in the correlation function.

No nontrivial internal operator exchange



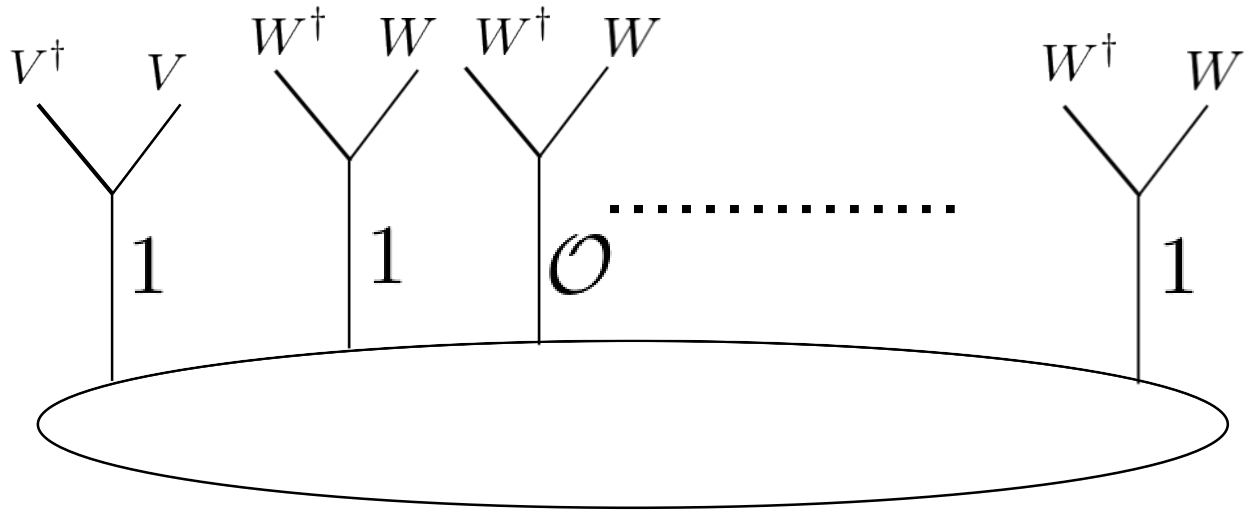
The first term in the expansion comes from the exchange of the n identity operators.

If we keep only this term, the correlation function gets factorized = $\langle V^\dagger V \rangle \langle W^\dagger W \rangle^{n-1}$

The contribution of this part in the relative entropy is vaninshing because

$$S_n(\rho_V || \rho_W) = \log \langle V^\dagger V \rangle_{\Sigma_n} - \log \langle W^\dagger W \rangle_{\Sigma_n} \rightarrow 0$$ in the n->1 limit

Single operator exchange



The next contribution comes from the single lightest operator exchange.

However, this term is zero because one point functions in a CFT is vanishing.

First non trivial contribution comes from double operator exchange.

After the summation the leading part of the 2n point function is given by

$$G_{2n} = \left(C_{\mathcal{O}VV} C_{\mathcal{O}WW} + \frac{(n-2)}{2} C_{\mathcal{O}WW}^2 \right) f(\Delta_{\mathcal{O}}, n)\, x^{2\Delta_{\mathcal{O}}}$$

$f(\Delta_{\mathcal{O}}, n)$ comes from the sum of two point functions of the internal operators ,

$$f(\Delta_{\mathcal{O}}, n) = \sum_{k,m} \langle \mathcal{O}(\tau = 2\pi k) \mathcal{O}(\tau = 2\pi m) \rangle_{S^1 \times H^{d-1}}$$

One can analytically continue this,

$$f(\Delta_{\mathcal{O}}, n) \to (n-1) \frac{\Gamma(\frac{3}{2}) \Gamma(\Delta_{\mathcal{O}} + 1)}{2\Gamma(\Delta_{\mathcal{O}} + \frac{3}{2})}$$

From this, we obtain the formula.

Example(1):Relative entropy in free boson theory

- Lashkari computed the relative entropy between two chiral vertex operators $\mathcal{V}_\alpha = e^{i\alpha X(z)}$, $\mathcal{V}_\beta = e^{i\beta X(z)}$ in 2d free boson theory,

$$S(\rho_{\mathcal{V}_\beta}||\rho_{\mathcal{V}_\alpha}) = (\alpha-\beta)^2(1-\pi x\cot\pi x) = \frac{1}{3}(\alpha-\beta)^2(\pi x)^2 + o(x^4)$$

・By using our formula, this leading behavior is explained by the exchange of the U(1) current operator, $J_z = i\partial X(z)$ with the conformal dimension $\Delta_J = 1$ $C_{J\mathcal{V}_\alpha\mathcal{V}_{-\alpha}=\alpha}$ and the gamma function factor=1/3.

