Supersymmetric Rényi Entropy and Defect Operators

based on:

T. Nishioka and IY - 1306.2958 + 1612.02894

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Entanglement and Rényi entropy

We split the Hilbert space

$$\mathcal{H} = \mathcal{H}_V \otimes \mathcal{H}_{-V}$$

Define a density matrix

$$\rho_V = \mathrm{tr}_{-V} |0\rangle\langle 0|, \quad \mathrm{tr}_V (\rho_V) = 1.$$

The entanglement entropy

$$S(V) = -\mathrm{tr}(\rho_V \log \rho_V).$$

Rényi entropy

$$S_n(V) = \frac{1}{1-n} \log \mathrm{tr} \rho_V^n,$$

$$\lim_{n \to 1} S_n(V) = S(V).$$

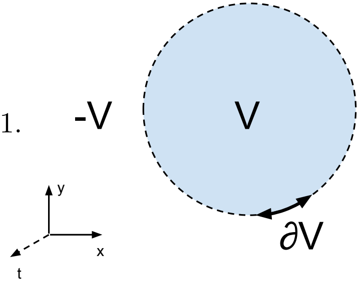


Figure: The entangling region (V), its outside (-V), and the "entangling surface" (∂V)

Euclidean path integral approach (replica trick)

$$S_n = \frac{1}{1-n} \log \left(\frac{Z_n}{(Z_1)^n} \right)$$

- Left: construct the ground state wave-function by integrating from $t = -\infty$.
- Right: identify spacial regions to obtain powers then sew up to obtain the trace.

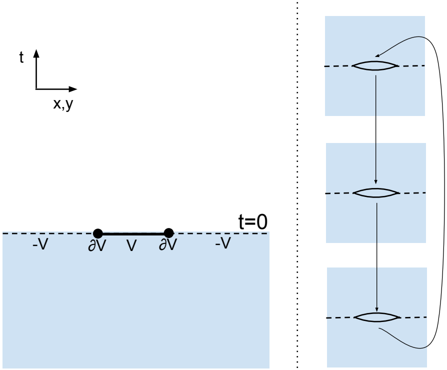


Figure: The "replica trick": constructing Z_n using a multi-sheet path integral.

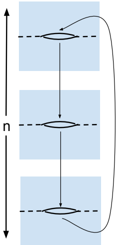
Free fields

For free fields and $n \in \mathbb{N}$ there is an alternative to the multi-sheet integral: use a field redefinition and defects

$$\beta_k = e^{2\pi i \frac{k}{n}}, \quad \begin{cases} k \in \{0, \cdots, n-1\} \ , & \text{bosons} \\ k \in \left\{-\frac{n-1}{2}, \cdots, \frac{n-1}{2}\right\} \ , & \text{fermions} \end{cases}$$

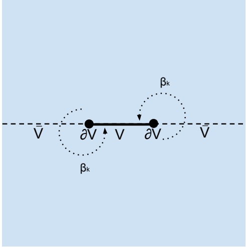
Option 1

Path integral on a branched cover



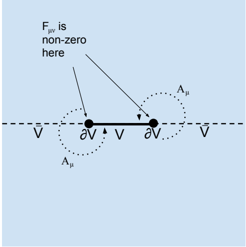
Option 2

n fields with prescribed monodromies



Option 3

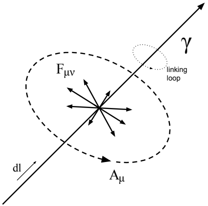
n fields coupled to flat connections



Vortex type defect operator

We use co-dimension 2 vortex operators[1]

- supported on a submanifold γ.
- data given by the holonomy β of a connection A.
- flux restricted to γ: $F = \beta \star [\gamma]$.



They are interesting objects

- 2d twist and winding state operators are of this type.
- supersymmetric versions are available in 2[2],3[3],4[4],5[5] dimensions.
- duality transformation properties are known in some cases.

[1] Witten 1988, Seiberg and Moore 1989
[2] Hosomoichi (2015) Okuda (2015)
[3] Kapustin, Willett, IY (2012), Drukker, Okuda, Passerini (2012)
[4] Gukov, Witten (2006)
[5] Bullimore and Kim (2014)

Reinterpretation: $\mathbb{Z}_n$ gauge theory

Consider the tensor product of n copies of the theory

- There is an $\mathcal{S}_n$ "replica symmetry".
- The action is a sum before, but not after, the field redefinition.

