駒場集中講義中間

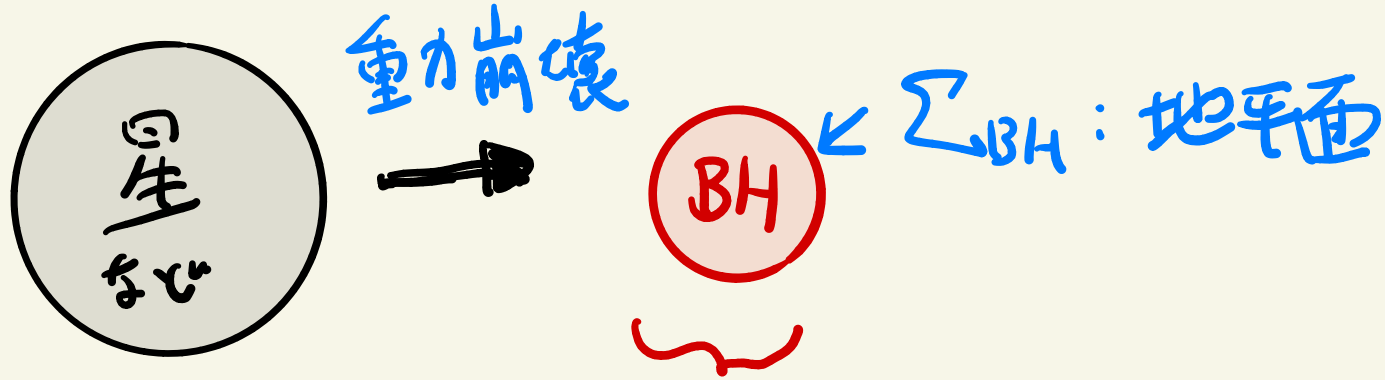
2020年10月

⑤ AdS/CFT対応

(5-1) ホログラフィー原理

BHのエントロピー $S_{BH} = \dfrac{A(\Sigma_{BH})}{4G_N}$

Bekenstein-Hawking.



隠れた情報量 $= S_{BH}$

もっと一般的に、

の内部の情報量 $\leq \dfrac{A(\Sigma)}{4G_N}$

エントロピーバウンド (Bousso)

→ 重力理論の自由度 ～ $\frac{A}{4G_N}$ → 面積に比例!

(cf. 熱力学 $S \propto$ 体積)

"ブラックホール熱力学" は 1次元低い物質の熱力学!

ホログラフィー原理 ('t Hooft, Susskind)

(d+2)次元の重力理論 $\simeq$ (d+1)次元の"物質"の理論

↓ 重力相互作用なし

例. QFT, Spin系, 行列模型…

(5-2) AdS/CFT 対応

ホログラフィーの代表例が AdS/CFT対応

AdS時空 (Anti de Sitter Space, 反ドジッター時空)

$\mathbb{R}^{2,d+1}$: $ds^2 = -(dX_0)^2 - (dX_{d+2})^2 + (dX_1)^2 + \cdots + (dX_{d+1})^2$

$(X^0, X^1, \cdots, X^{d+2})$

の内部の超曲面 (これが、AdS_{d+2})

$$(X_0)^2 + (X_{d+2})^2 = (X_1)^2 + \cdots + (X_{d+1})^2 + R^2,$$

↑ AdS半径

Global 座標 / Poincare 座標

$$\begin{cases} X_0 = R\cosh\rho \cdot \cos t = \dfrac{z}{2}\left(1 + \dfrac{R^2 + |\vec{x}|^2 - x_0^2}{z^2}\right) \\ X_{d+2} = R\cosh\rho \cdot \sin t = R \cdot \dfrac{x_0}{z} \\ X_i = R\sinh\rho\, \Omega_i = R \cdot \dfrac{x_i}{z} \\ X_{d+1} = R\sinh\rho\, \Omega_{d+1} = \dfrac{z}{2}\left(1 - \dfrac{R^2 - |\vec{x}|^2 + x_0^2}{z^2}\right) \end{cases}$$

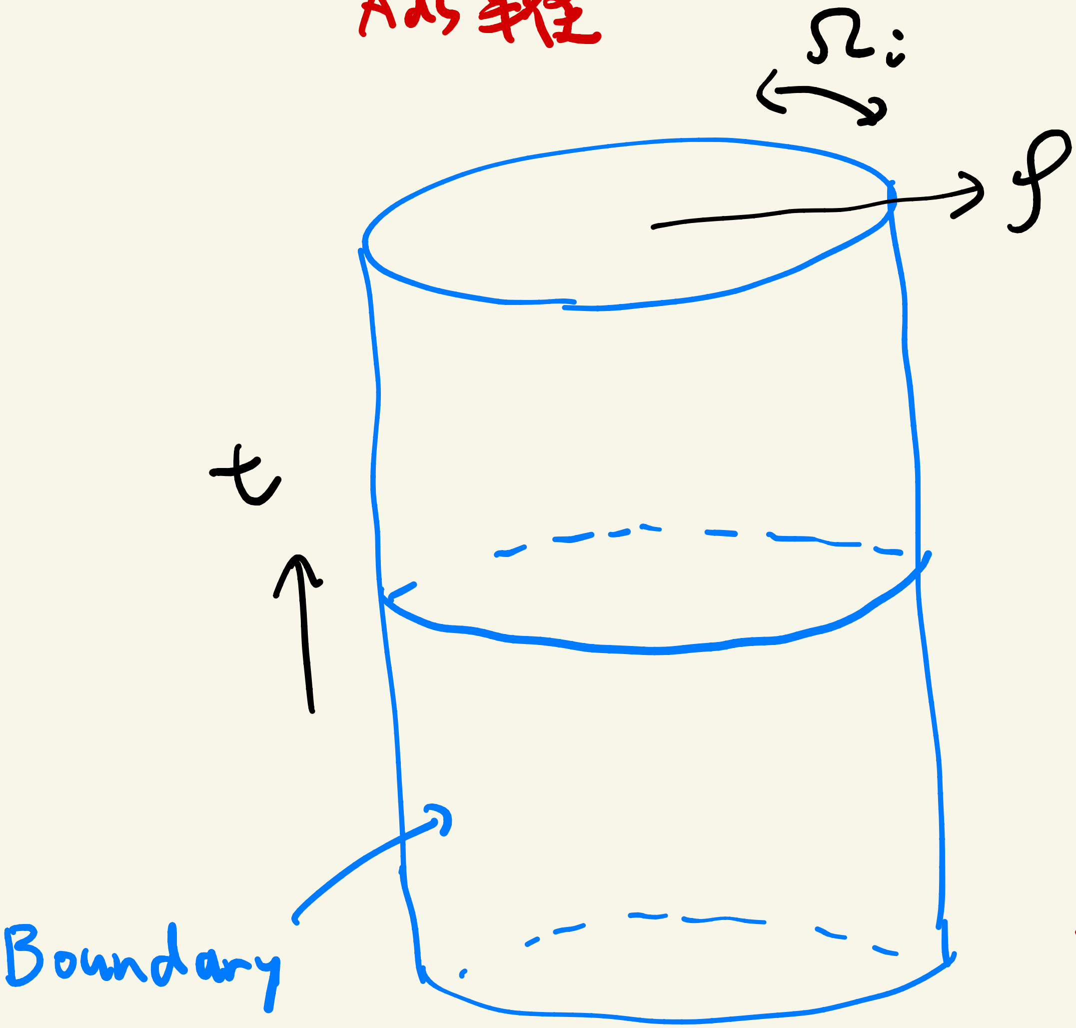
$i = 1, 2, \cdots, d$ 　 $\Omega \in S^d$

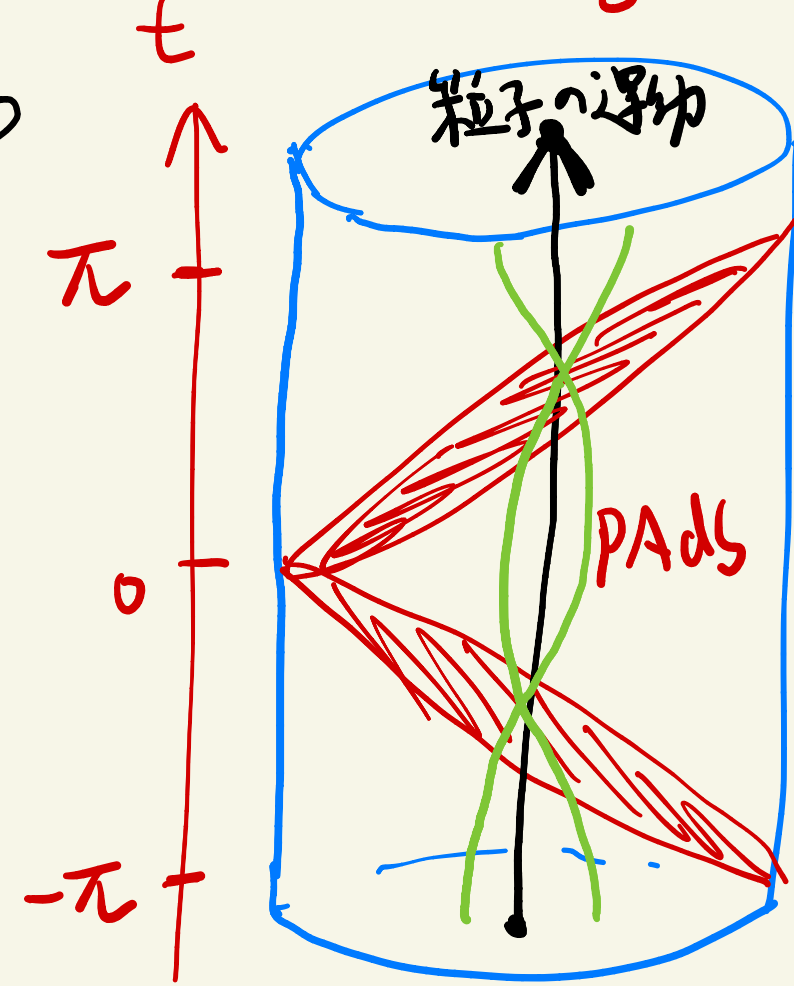
$\left(\sum_{i=1}^{d+1} (\Omega_i)^2 = 1 \right)$

Global AdS

$$ds^2 = R^2\left(-\cosh^2\rho\, dt^2 + d\rho^2 + \sinh^2\rho\, (d\Omega_d)^2\right)$$

AdS半径 (R) S^d ($d\Omega_d$)