Ex(2):Relative entropy from the modular Hamiltonian

- The relative entropy between an excited states V and the ground state in 2d has been calculated in the somewhat different way by using the modular Hamiltonian, ex [Blanco Casini Hung Myers]..

$$S(\rho_V||\rho_0) = \Delta\langle K\rangle - \Delta S$$

$$K = -\log\rho_0 = 2\pi\int_0^{2\pi x} d\phi \left[\frac{\cos(\phi-\pi x)-\cos(\pi x)}{\sin(\pi x)}\right] T_{00}$$

$$\Delta\langle K\rangle = \langle V|K|V\rangle - \langle 0|K|0\rangle$$

ΔS is the difference of entanglement entropy. When there are no primary operator lighter than the stress tensor ,we can compute ΔS in the small x limit from the vacuum conformal block. Then

$$S(\rho_V||\rho_0) = \frac{4}{15c}\Delta_V^2(\pi x)^4$$

This is a special case of our formula, $S(\rho_V||\rho_W) = \frac{4}{15c}(\Delta_V - \Delta_W)^2(\pi x)^4$

Ex(3):Relative entropy between two generalized free fields

Generalized free fields =The CFT dual of bulk free scalar fields.

The correlation functions of the generalized free fields are calculated as if they are free fields (ie Wick contractions).

This property allows us to compute the relative entropy between two generalized free fields without using the general formula,

$$S(\rho_V||\rho_W) = \frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{\Gamma(2\Delta+\frac{3}{2})}\left[1-|\langle V|W\rangle|^2\right](\pi x)^{4\Delta}$$

We checked this result can be reproduced from the general formula.

Comparison with holographic results

- When one of the states is vacuum, we can compare our results with the holographic predictions ie, area of the RT surface on corresponding dual geometry.

- When the stress tensor is the lightest operator with non zero expectation value, the dual space time is AdS black brane,

$$ds^2 = \frac{L^2}{z^2}\left[dz^2 + T(z)(dr^2 + r^2 d\Omega^2)\right]$$

$$T(z) = \left(1 + \frac{z^d}{4z_h^d}\right)^{\frac{4}{d}}$$

$$z_h^d = \frac{d-1}{2}\frac{L^{d-1}}{l_p^{d-1}}\frac{1}{\epsilon_V}$$

Comparison with holographic results

- ϵ_V is the energy density of the black brane

- We can calculate the area of the bulk minimal surface perturbatively in ϵ_V,

$$S_A = S_0 + \epsilon_V S_1 + \epsilon_V^2 S_2 + \cdots \qquad S_2 = -\frac{\pi^{3/2}}{2^{d+1}} \frac{d\Omega_{d-2}}{(d+1)} \frac{\Gamma(d-1)}{\Gamma(d+1)} \frac{l_p^{d-1}}{L^{d-1}} \epsilon_V^2 \theta_0^{2d}$$

By using

$$C_T = \frac{d+1}{d-1} \frac{\Gamma(d+1)}{\pi^{d/2}\Gamma(d+\frac{3}{2})} \frac{L^{d-1}}{l_P^{d-1}}$$

We found an agreement with the CFT result.

Comparison with holographic results

- In order to derive this result, we need to know the first non linear part of the bulk gravity equation. This is in contrast to the fact that the first law

is related to the linearized EOM.

- Our computation suggests that somehow CFT correctly knows about the

non linear part of the bulk equations.

We also found similar agreement when the lightest operator is spin 0 /spin 1, which corresponds to nontrivial scalar/ vector hair in the bulk.

Rewriting CFT entropy holographically.[work in progress with Agon, Sarosi ?]

- In the computation so far, we only focused the effect of lightest primary operator.

▪It is also possible to include the contributions of descendants.

▪Using OPE block / Bulk integral correspondence we can interpret the result as an integral of the bulk two point function on the RT surface of Pure AdS [=> Talk on Wednesday].

Relative entropy of two disjoint intervals

Relative entropy of two disjoint intervals

- It is also possible to calculate the relative entropy of two disjoint intervals.

- The replica trick used previously is not efficient to derive this.

- Instead we use what we call first law trick, which enable us to read off the form of the modular Hamiltonian from the entanglement entropy.

- The resulting relative entropy is roughly given by the difference of

the connected part of the two point function of the lightest operator.