Gauge a $\mathbb{Z}_n$ subgroup and introduce the monodromy around ∂V in *replica space*

$$\mathcal{M} = \begin{pmatrix} 0 & 1 & 0 & \dots \\ 0 & 0 & 1 & \dots \\ \vdots & \vdots & 0 & \vdots \\ 1 & 0 & 0 & 0 \end{pmatrix}.$$

- resides in the group: $\mathcal{M}^n = \mathbb{1}_n$.
- can be diagonalized to the β_k using the field redefinition.
- implements the calculation for the n'th Rényi entropy.

Lagrangian $\mathbb{Z}_n$ gauge theory

Starting with n copies of the free theory, couple to a connection A

1. Consider an associated $\mathbb{Z}_n$ gauge theory with action

$$S_{\mathsf{BF}} = \frac{in}{2\pi}\int F \wedge B, \tag{1}$$

   - $F = dA$ is a gauge field constrained to produce flat $\mathbb{Z}_n$ connections.
   - B is a $d-2$ form field that acts as a Lagrange multiplier.

2. Add co-dimension 2 "electric" operators which carry the defect parameters

$$\exp\left(\frac{ik}{2\pi}\oint_{\partial\Sigma} B\right), \qquad k \in \{1 \ldots n-1\}\,.$$

3. Integrate out B to produce the right holonomy.

SCFT Rényi entropy

In a CFT, and a spherical entangling surface, one may use a (large) conformal transformation to calculate S_n as [6]

- Partition function on a branched S^d (conical singularity)

$$ds^2 = d\theta^2 + n^2 \sin^2\theta d\tau^2 + \cos\theta d\Omega^{d-2}, \quad \tau \in [0, 2\pi).$$

$$R^{(\text{3d})} = \text{const } + 2\frac{n-1}{n}\frac{\delta(\theta)}{\theta}.$$

- Thermal partition function on hyperbolic space ($\mathbb{H}^{d-1} \times S^1$) with singularity at the boundary.

We must decide how to treat the singularity in order to do QFT

1. Excise: cut and impose boundary conditions.
2. Smooth: we make choices based on supersymmetry.

[6] Casini, Huerta, Myers (2011)

Coupling to supergravity - 3d example

The easy way to supersymmetric actions on a curved 3-manifold: couple to rigid $\mathcal{N}=2$ supergravity

$\mathcal{H}_\mu$	$g_{\mu\nu}$	$H \equiv \frac{i}{2} \star dB$	$V_\mu \equiv -i\left(\star dC\right)_\mu$	$A_\mu^{(R)}$	fermions

The multiplet $\mathcal{R}_\mu$ can be used when the theory has a conserved $U\left(1\right)_R$ R-charge.

R-multiplet $\mathcal{R}_\mu$	$T_{\mu\nu}$	$J^{(Z)}$	$j_\mu^{(Z)}$	$j_\mu^{(R)}$	fermions

The linearized coupling is

$$\mathcal{L} = T_{\mu\nu} g^{\mu\nu} - j_\mu^{(R)} \left(A^{(R)\mu} - \frac{3}{2} V^\mu \right) + i j_\mu^{(Z)} C_\mu - J^{(Z)} H.$$

To preserve a supercharge[7]

$$\left(\nabla_\mu \pm i A_\mu^{(R)} \right) \zeta = -\frac{1}{2} H \gamma_\mu \zeta - i V_\mu \zeta - \frac{1}{2} \varepsilon_{\mu\nu\rho} V^\nu \gamma^\rho \zeta$$

[7] Closset, Dumitrescu, Festuccia, Komargodski (2012)

A smooth supersymmetric background

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There is still a singularity, but now its supersymmetric. We introduce a supersymmetry preserving resolution

$$ds^2 = \frac{1}{f_\epsilon(\theta)} d\theta^2 + n^2 \sin^2\theta d\tau^2 + \cos^2\theta d\phi^2,$$

$$f_\epsilon(\theta) = \begin{cases} \frac{1}{n^2} & \theta \to 0 \\ 1 & \epsilon < \theta \leq \frac{\pi}{2} \end{cases}$$

$$H = -i\sqrt{f_\epsilon(\theta)}, \qquad A^{(\mathrm{R})} = \frac{1}{2}\left(n\sqrt{f_\epsilon(\theta)} - 1\right) d\tau, \qquad V = 0$$