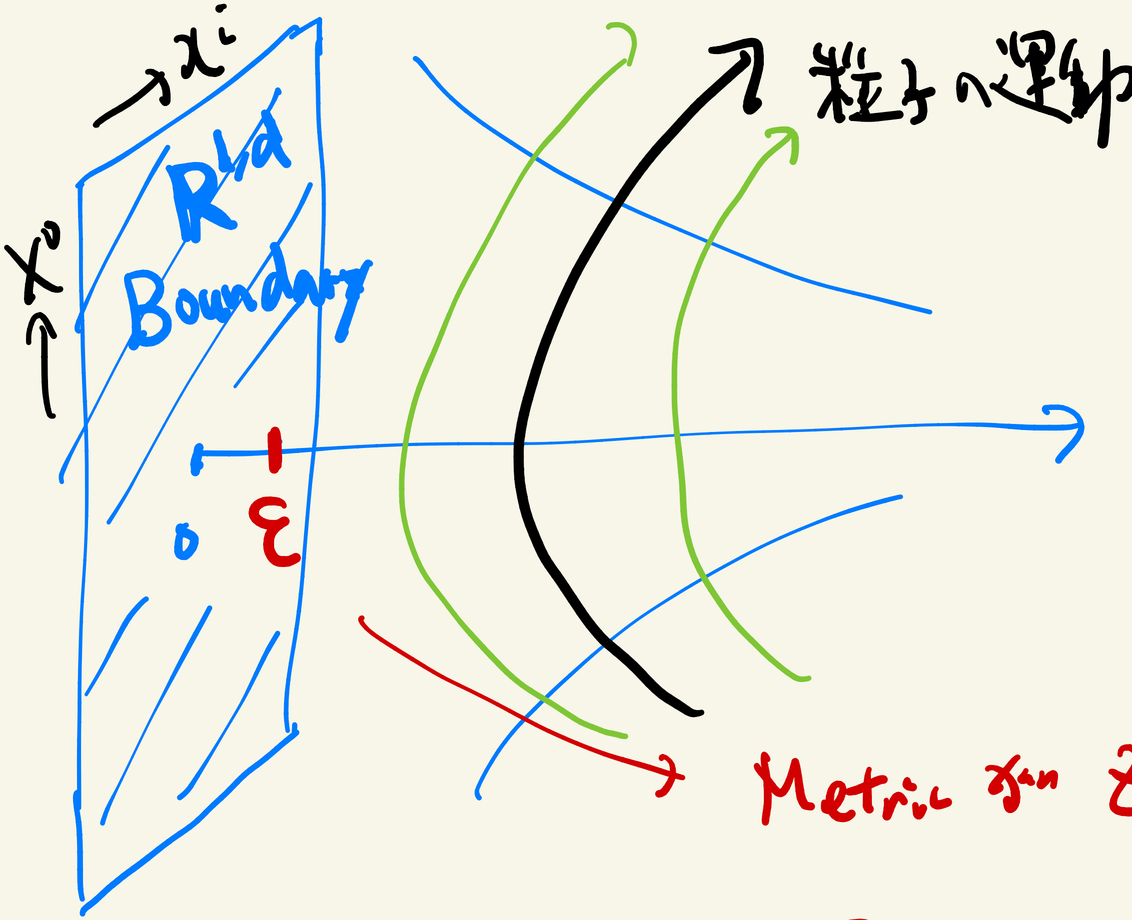


$$\partial(gAdS_{d+2}) = \mathbb{R}\times S^d$$

Poincare AdS

$$ds^2 = R^2\left(\frac{dz^2 + d\vec{x}\cdot d\vec{x} - dx^0 dx^0}{z^2}\right)$$

スケール不変性
$(z, \vec{x}, t)$
$\updownarrow$
$\lambda(z, \vec{x}, t)$



RG flowの
z = 長さのスケール

Metric が $z=0$ で発散

→ $z \geq \varepsilon$ のカットオフ

↳ CFTの紫外カットオフと同じ

AdS/CFT対応 (Maldacena 1997)

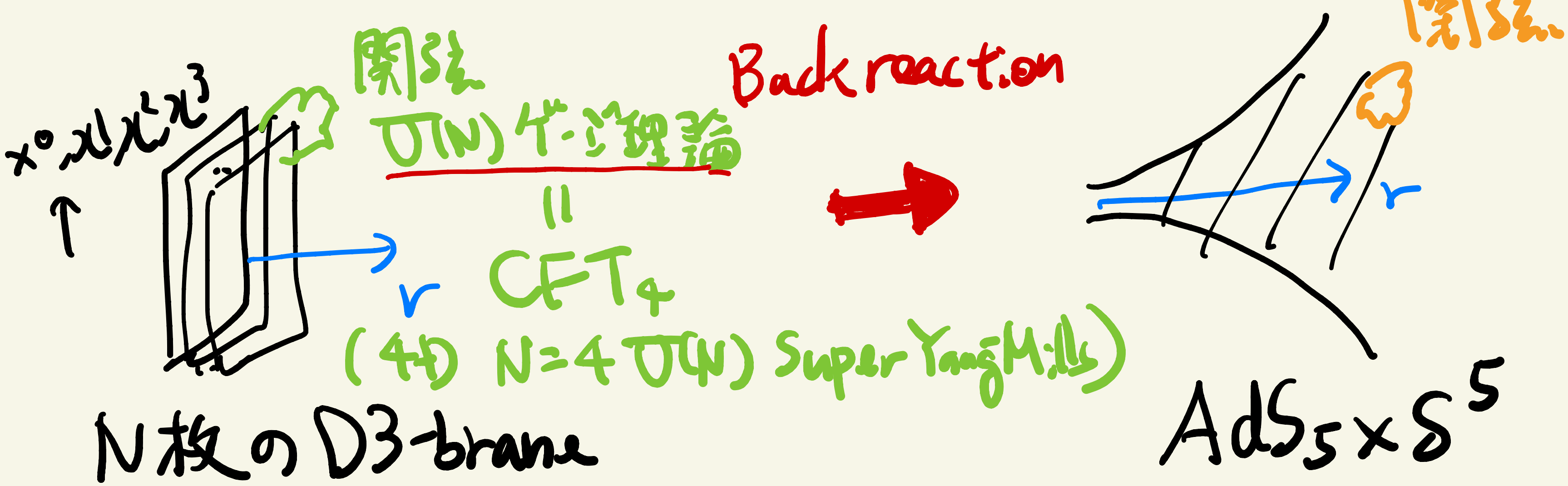
AdS_{d+2}の重力理論 $=$ $\partial(AdS_{d+2})$上のCFT_{d+1}

(境界 — ∂; ラージNゲージ理論 — CFT_{d+1})

幾何学的対称性 $SO(2,d+1)$ $=$ 共形対称性 $SO(2,d+1)$

一致する.

※超弦理論からのAdS/CFT



D3-brane解（ブラックブレーン解）

$\alpha' = l_s^2$

$$ds^2 = \frac{1}{\sqrt{H(r)}}\sum_{\mu=0}^{3} dx^\mu dx_\mu + \sqrt{H(r)}\left(dr^2 + r^2 d\Omega_5^2\right)$$

$$H(r) = 1 + \frac{R^4}{r^4}, \quad R^4 = 4\pi(\alpha')^2 N g_s$$

$\rightarrow$ ($r \to 0$ Near horizon limit)

$$ds^2 \simeq \underbrace{\frac{R^2}{r^2}\left(dx^\mu dx_\mu\right) + R^2 \frac{dr^2}{r^2}}_{AdS_5} + \underbrace{R^2 (d\Omega_5)^2}_{S^5}$$

$AdS_5 \times S^5$

パラメータの対応

$g_s = g_{YM}^2, \quad \mathcal{L} = -\frac{1}{4 g_{YM}^2} \mathrm{Tr}\left(F^{\mu\nu} F_{\mu\nu}\right)$

't Hooft結合定数

古典重力近似を適用できる

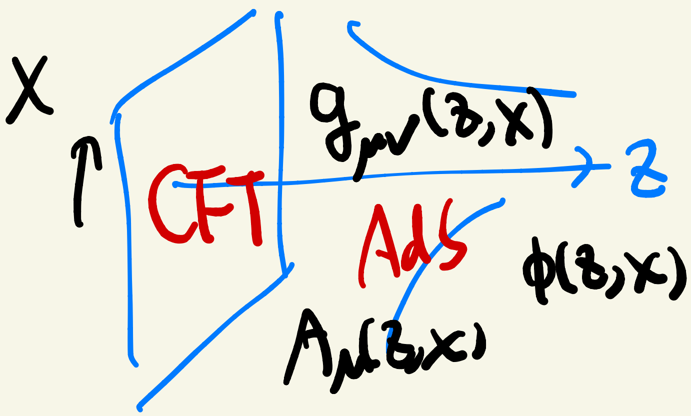
- $R/l_s \gg 1 \rightarrow \lambda \equiv N g_{YM}^2 \gg 1$ 強結合
- $R/l_{pl} \gg 1 \rightarrow N \gg 1$ ラージN

→ ホログラフィック CFTと呼応

AdS/CFTの対応関係 (Bulk-Boundary Relation)
GKPW

$$Z_{重力}(g^{(0)}_{ab}, A^{(0)}_a, \phi^{(0)}) = Z_{CFT}(g^{(0)}_{ab}, A^{(0)}_a, \phi^{(0)})$$

Bdyでの値　　外場



$$\lim_{z\to 0} g_{ab}(z,x) = g^{(0)}_{ab}$$

古典重力近似で

$$Z_{重力} \simeq e^{-I^{\text{on-shell}}_{重力}[g^{(0)}_{ab}, A^{(0)}_a, \phi^{(0)}]}$$

$$Z_{CFT} = \int_{CFT場} D\Phi \, e^{-S_{CFT}[\Phi;\, g^{(0)}_{ab}, A^{(0)}_a, \phi^{(0)}]}$$

$$\downarrow$$

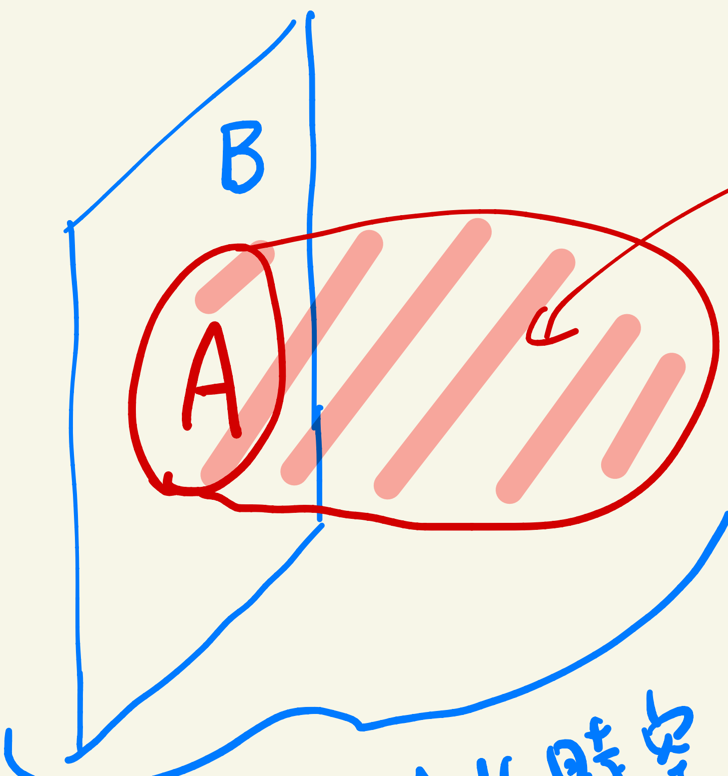
$$S_0[\Phi] + \int dx^{d+1} \phi^{(0)} O(x)$$

$+ \cdots$

⑥ ホログラフィック・エンタングルメント・エントロピー(HEE)