First law trick in the single subsystem case [Jafferis Maldacena Lewkowitz Suh] [Dong Harlow Wall] [Leichnauer: private communication]

We start from the entanglement entropy of ρ_W : $S_A(\rho_W) = -c_A(l)\mathrm{tr}[\rho_W \mathcal{O}]^2$

Slightly deform the density matrix : $\rho_W \to \rho_W + \delta\rho$

$$\delta S_A(\rho_W) = -c_A(l)\mathrm{tr}[\delta\rho\mathcal{O}]\langle\mathcal{O}\rangle_W = \mathrm{tr}[K_A^W \delta\rho]$$

Since this is true for any deformation $\delta\rho$, we can read off the modular Hamiltonian from it

$$K_A^W = -c_A(l)\langle\mathcal{O}\rangle_W \, \mathcal{O}_A$$

$$S(\rho_V||\rho_W) = \Delta\langle K_W\rangle - \Delta S = c_A(l)(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2$$

First law trick in the single subsystem case(2)

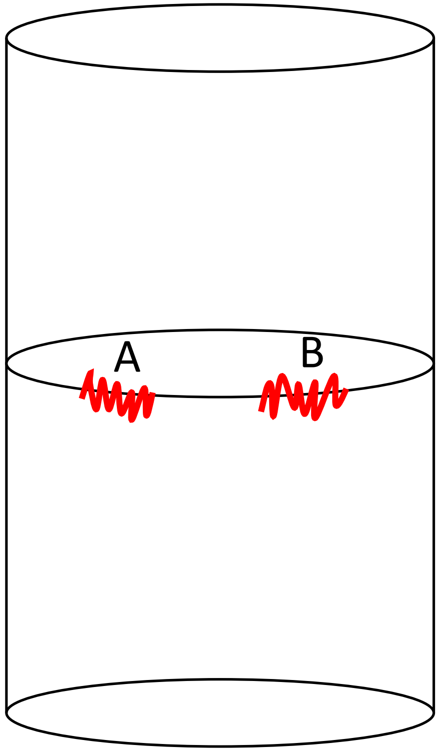
- By using this trick, it is possible to identify the form of the first asymmetric part of the relative entropy,

$$S(\rho_V||\rho_W) = S^{(2)}(\rho_V||\rho_W) + c_{\Delta}^{(3)}\theta_0^{3\Delta}\left(\langle V|\mathcal{O}|V\rangle - \langle W|\mathcal{O}|W\rangle\right)^2\left[\langle V|\mathcal{O}|V\rangle + 2\langle W|\mathcal{O}|W\rangle\right]$$

where $S^{(2)}(\rho_V||\rho_W)$ is the leading symmetric result.

- We can check this against the exact result for a class of states in 2d CFTs.

Relative entropy of two disjoint subsystems



$$S_{A\cup B}(\rho_W) = S_A(\rho_W) + S_B(\rho_W) - I_W(A,B)$$

We find that in the small subsystem limit, $l_A, l_B \to 0$
the mutual information of the excited state is given by

$$I_W(A,B) = (l_A)^{2\Delta}(l_B)^{2\Delta}\frac{\Gamma(\frac{3}{2})\Gamma(2\Delta+1)}{\Gamma(2\Delta+\frac{3}{2})}\Big(\langle W|\mathcal{O}_A\mathcal{O}_B|W\rangle - \langle W|\mathcal{O}_A|W\rangle\langle W|\mathcal{O}_B|W\rangle\Big)^2$$

$$\equiv C_{AB}(l_A,l_B)(M^W_{AB})^2$$

Cf. [Wolf Verstraete Hastings Cirac]

Relative entropy of two disjoint subsystems

$$K_{A\cup B} = K_A + K_B$$

$$-2c_{AB}M^{W}_{AB}\Big[\mathcal{O}_A\mathcal{O}_B - \langle\mathcal{O}_A\rangle_W\mathcal{O}_B - \langle\mathcal{O}_B\rangle_W\mathcal{O}_A\Big]$$

Relative entropy of two disjoint subsystems

$$S_{A\cup B}(\rho_V||\rho_W) = S_A(\rho_V||\rho_W) + S_B(\rho_V||\rho_W)$$

$$+c_{AB}\left[(M_{AB}^V - M_{AB}^W)^2 - 2M_{AB}^W(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2\right]$$

Relative entropy of two disjoint subsystems

$$S_{A\cup B}(\rho_V||\rho_W) = S_A(\rho_V||\rho_W) + S_B(\rho_V||\rho_W)$$

$$+c_{AB}\left[(M_{AB}^V - M_{AB}^W)^2 - \underline{2M_{AB}^W(\langle\mathcal{O}\rangle_V - \langle\mathcal{O}\rangle_W)^2}\right]$$

Asymmetric under the exchange $V \leftrightarrow W$

Although the relative entropy for a single interval starts from the symmetric term, the part involving both A and B contains the asymmetric part from the beginning.

Conclusions

- We found a formula for relative entropy in small subsystem size limit

- The formula reproduces several known results, including holographic

Ones.

We also derived an expression of relative entropy of two disjoint interval.

Several works in progress: Relative entropy in 2D YM, 3D CS, between

Conformal interfaces[with Numasawa Ryu Wen], Numerical computation [with Nakagawa], the effects of descendants[with Agon].