Everything is now smooth, but still preserves the same Killing spinors! We define a partition function as

$$Z_{\text{singular space}}(n) \equiv \lim_{\epsilon \to 0} Z_{\text{resolved background}}(n).$$

If $n = 1$ everything is already smooth and

$$Z_{\text{singular space}}(1) = Z_{S^3}.$$

The supersymmetric Rényi entropy (SRE)

Define the **supersymmetric Rényi entropy**[8]

$$S_n^{\text{susy}} \equiv \frac{1}{1-n} \Re \left[\log \left(\frac{Z_{\text{singular space}}(n)}{(Z_{S^d})^n} \right) \right]$$

- Possible non-universal terms are removed by taking the real part.[9]
- We concentrate on the finite part, which is universal for the theories we consider.
- The higher dimensional analogues were defined in
  - 4d $\mathcal{N}=2$ gauge theories: [10]
  - 5d $\mathcal{N}=1$ gauge theories.[11]
- Exact calculation by localization reduces the result to a matrix model (next section).

[8] Nishioka and IY (2013)
[9] Closset, Dumitrescu, Festuccia, Komargodski, Seiberg (2012)
[10] Huang and Zhou (2014), Crossley, Dyer, Sonner (2014)
[11] Alday, Richmond, Sparks (2014), Hama, Nishioka, Ugajin (2014)

Squashed sphere and behavior near $n = 1$

As it turns out, supersymmetric Rényi entropy is related to the partition function on the squashed sphere (S_b^d)

$$S_n^{\text{susy}} = \frac{1}{1-n}\Re\left[\log\left(\frac{Z_{S_b^d}\left(b=\sqrt{n}\right)}{\left(Z_{S^d}\right)^n}\right)\right].$$

- The relationship is due to deformation invariance.

The expansion in 3d around $n = 1$

$$S_n^{\text{susy}} = S_{\text{Entanglement}} + \frac{\pi^2}{16}\tau_{rr}\left(n-1\right) + O\left(\left(n-1\right)^2\right) \ ,$$

where τ_{rr} appears in flat space CFT correlators[12]

$$\langle T_{\mu\nu}\left(x\right) T_{\rho\sigma}\left(0\right)\rangle = -\frac{\tau_{rr}}{64\pi^2}\left(\delta_{\mu\nu}\partial^2 - \partial_\mu\partial_\nu\right)\left(\delta_{\rho\sigma}\partial^2 - \partial_\rho\partial_\sigma\right)\frac{1}{x^2} + \dots$$

[12] Closset, Dumitrescu, Festuccia, Komargodski (2012)

Supersymmetric vortex operators -3d example

The 3d $\mathcal{N}=2$ $1/2$ BPS Wilson loop

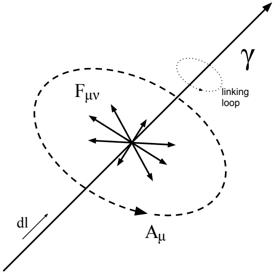
$$\exp\left(iq\oint_{\gamma}(A-i\sigma d\ell)\right)$$

has an associated supersymmetric vortex loop, given by the background

$$F=dA=2\pi q\delta_{\gamma},\quad \star D=-2\pi iq\delta_{\gamma}\wedge d\ell$$

which solves the BPS equation on S^3

$$\left(-i\gamma^{\mu}\left(\star F\right)_{\mu}+D\right)\epsilon=0.$$



They are $SL\left(2,\mathbb{Z}\right)$ buddies!

Supersymmetric BF couplings - $5d$ example

A vector multiplet gauging a flavor symmetry

$$\left\{A_{\mu}, \sigma, Y_{IJ}, \text{fermions}\right\},$$

couples to a $3-$form multiplet

$$\left\{B_{\mu\nu\rho}, N, L^{IJ}, \text{fermions}\right\},$$

using a supersymmetric BF term

$$S_{\text{5d BF}} = \frac{in}{2\pi}\int_{\mathbb{S}^5}\left(\frac{1}{4}B\wedge F_A - \sqrt{g}\frac{1}{2}\sigma N + \sqrt{g}L^{IJ}Y_{IJ} + \text{fermions}\right)$$

and we introduce a supersymmetric Wilson volume operator using some fixed K^{IJ}

$$W\left(k\right) = \exp\left(ik\oint_{\mathbb{S}^3}\left[B + iK^{IJ}L_{IJ}\right]\right).$$

The basics of localization

Deformation

- Identify an appropriate conserved supercharge Q.
- Choose V such that $\{Q, V\}$ is a positive semi-definite functional and $Q^2 V = 0$.
- Path integral deformed by total Q variation $S \to S + t\{Q, V\}$ is independent of t.
- Add Q closed operators (Wilson loops, defect operators).