(6-1) Motivation

Question：CFTの部分領域A内の情報は、
AdSのどの領域の情報に対応するか？



漸近的AdS時空

こんな感じ？ ← 最近では、エンタングルメント・ウェッジと呼ばれる。

↓

定量化するには、物理量が必要！

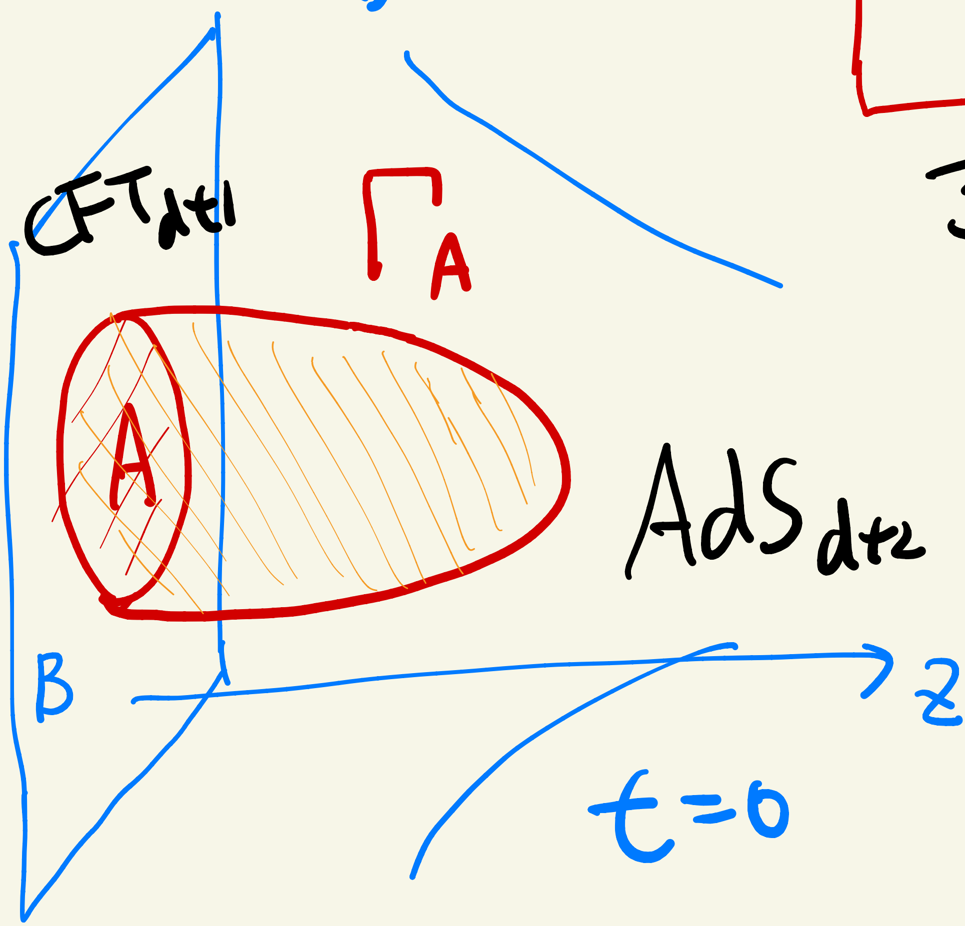
$$S_A = -\mathrm{Tr}\,\rho_A \log \rho_A$$

↓

EEを考えよう！
(量子情報量)

(6-2) HEE公式 Ver 1. 笠・高柳 2006

※ Staticな漸近的AdS時空の場合



時間一定面をとって

$$S_A = \underset{\Gamma_A}{\mathrm{Min}} \left[\frac{A(\Gamma_A)}{4G_N} \right]$$

極小曲面

ここで Γ_A は、d次元曲面で、（余次元2）

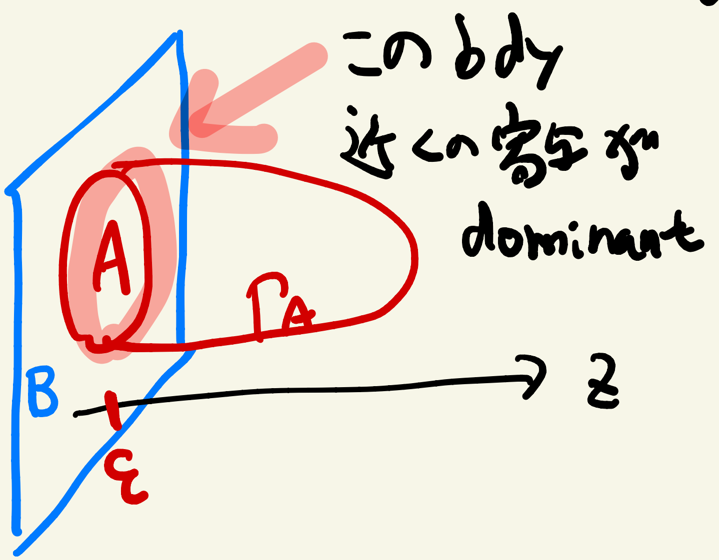
$\partial \Gamma_A = \partial A$ かつ、

$\Gamma_A \sim A$（ホモロジー同値）

を満たす。

(6-3) HEEの基本的性質

(a) 面積則



$$ds^2 = R^2\left(\frac{dz^2 + dx^\mu dx_\mu}{z^2}\right)$$

$$A(\Gamma_A) \simeq R^d \int_\varepsilon \frac{dz}{z^d} \int_{\partial A} dx^{d-1}$$

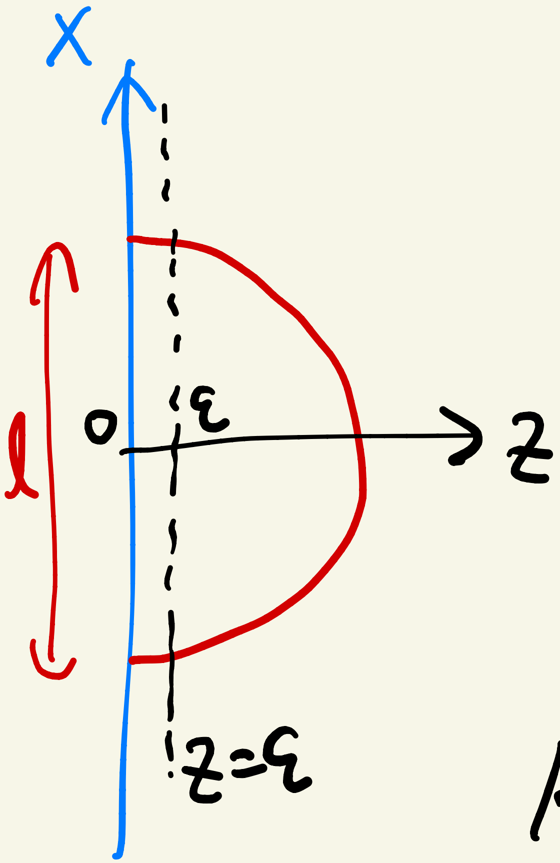
$$S_A \simeq \frac{R^d}{4G_N} \cdot \frac{A(\partial A)}{(d-1)\varepsilon^{d-1}}$$

$\frac{R^d}{4G_N}$: $\sim C$　　$\frac{A(\partial A)}{(d-1)\varepsilon^{d-1}}$: Area law!

同時、$S_A = S_B$（全体がpure stateの場合）

も明らか。

(b) CFT_2 の場合



極小曲面 (Minimal surface)

$$\rightarrow x^2+z^2=\frac{l^2}{4} \qquad x=\sqrt{\frac{l^2}{4}-z^2}$$

$$ds^2=R^2\left(\frac{dz^2+dx^2}{z^2}\right)$$

$$ds^2\Big|_{\Gamma_A}=R^2\times\frac{l^2}{z^2(l^2-4z^2)}dz^2$$

$$A(\Gamma_A)=2R\times\int_{\varepsilon}^{l/2}\frac{dz}{z}\times\frac{l}{\sqrt{l^2-4z^2}}=2R\log\left(\frac{l}{\varepsilon}\right)$$

よって.

$$S_A=\frac{c}{3}\log\left(\frac{l}{\varepsilon}\right)$$

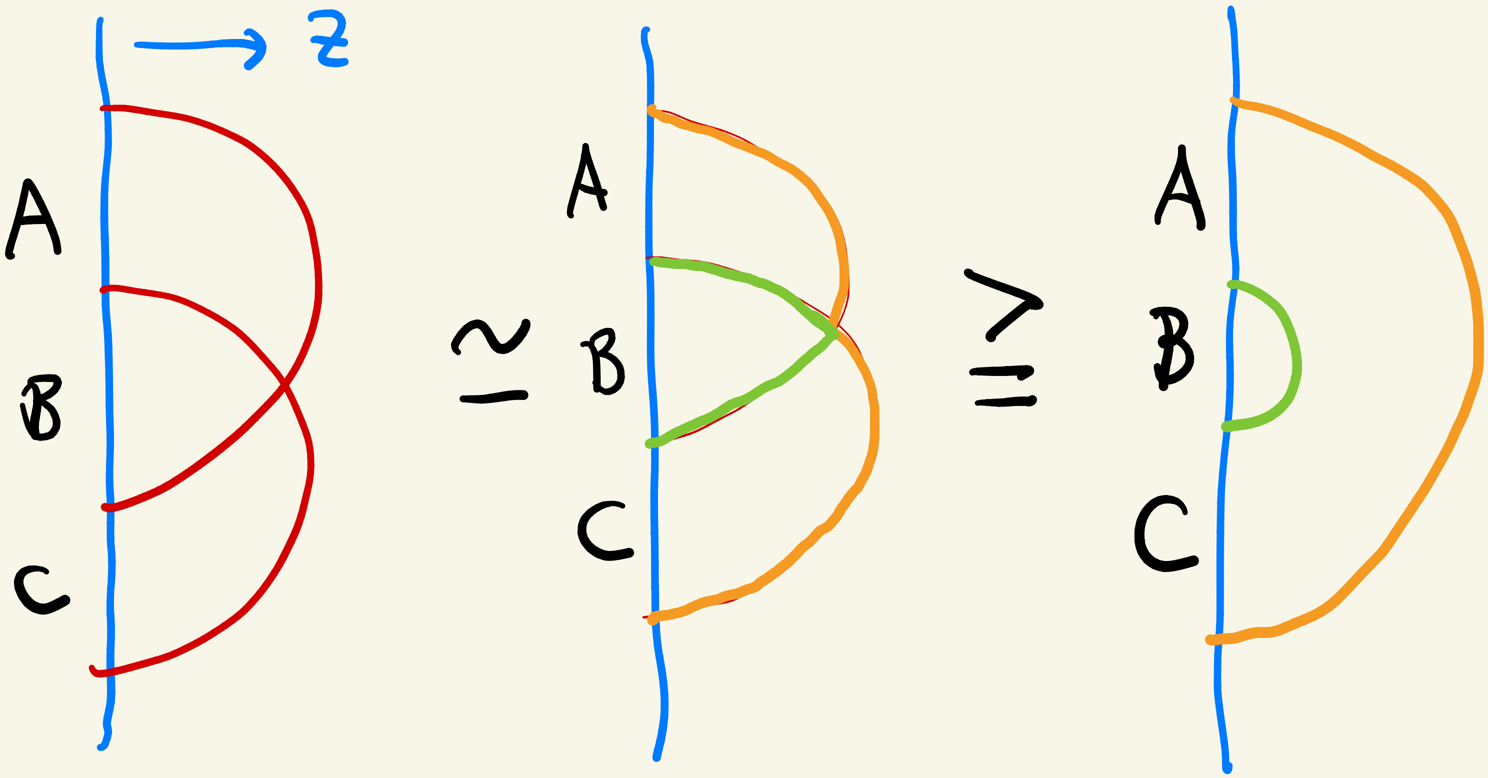
↓

2次 CFTの結果と一致

ここで Brown-Henneaux 関係式

$$c=\frac{3R}{2G_N}$$ を用いた.