Localization

- Take the limit $t \to \infty$.
- The measure e^{-S} is very small for $\{Q, V\} \neq 0$.
- The semi-classical approximation becomes exact, but there may be many moduli (saddle points) to sum over.
- Exact answer: integrate the classical action and a one loop contribution over the moduli space.

Computing SRE with localization

Partition function on n-branched or $b=\sqrt{n}$ squashed sphere[13]

$$Z^{\text{susy}}_{(n)} = \underbrace{\frac{\sqrt{n}}{|W|}\int d\sigma}_{\text{Coulomb moduli}} \underbrace{e^{n\sum_i c_i(\kappa,g_{\text{YM}})\,\text{Tr}(\sigma^i)}}_{\text{classical}}$$

$$\prod^{\text{fixed}}_{\text{points}} \underbrace{\left(\underbrace{Z_{\text{pert}}(n,\sigma,m)}_{\Gamma_r\ \text{functions}} \underbrace{\sum_{\text{partitions }\vec{Y}} Z_{\vec{Y}}(\tau/\bar{\tau},\kappa,n,\sigma,m)}_{\text{Nekrasov instanton partition function}} \right)}_{\text{holomorphic block}}$$

- Parameters: real masses m, Chern-Simons κ and holomorphic gauge couplings τ.

[13] Hama and Hosomichi (2012,2013), Källén and Zabzine (2012), Imamura (2012), Pasquetti (2016)

Defects on the round sphere with localization

Couple the theory on the round sphere $Z^{\text{susy}}_{(1)}$ to defects using the vector multiplet of the $\mathbb{Z}_n$ theory

- Additional scalar moduli σ_V and N.
- σ_V couples like a dynamical real mass term.

Making the extra multiplets dynamical adds

$$\int d\sigma_V dN \cdot$$

The BF term and the "electric" Wilson operator localize to

$$e^{-S_{\text{BF}}} \to e^{2\pi i n \sigma_V N}, \quad W(k) \to e^{2\pi k N},$$

Integrating out both multiplets yields an imaginary mass

$$\int d\sigma_V \, Z^{\text{susy}}_{(1)} (\sigma_V \ldots) \int dN \, e^{2\pi i n \sigma_V N} e^{2\pi k N}$$

$$\to \int d\sigma_V \, Z^{\text{susy}}_{(1)} (\sigma_V \ldots) \, \delta \left(\sigma_V - i \frac{k}{n} \right)$$

Example: 3d Mirror symmetry

IR duality exchanges $U(1)$ $\mathcal{N}=4$ gauge theory with a single charged flavor and a free twisted hypermultiplet (and $U(1)_J$ with $U(1)_{\text{flavor}}$ and the FI term with the real mass)[14]

$$Z^{\mathcal{N}=4}_{\text{free twisted hyper}} = \frac{1}{\cosh \pi m}$$

$$\overleftrightarrow{\eta \leftrightarrow m} \quad Z^{\mathcal{N}=4}_{U(1),N_f=1} = \int d\sigma \frac{e^{2\pi i \sigma \eta}}{\cosh \pi \sigma}.$$

Duality exchanges Wilson loops and (flavor) defect operators[15]

$$W_q Z^{\mathcal{N}=4}_{U(1),N_f=1}(\eta) = \int d\lambda \frac{e^{2\pi i \eta \lambda} e^{2\pi q \lambda}}{2\cosh(\pi\lambda)}$$

$$= \frac{1}{2\cosh \pi(\eta - iq)} = D_q Z^{\mathcal{N}=4}_{\text{free twisted hyper}}(\eta)$$

[14] Kapustin,Strassler (1999)

[15] Kapustin, Willett, IY (2012)

What we aim to show

We wish to show that SRE can be calculated in two ways

1. The squashed sphere partition function of the original theory $Z_{(n)}$.
2. The round sphere with n copies coupled to defects

$$Z^{\text{round sphere with defects}}(n) \equiv$$

$$\sum \int_{n \text{ moduli}} \left[\delta_{\text{moduli}} \prod_{k=0}^{n-1} Z_{(1)}^{\text{classical}} Z_{(1)}^{\text{perturbative}} (k, \text{moduli}) \right].$$

We need to show

- The n sets of moduli can be sewn up (using δ_{moduli}) into the original set. The fractionalization and sewing reflect the boundary conditions applied to the non-free modes.
- The classical contributions of the copies add up.
- The defect deformed perturbative contributions magically multiply to yield the original ones.