(c) 強劣加法性の証明 [Headrick-高柳 2007]

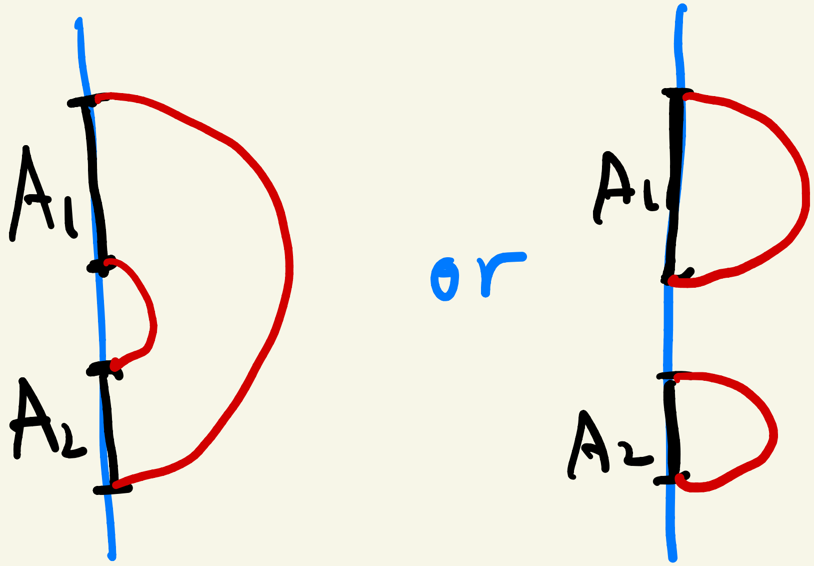


$$S_{AB} + S_{BC} \geq S_{ABC} + S_B$$

(d) 相転移

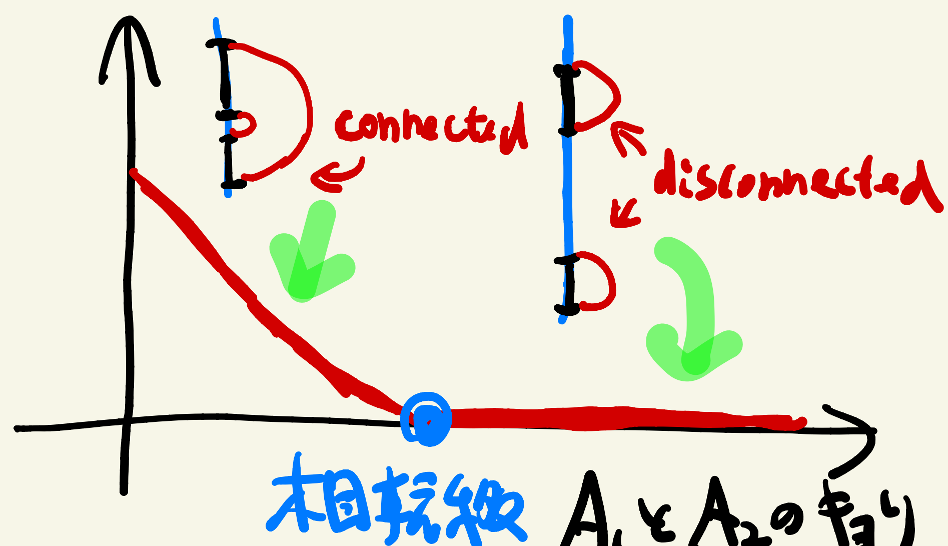
[Headrick 2010]

$A = A_1 \cup A_2$ (disconnected sum) の場合



→ 面積の小さい方をとる.

$$I(A_1:A_2) = S_{A_1} + S_{A_2} - S_{A_1A_2}$$



この相転移は、
ホログラフィックCFT
ラージN・強結合
に特有である!

Monogamy of Mutual Information [Hayden-Headrick-Maloney 2011]

$$I_3(A:B:C) = I(A:B) + I(A:C) - I(A:BC)$$
$$= S_A + S_B + S_C - S_{AB} - S_{BC} - S_{CA} + S_{ABC}$$

Monogamy: "$I(A:BC) \geqq I(A:B) + I(A:C)$"

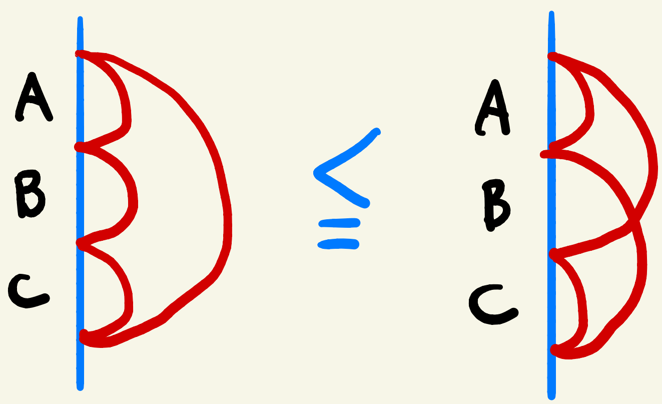
理想的なエンタングルメント測度が満たすべき性質

しかし、MIは一般にMonogamyを破る（古典相関もあるから）

例、GHZ $|\psi\rangle = \frac{1}{\sqrt{2}}(|0000\rangle + |1111\rangle)$

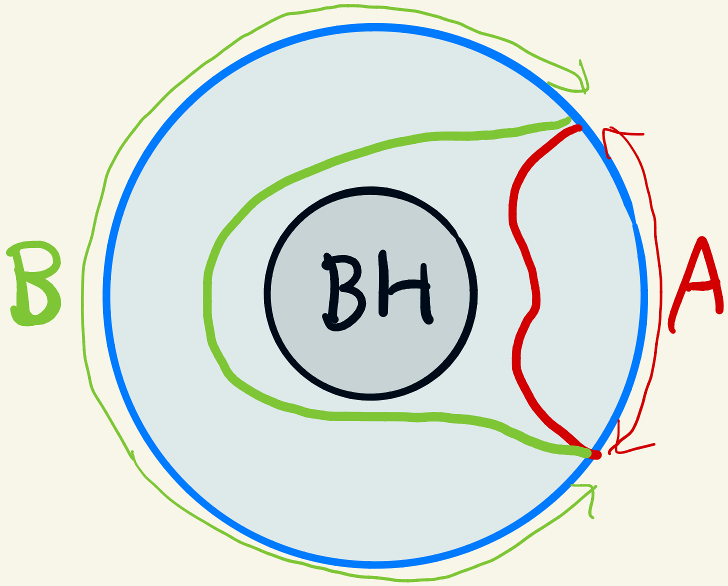
しかし、HEE（古典重力のAdS/CFT）では、Monogamyが成立。

$$I_3(A,B,C) \leqq 0$$



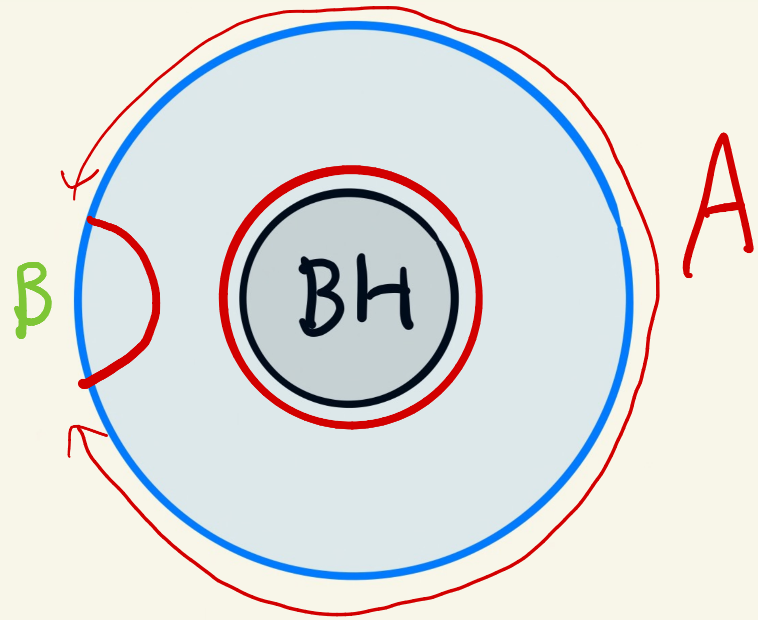
$\Longleftrightarrow$ $S_A + S_B + S_C + S_{ABC} \leqq S_{AB} + S_{BC} + S_{CA}$ //

(e) Pure State vs. Mixed State



$$S_A \neq S_B$$

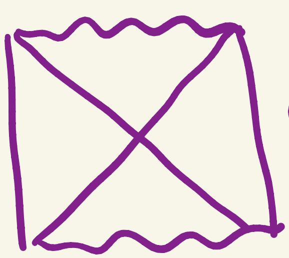
(Mixed State であることと適合)



$$\lim_{|B|\to 0} S_A = S_{BH}$$

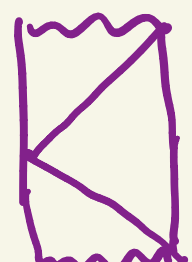
HEEは、BH entropyの拡張.

Mixed State BH (Eternal BH)



$CFT_1 \to S_A \neq S_B$

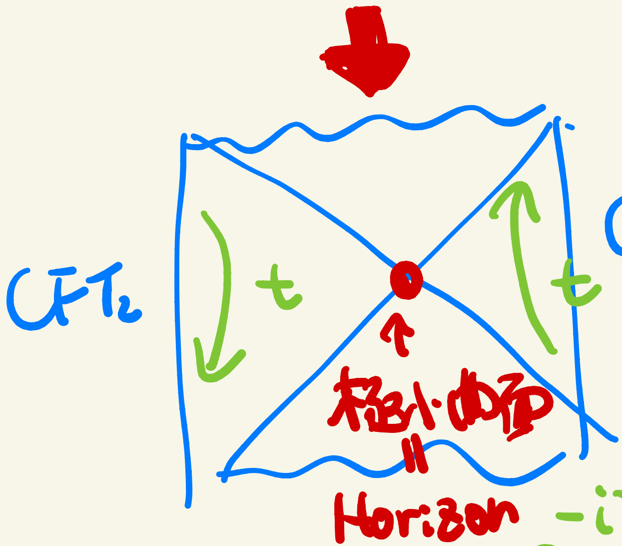
Pure State BH (Single sided BH)



$CFT \to S_A = S_B$

(f) Thermofield Double 状態 (TFD状態)

$$|TFD\rangle = \frac{1}{Z}\sum e^{-\frac{\beta E_n}{2}} |n\rangle_1 \otimes |n\rangle_2$$

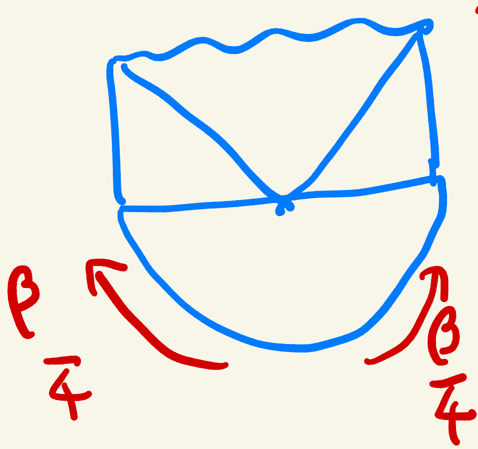


$$\downarrow$$

$$\rho_1 = \mathrm{Tr}_2 |TFD\rangle\langle TFD| = \frac{1}{Z}\sum_n e^{-\beta E_n} |n\rangle\langle n|$$

$$e^{-it(H_1 - H_2)}|TFD\rangle = |TFD\rangle$$

$$S_1 = S(\rho_1) = S_{BH}$$

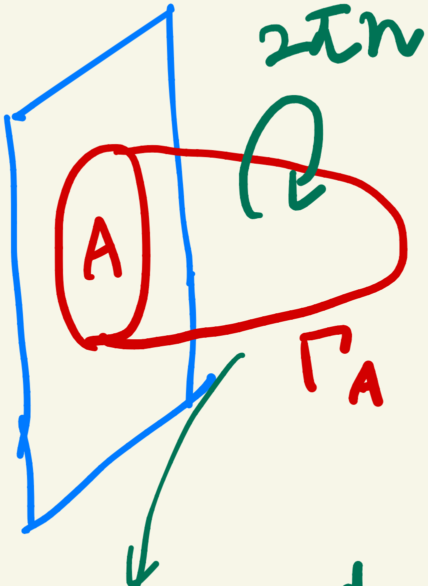


} Lorentzian (Real time evolution)

} Euclid → $e^{-\frac{\beta}{4}(H_1 + H_2)} \sum |n\rangle_1 |n\rangle_2$

$$= |TFD\rangle$$

(6-4) HEEの導出 [Lewkowycz-Maldacena 2013]



$$\mathrm{Tr}(\rho_A)^n = \frac{Z_n}{(Z_1)^n}, \quad Z_n \simeq e^{-I_G^{(n)}}$$

$$I_G^{(n)} = -\frac{1}{16\pi G_N}\int\sqrt{g}\,(R-2\Lambda)+\cdots$$

$$\underset{(n-1)\to 0}{\simeq} \frac{n-1}{4G_N}\int_{\Gamma_A}\sqrt{g} + n\times \text{(///)}$$

$$R = 4\pi(1-n)\cdot\delta^d_{\Gamma_A}(x)$$

↑ Singularだが、Smoothにしても同じ。

$$S_A = -\frac{\partial}{\partial n}\log\frac{Z_n}{(Z_1)^n}\Big|_{n=1} = \frac{1}{4G_N}\int_{\Gamma_A}\sqrt{g}$$