Scalar moduli and classical contributions

The n scalar moduli σ_k are constant modes on the sphere, so the boundary conditions simply imply

$$\delta_{\text{scalar moduli}} = \prod_{k=0}^{n-1} \delta\left(\sigma_k - \sigma_{k+1}\right).$$

Classical contributions are of the form

$$e^{c(\kappa, g_{\mathsf{YM}})\,\mathrm{Tr}(\sigma_k^2)}.$$

They simply collapse

$$\int \prod_{k=0}^{n-1} \left[\delta\left(\sigma_k - \sigma_{k+1}\right) d\sigma_k\right] e^{\sum_i c_i(\kappa, g_{\mathsf{YM}})\,\mathrm{Tr}(\sigma_k^i)}$$

$$\to \int d\sigma e^{n \sum_i c_i(\kappa, g_{\mathsf{YM}})\,\mathrm{Tr}(\sigma^i)}$$

Non-instanton perturbative contributions

One-loop contributions are built out of multiple Γ-functions

$$\Gamma_r\left(z,\vec{\omega}\right) \equiv \exp\left[\partial_s \zeta_r\left(s,z,\vec{\omega}\right)\right]_{s=0},$$

$$\zeta_r\left(s,z,\vec{\omega}\right) \equiv \int_0^\infty \frac{e^{-zt}}{\prod_{i=1}^r\left(1-e^{-\omega_i t}\right)} t^{s-1} dt.$$

They satisfy a multiplication identity

$$\prod_{k=0}^{n-1} \Gamma_r\left(z+\frac{k}{n},\vec{\omega}\right) = \Gamma_r\left(z,\omega_1,\ldots\omega_{r-1},\frac{\omega_r}{n}\right).$$

- In the sphere partition functions $z = im$, with m a mass or Coulomb branch parameter.
- The shift by k/n is an imaginary mass or, equivalently, a coupling to a co-dimension 2 defect.

The instanton partition function for $SU(N)$

The $4d$ and $5d$ calculations include instanton/contact-instanton contributions

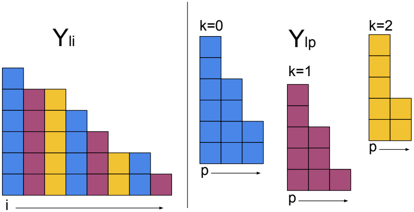
$$Z_{\text{inst}} = \sum_{\vec{\mathbf{Y}} \in \text{partitions}} \mathfrak{q}^{|\vec{\mathbf{Y}}|} Z^{\text{5d-CS}}_{\vec{\mathbf{Y}},\kappa}\left(\vec{a}, \epsilon_1, \epsilon_2, \beta\right) Z_{\vec{\mathbf{Y}}}\left(\vec{a}, \vec{m_f}, \epsilon_1, \epsilon_2, \beta\right),$$

- The equivariant parameters ϵ_1, ϵ_2 are related to the squashing parameter on the squashed $S^{4,5}$ and hence to n. β is the size of the 5d circle fiber, and can also be related to n. $\beta \to 0$ is the 4d limit.
- In the singular space limit one has to count the instantons on a ramified covering.

We need to check

1. How do the partitions, which are moduli, fractionalize and glue?
2. Do the fluctuations $Z_{\vec{\mathbf{Y}}}$ satisfy multiplication identities?

Instanton gluing I - Partitions



lhs: a partition in the full squashed/branched theory.
rhs: partitions in each of the decoupled n-copies.

We explicitly glue the instanton moduli back together.