Einstein eq. $\Longleftrightarrow$ $\delta I_G = 0$ $\rightarrow$ $\delta S_A = 0$ (極小曲面)

(6-5) HEE公式 Ver2. [Hubeny-Rangamani-高柳 2007]

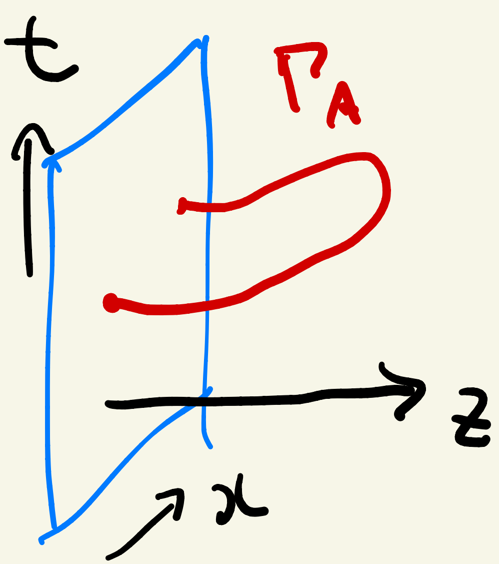
一般に時間に依存する漸近的AdS時空のHEEを計算する公式 (Covariant HEE):

$$S_A = \underset{\Gamma_A^{ext}}{\mathrm{Min}}\ \underset{\Gamma_A}{\mathrm{Ext}}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

非平衡 → 過程への応用 (Kerr BHも)

↓ Extremal surfaceが複数ある時には、その中で面積が最小のものを選ぶ

↳ Lorentzian AdSの中での極値曲面 (Extremal surface)

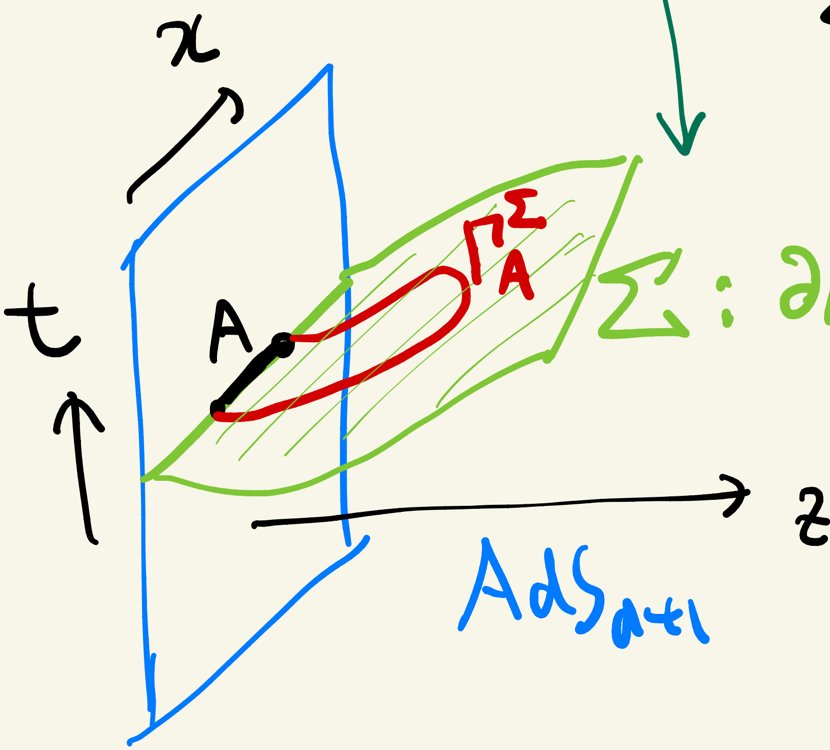


HEE 公式 Ver 2′ → Ver 2 と等価

[Wall 2012]

$$S_A = \underset{\Sigma}{\mathrm{Max}}\ \underset{\Gamma_A^\Sigma}{\mathrm{Min}} \left[\frac{A(\Gamma_A^\Sigma)}{4G_N} \right]$$

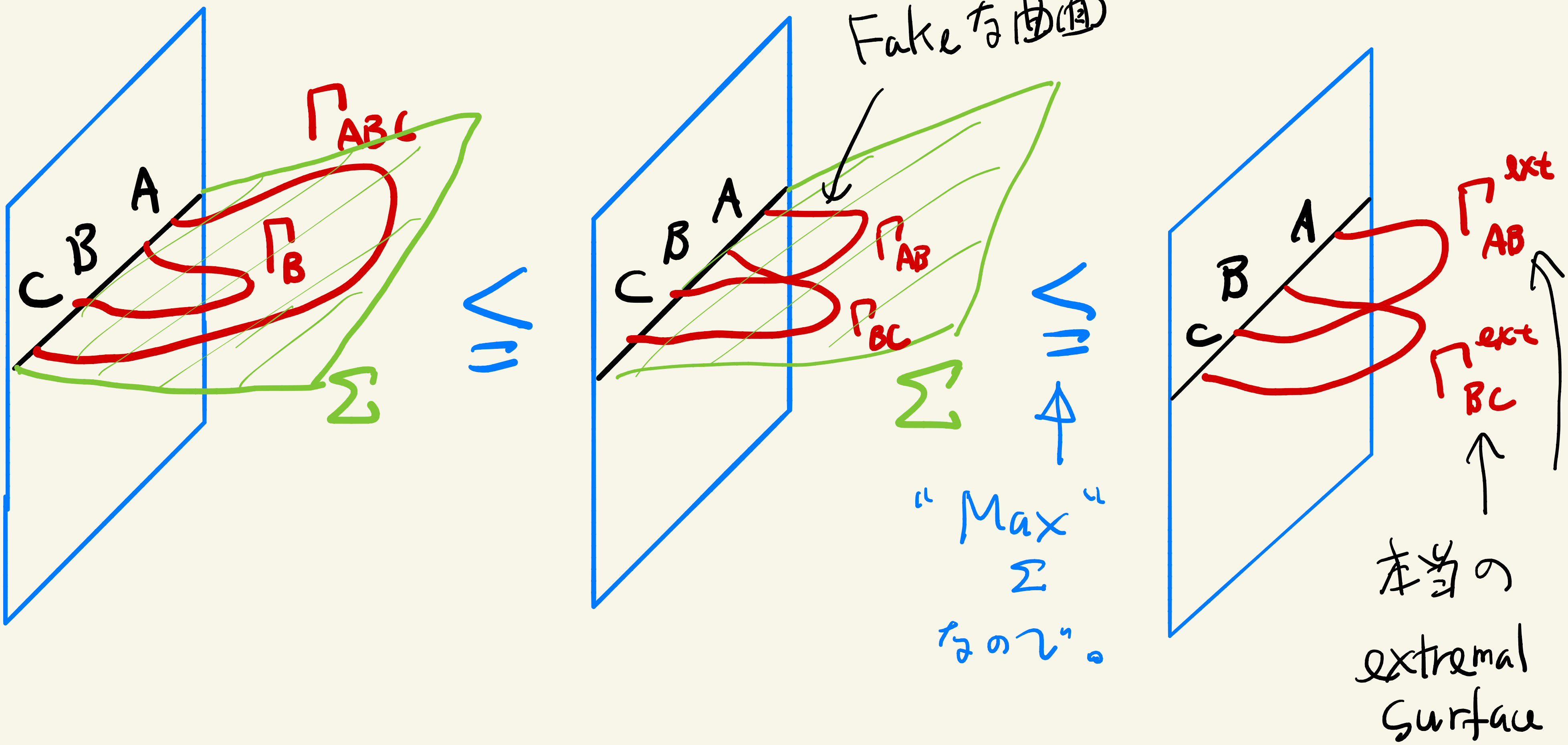
Σ: fixed



Σ: ∂A を含む任意の time-slice

d+1 次元

強劣加法性の証明　Ver 2' を用いる.



$\Rightarrow$ $S_{ABC} + S_B \leq S_{AB} + S_{BC}$ //

(6-6) HEE公式 Ver3

[魏-瀧-玉岡-中田-高柳 2020]

$t \to it$

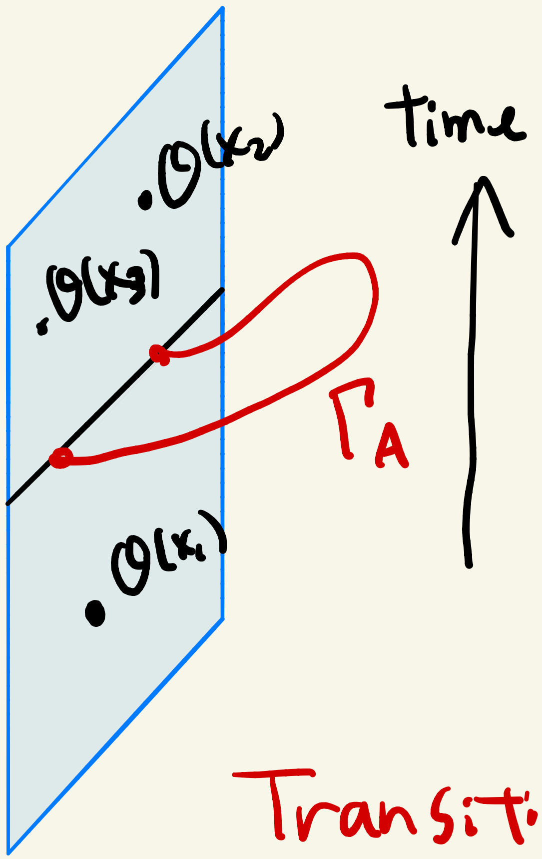
Static な漸近的 AdS 時空 → ユークリッド化は自明

(d+1)次元の Timeslice 上の極小曲面 → (d+2)次元全体の極小曲面
$t=0$

では、より一般に、

ユークリッド時間に依存する漸近的ユークリッド AdS 空間の極小曲面は何と等価？

実は、この答えは、EE の定義を拡張したもの。
(Pseudo Entropy と呼ぶ)



Final State $|\Psi_2\rangle$

Initial State $|\Psi_1\rangle$

Not Hermitian!

Transition Matrix $\tau_A = \mathrm{Tr}_B\left[\frac{|\Psi_1\rangle\langle\Psi_2|}{\langle\Psi_2|\Psi_1\rangle}\right]$

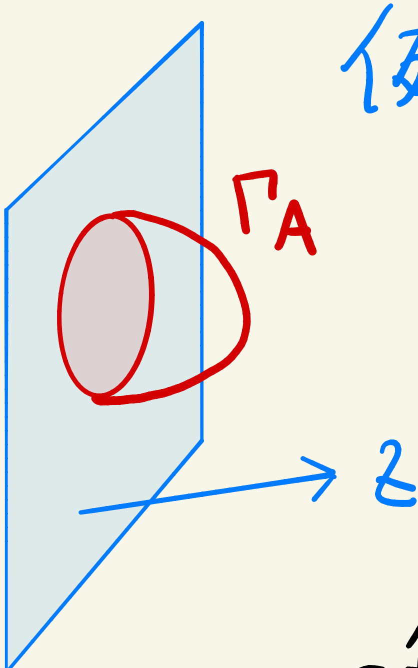
Pseudo Entropy $S'_A = -\mathrm{Tr}[\tau_A \log \tau_A]$

を導入すると、

$$S'_A = \min_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

(6-7) 高次元CFTのHEE

例：$\partial A = S^{d-1}$ in CFT_{d+1} on $\mathbb{R}^{d+1}$



$$ds^2\Big|_{x_0=0} = R^2 \frac{dz^2 + dx_1^2 + \cdots + dx_d^2}{z^2}$$

$$l^2 = x_1^2 + x_2^2 + \cdots + x_d^2$$

$$S_A = a \cdot \int_{\varepsilon/l}^{1} dy \frac{(1-y^2)^{\frac{d-2}{2}}}{y^d} \qquad \left(a = \frac{2\pi^{d/2} R^d}{\Gamma(\frac{d}{2}) \cdot 4G_N} \right)$$

F関数

$$= \underbrace{P_0 \cdot \left(\frac{l}{\varepsilon}\right)^{d-1}}_{\text{面積則}} + P_2 \left(\frac{l}{\varepsilon}\right)^{d-3} + \cdots + \begin{cases} d+1=\text{奇} & + P_{d-2}\left(\frac{l}{\varepsilon}\right) + P_{d-1} \\ d+1=\text{偶} & + \underbrace{q}_{\text{共形アノマリー}} \cdot \log\left(\frac{l}{\varepsilon}\right) + \text{const.} \end{cases}$$

(6-8) HEEへの補正

$$S_A = \min_{\Gamma_A}\left[\frac{A(\Gamma_A)}{4G_N}\right]$$

$\min_{\Gamma_A}$：古典作用の運動方程式の解

$4G_N$ → Einstein重力

$R/l_s \gg 1$, $R/l_{pl} \gg 1$ での近似！

補正を入れると、一般に、

$$\tilde{S}_A = \min_{\Gamma_A}\left[\frac{1}{4G_N}\int_{\Gamma_A}\sqrt{g}\, f(\alpha' R, \alpha' K^2) + O(1) + O(G_N) + \cdots\right]$$

Γ_A ← Quantum Extremal Surface

$\frac{1}{4G_N}\int_{\Gamma_A}\sqrt{g}\, f(\alpha' R, \alpha' K^2)$：弦理論の補正

$O(1) + O(G_N) + \cdots$：量子重力補正（非局所的）

(i) 高階微分項の補正 (弦理論効果の補正)

$$I_G = -\frac{1}{16\pi G_N}\int\sqrt{g}\,[R - 2\Lambda + \alpha' R^2 + \cdots\cdots]$$

時に、ガウスボネ型

$$\lambda\,(R^{abcd}R_{abcd} - 4R^{ab}R_{ab} + R^2)$$

の時は、

$$\rightarrow\ S_A = \frac{1}{4G_N}\,\underset{\Gamma_A}{\mathrm{Min}}\int_{\Gamma_A} dx^d\sqrt{g}\,(1 + 2\lambda R)$$

(R: Γ_A 上の instrinsic な曲率)

(ii) 量子重力の補正

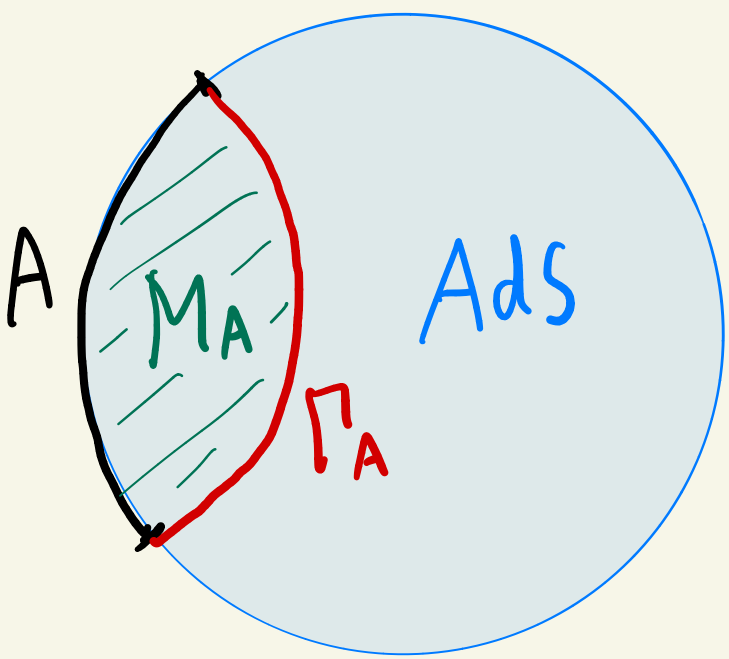
One-loop $\mathcal{O}(1)$ までの補正は、

$$S_A = \underbrace{\frac{A(\Gamma_A^{min})}{4G_N}}_{\mathcal{O}(G_N^{-1})} + \overbrace{S^{bulk}_{M_A}}^{\mathcal{O}(1)}$$

と書ける。

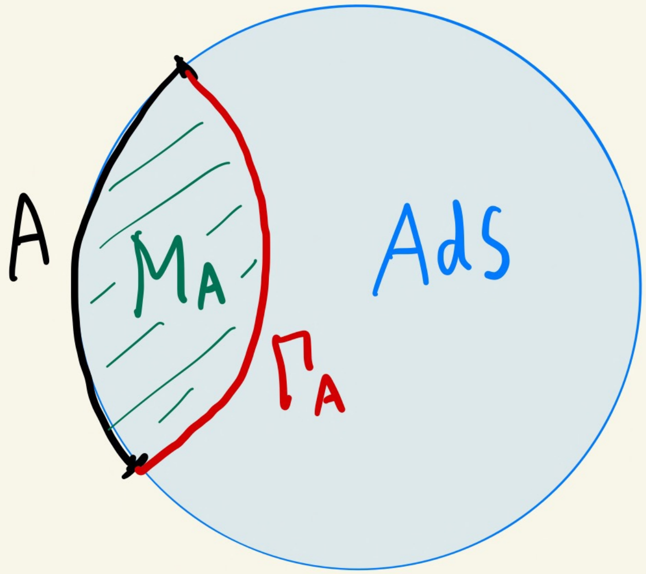
[Faulkner-Lewkowycz-Maldacena 2013]

$S^{bulk}_{M_A}$: Bulk 重力を QFT と見た時の EE



この M_A をエンタングルメント・ウェッジと呼ぶ。 Entanglement Wedge

(6-9) Entanglement Wedge (EW)



CFT の領域 A ⟷ ρ_A

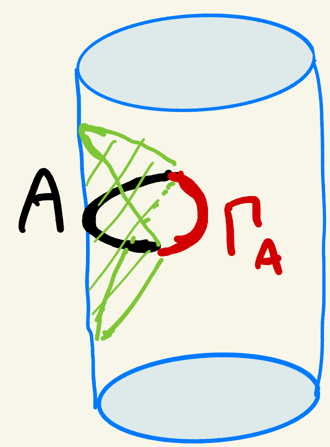
↕ 対応 ↕

AdS の領域 M_A ⟷ $\rho^{bulk}_{M_A}$

M_A 内の重力理論の低エネルギー情報は、ρ_A に記録されている。

これを、Entanglement Reconstruction と呼ぶ。

※但し、時間に依存する背景では、Covariant に、右図のように、EW を定義する。



$$S_A^{CFT} = \frac{A(\Gamma_A)}{4G_N} + S_{M_A}^{bulk}$$

$$\rho_A^{CFT} = e^{-H_A^{CFT}}$$

$$\to \quad H_A^{CFT} = \frac{\hat{A}}{4G} + H_{M_A}^{bulk}$$

重力のエッジモード → Area Operator

Bulk 重力の Modular Hamiltonian

従って

$$S(\rho_A | \rho_A') = S(\rho_{M_A}^{bulk}, \rho_{M_A}'^{bulk})$$

が成り立つ.

ρ_A と ρ_{M_A} の識別は同じ!

[Jafferis-Lewkowycz-Maldacena-Suh 2015]

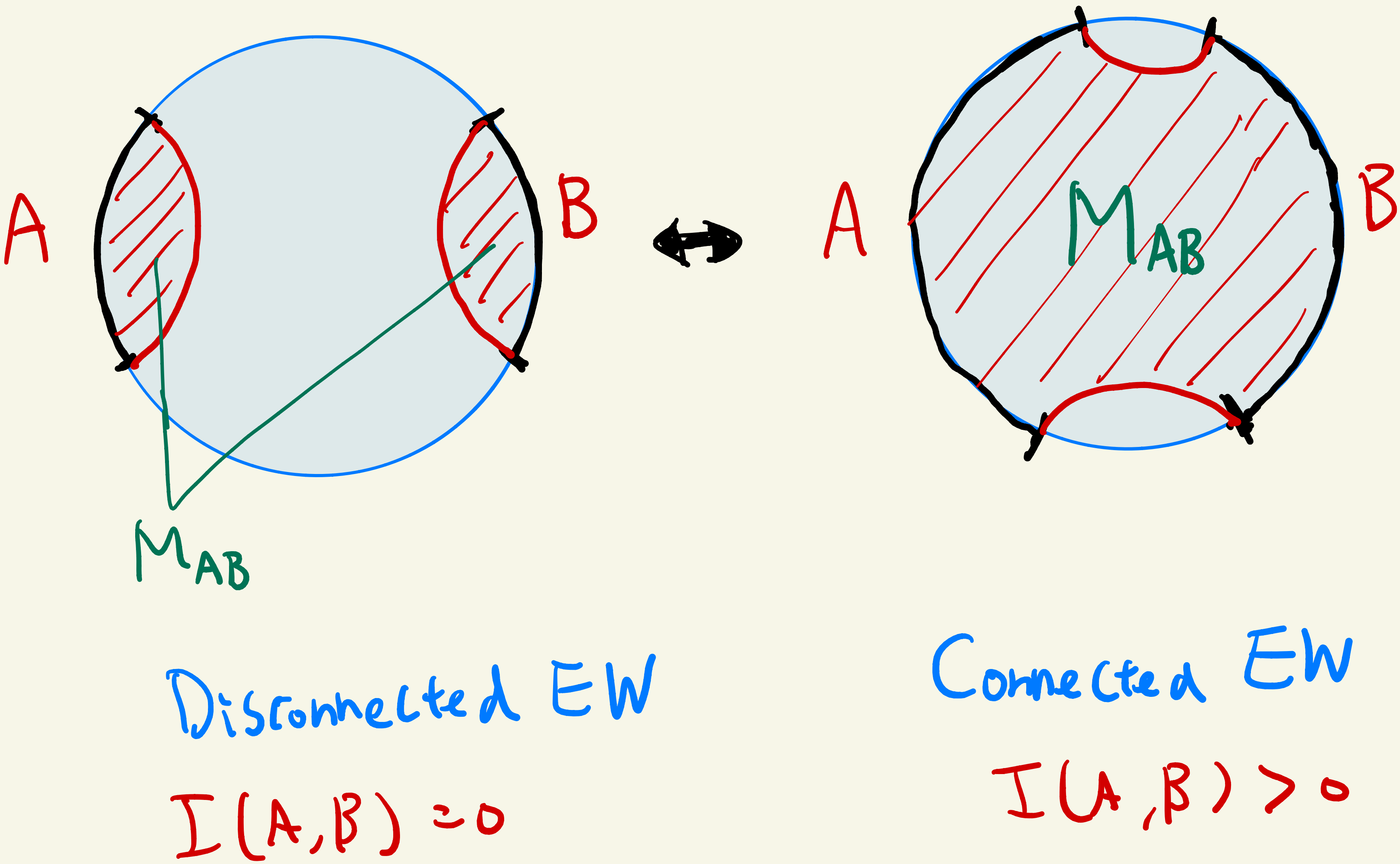
$$\left(\because \; S(\rho_A | \rho_A') = \Delta H_A^{CFT} - \Delta S_A^{CFT} \right.$$

$$= \frac{\Delta A}{4G_N} + \Delta H_{M_A}^{bulk} - \left(\frac{\Delta A}{4G_N} + \Delta S_{M_A}^{bulk} \right)$$