- Every box has a well defined weight j under the equivariant action and we decompose $i = np + k$.
- The "gluing" of the moduli $\vec{Y}$ is related to orbifold partitions.[16]

[16] Dijkgraaf and Sulkowski (2007)

Instanton gluing II - third point in $5d$ calculation

$5d$ calculation: contact-instantons localized at 3 points on $\mathbb{C}P^2$. At two of these we have $\beta = 2\pi$ and the orbifold partitions from the previous slide. On the third we have $\beta = 2\pi n$ (a larger circle fiber)

$$Z^{\text{vector}}_{\vec{\mathbf{k}}} = \prod_{l,m}^{N_c} \prod_{s\in Y_l} \left(1 - e^{i\beta\left(\ell_{Y_m}(s)\epsilon_1 - \left(a_{Y_l}(s)+1\right)\epsilon_2 + a_l - a_m\right)}\right) \prod_{t\in Y_m} \left(1 - e^{i\beta\left(-\left(\ell_{Y_l}(t)+1\right)\epsilon_1 + a_{Y_m}(t)\epsilon_2 + a_l - a_m\right)}\right),$$

It glues back together by decomposing the KK modes as $m = np + k$ in

$$\prod_{m=-\infty}^{\infty} (m + a) = 1 - e^{2\pi i a}, \quad \prod_{k=0}^{n-1} \left(1 - e^{2\pi i\left(a + \frac{k}{n}\right)}\right) = 1 - e^{2\pi i n a}.$$

Example: $5d$ hypermultiplet fluctuations

The fluctuation part of the instanton partition function for a $5d$ hypermultiplet

$$q \equiv e^{i\beta\epsilon_2}, \quad t \equiv e^{-i\beta\epsilon_1}, \quad Q_l \equiv e^{i\beta a_l}, \quad Q_{m_f} \equiv e^{i\beta m_f}.$$

$$Z^{\text{5d}}_{\text{hyper}}\left(q, a, m_f, \epsilon_1, \epsilon_2\right) = \prod_{l=1}^{N}\prod_{j=1}^{\infty} \frac{\left(Q_{m_f}^{-1} Q_l q t^{-j}; q\right)_\infty}{\left(Q_{m_f}^{-1} Q_l q^{Y_{lj}+1} t^{-j}; q\right)_\infty},$$

- $(x, q)_\infty \equiv \prod_{p=0}^{\infty}(1 - xq^p)$ is the q-Pochhammer symbol.
- a is the Coulomb branch modulus. On the sphere it is set to $i\sigma$.
- m_f is a "real mass". It should be set to a specific value for a CFT, with real part m_0.

Example: $5d$ hypermultiplet fluctuations

Adding some imaginary masses makes the squashed sphere version look like the round sphere. These "masses" are the defects. In addition, the k'th fluctuation determinant can only see the k'th partition.

$$Z_{\vec{\mathbf{Y}}}^{\text{fund hyper}}\left(\vec{a}, m_f^0, \frac{1}{n}, 1, 2\pi\right) = \prod_{l=1}^{N}\prod_{j=1}^{\infty}\frac{\left(Q_l q t^{-\frac{j-1}{n}}; q\right)_\infty}{\left(Q_l q^{Y_{lj}+1} t^{-\frac{j-1}{n}}; q\right)_\infty}$$

$$= \prod_{k=0}^{n-1}\left[\prod_{l=1}^{N}\prod_{j=1}^{\infty}\frac{\left(Q_l q t^{-\frac{j-1}{n}}; q\right)_\infty}{\left(Q_l q^{Y_{lj}+1} t^{-\frac{j-1}{n}}; q\right)_\infty}\right]_{j=np-k}$$

$$= \prod_{k=0}^{n-1}\left[\prod_{l=1}^{N}\prod_{p=1}^{\infty}\frac{\left(Q_{m_f}^{-1}(k)\, Q_l q t^{-(p-1)}; q\right)_\infty}{\left(Q_{m_f}^{-1}(k)\, Q_l q^{Y_{l,np-k}+1} t^{-(p-1)}; q\right)_\infty}\right]_{Q_{m_f}^{-1}(k)\equiv t^{-\frac{k}{n}}}$$

Summary and questions

We proved that supersymmetric defects compute SRE by showing

$$Z_n^{\text{squashed sphere}} = Z_n^{\text{round sphere with defects}}$$

Some interesting directions to explore

- The examples in $3d$ and $4d$ show a strange feature

$$Z_n^{\text{susy}} = Z_{1/n}^{\text{susy}}$$

- Is there some direct physical interpretation of the "super entanglement spectrum"? is it computable? Does it represent the spectrum of a real Hamiltonian?
- Is there a computable analogue for $4d$ $\mathcal{N} = 1$?
- What can one say about the holographic dual of the defect computation?
- Can we use some generalizations of the discrete gauge theory?

Thank you