$$\left. = S(\rho_{M_A}^{bulk} | \rho_{M_A}'^{bulk}) \quad // \right)$$

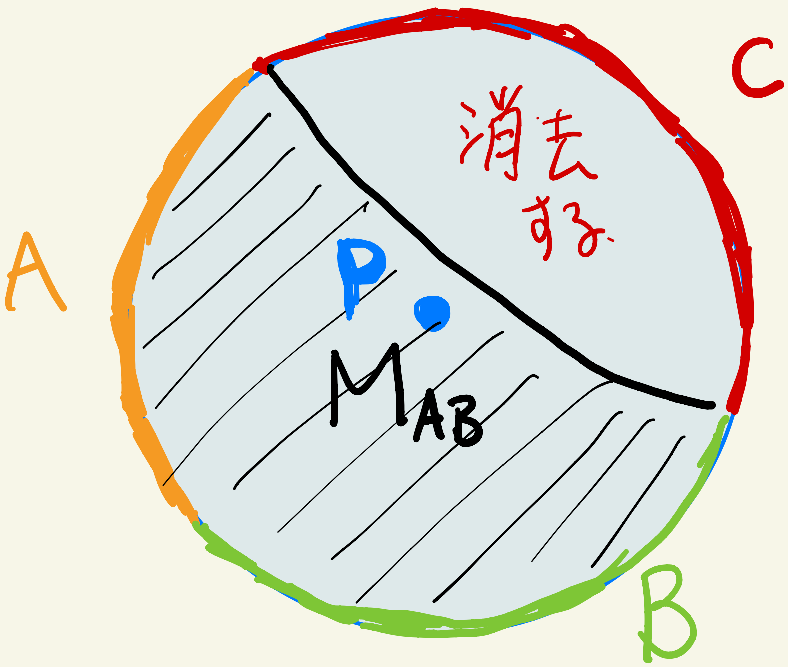
(6-10) EWの相転移

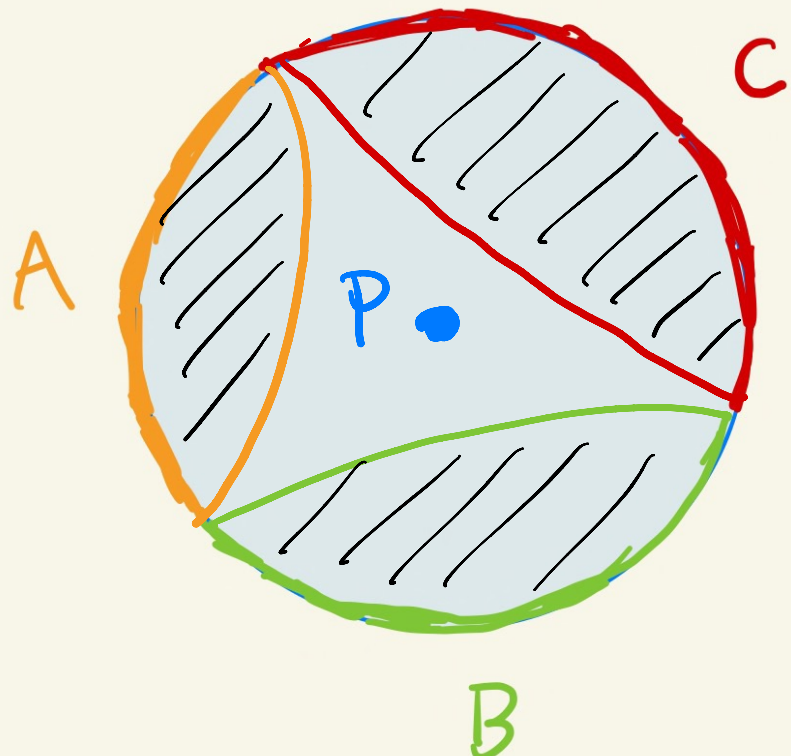
ρ_{AB}に対応するEWを考える.



(6-11) EW と QEC

ρ_{ABC} の EW を考える.



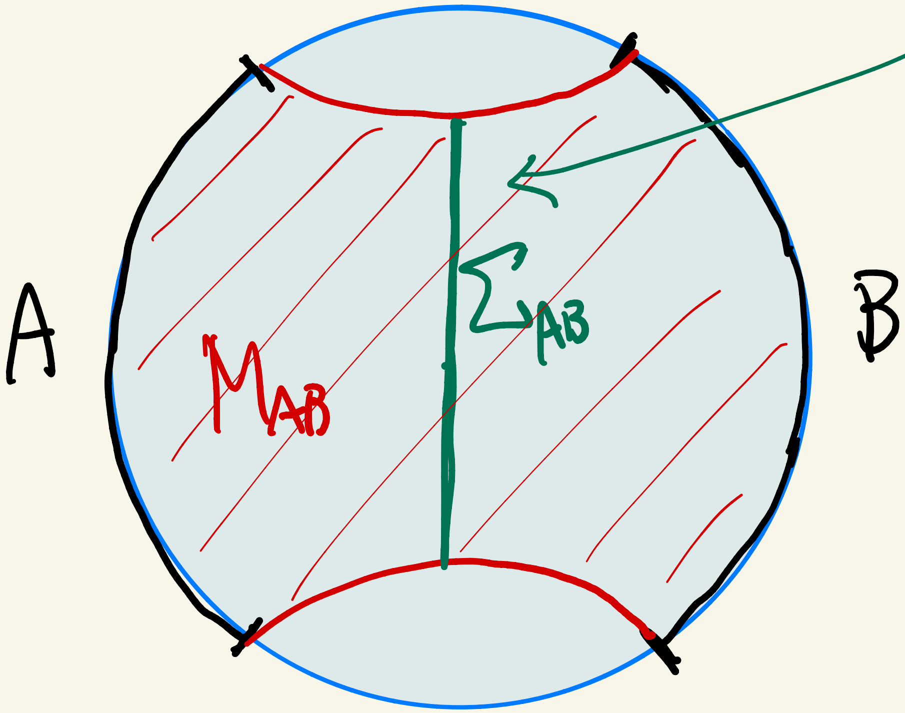


P の情報は、ρ_{AB} から再構成できるが、$\{\rho_A, \rho_B, \rho_C\}$ からは、再構成不可!

→ 量子誤り訂正符号 (Quantum Error Correcting Code) と類似!

(6-12) Holographic Entanglement of Purification

EWの最小断面積



$$E_W(\rho_{AB}) = \frac{A(\Sigma_{AB})}{4G_N}$$

Entanglement Wedge Cross Section

Conjecture

[梅本-高柳, Swingle et.al., 2017]

$$E_W(\rho_{AB}) = E_P(\rho_{AB})$$

Entanglement of Purification

Entanglement of Purification (EoP)

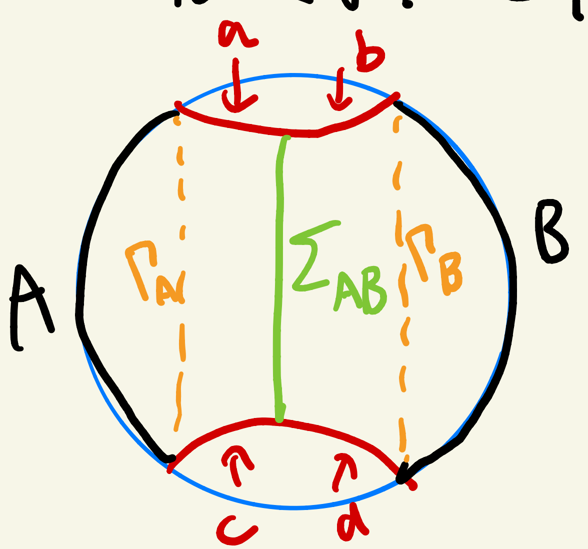
$$H_A \otimes H_B \rightarrow H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$$

ρ_{AB} $\qquad\qquad$ $|\psi\rangle_{A\tilde{A}B\tilde{B}}$: purification

$$E_p(\rho_{AB}) \equiv \min_{|\psi\rangle} \left[S_{A\tilde{A}} \right]$$

$$\text{s.t. } \mathrm{Tr}_{\tilde{A}\tilde{B}} |\psi\rangle\langle\psi| = \rho_{AB}$$

例えば、$E_p(\rho_{AB}) \geqq \frac{1}{2} I(A:B)$ が知られているが、



$$A(\Gamma_A) \leqq A(a) + A(c) + A(\Sigma_{AB})$$

$$A(\Gamma_B) \leqq A(b) + A(d) + A(\Sigma_{AB})$$

両辺足して、$S_A + S_B \leqq S_{AB} + 2 \cdot E_p(\rho_{AB})$

とHolographicに示せる。

(6-13) Einstein方程式とEEの第一法則

例として、AdS$_4$/CFT$_3$を考える.

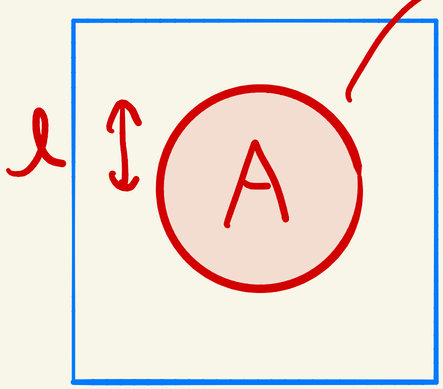
$$ds^2 = \frac{R^2}{z^2}\left(dz^2 + g_{\mu\nu}(x,z)\,dx^\mu dx^\nu\right) \qquad \mu=0,1,2$$

漸近的AdS計量

の1次振動 $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}(x,z)$ を考える.

微小

$S_A(t, x_1, x_2, \ell)$



- Einstein方程式(振動の1次)

$$\longleftrightarrow \left[\frac{\partial^2}{\partial \ell^2} - \frac{1}{\ell}\frac{\partial}{\partial \ell} - \frac{3}{\ell^2} - \frac{\partial^2}{\partial x_1^2} - \frac{\partial^2}{\partial x_2^2}\right]\Delta S_A = 0$$

- Hol Energy Stress Tensor

$$\longleftrightarrow \quad \Delta S_A \xrightarrow{\ell\to 0} \frac{\pi^2 \ell^3}{2} T_{tt}(t, x_1, x_2)$$

これらは、EEの第一法則

$$\Delta S_A \cong \Delta H_A$$

$$H_A(t,x_1,x_2) = 2\pi \int_{|\tilde{x}-x|\leq l} d\tilde{x}_1 d\tilde{x}_2 \left(\frac{l^2 - |x-\tilde{x}|^2}{2l} \right) T_{tt}(t,\tilde{x}_1,\tilde{x}_2)$$

↑ Aの中心点

から従う！

Einstein方程式の摂動 ＝ EEの第一法則

⑦ Entanglement Wedges from CFTs

梅本-鈴木-高柳　arXiv: 1908.09939 [PRL 123(2019)22, 221601]

梅本-楠亀-鈴木-高柳　arXiv: 1912.08423 [To appear in PTEP]

内容

[7-1] Introduction

[7-2] Bures Information Metric

[7-3] Single Interval Case

[7-4] Double Intervals Case

[7-5] AdS/BCFT Case

[7-6] HKLL States

[7-7] Conclusions

AdS/CFTを仮定せずに

CFTから、EWの存在

を導出したい!

Main claim

CFT Wedges = Shadows of Entanglement Wedges

Purely defined in CFTs from the information metric.
Exists only in holographic CFTs.

We also suggest

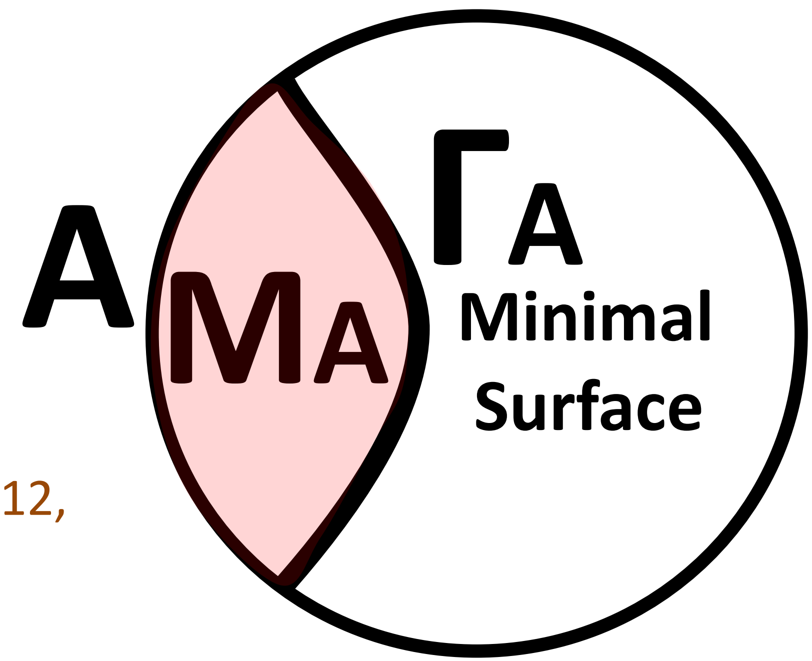
Bures Information Metric in CFT ~ Time Slice Metric of AdS

[7-1] Introduction

One of the basic mechanisms on how AdS/CFT works, is Entanglement Wedges (or Subregion-Subregion duality).

Which bulk region is dual to a given region A in CFT ?
⇒ Entanglement Wedge (EW): M_A

ρ_A in CFT
$\Leftrightarrow$ ρ_{MA} in AdS gravity

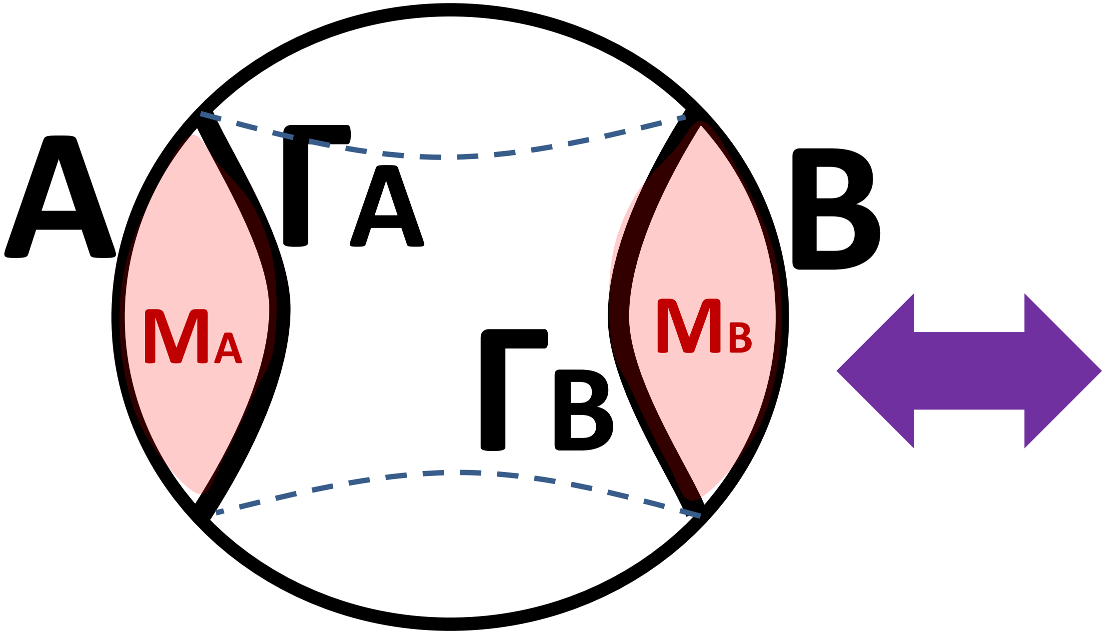


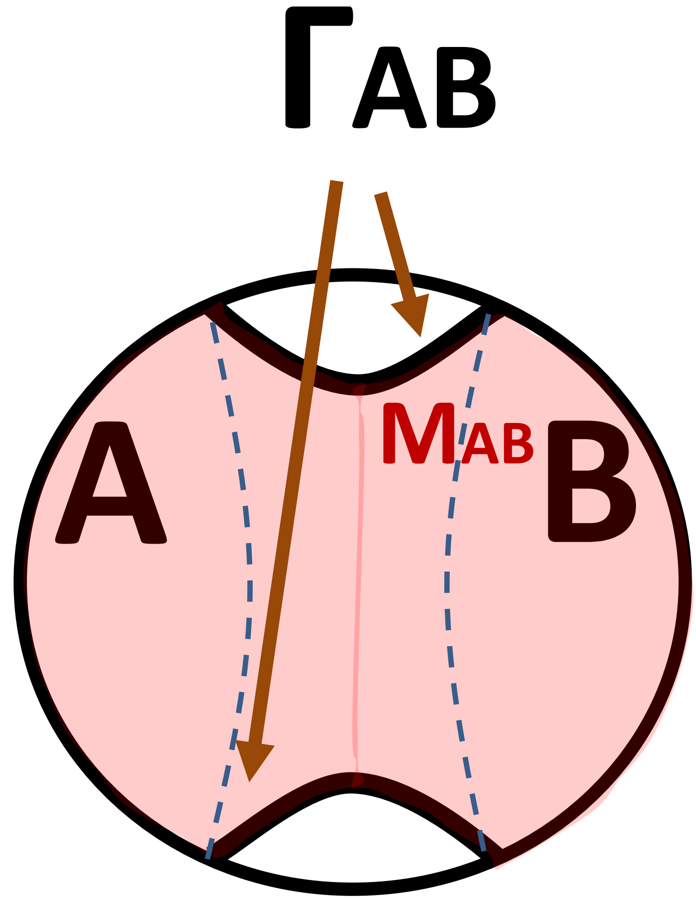
Time slice of AdS

[Czech-Karczmarek-Nogueira-Raamsdonk 2012, Wall 2012, Headrick-Hubeny-Lawrence-Rangamani 2014]

EW for Disconnected Subregions

$$I(A:B) = S_A + S_B - S_{AB} = 0$$





$$I(A:B) > 0$$

$M_{AB} = M_A \cup M_B$, $\Gamma_{AB} = \Gamma_A \cup \Gamma_B$

In this talk, we would like to start from (holographic) CFTs and derive the entanglement wedge geometry in the CFT language. (i.e. postpone the use of AdS/CFT to the end.)

Our argument will clarify how entanglement wedges emerge from reduced density matrices in CFTs.

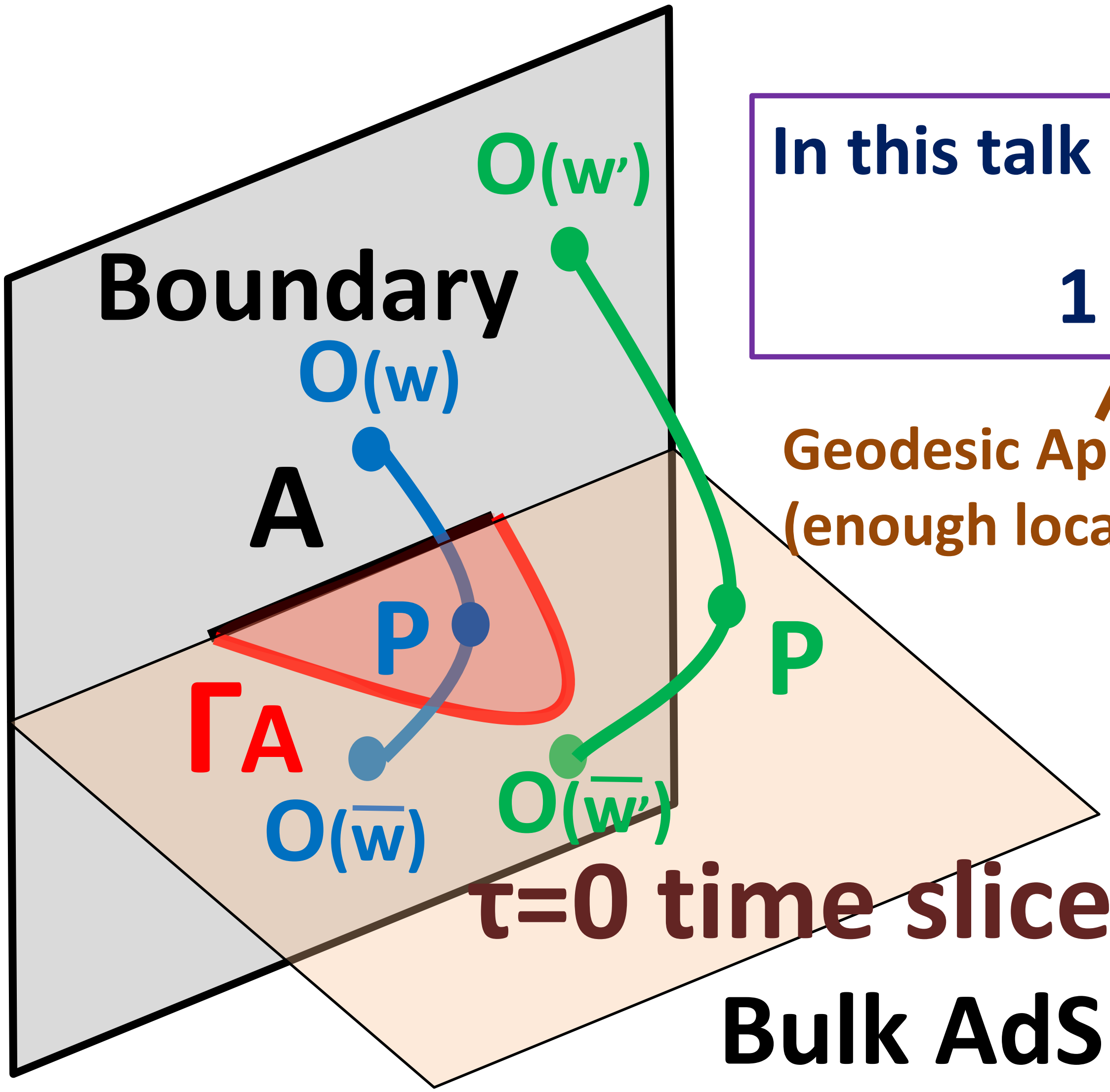
For this purpose we study locally excited states in 2d CFTs:

$$\boldsymbol{\rho_A(w,\bar{w}) = \mathrm{Tr}_B \left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w) \right]}$$

$w = x + i\tau$ is the coordinate of complex plane R^2.

$O(w,\bar{w})$: A primary in 2d CFT with (chiral) conformal dim. h

$\rho_A(w,\bar{w}) = \mathrm{Tr}_B \left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w) \right]$ and its Dual



In this talk we always assume

$1 \ll h \ll c$.

Geodesic Approximation
(enough localization)

Negligible
Back-reactions

ρ_A may include the information of O(w) but not of O(w').

[7-2] Bures Information Metric

To study distinguishability of quantum states, consider a distance measure between density matrices ρ and ρ'.

Especially we focus on the Bures distance:

$$D_B(\rho, \rho')^{\,2} = 2 - 2\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$$

When the states are pure, we have

$$D_B(|\Psi\rangle\langle\Psi|, |\Psi'\rangle\langle\Psi'|)^{\,2} = 2(1 - |\langle\Psi|\Psi'\rangle|).$$

Assume density matrices depend on parameters λi, denoted by ρ(λ). **The Bures metric** is defined as follows:

$$ds^2 \equiv D_B(\rho(\lambda), \rho(\lambda + d\lambda))^{\,2} \cong G_{ij} d\lambda^i d\lambda^j$$

Quantum Cramer-Rao Theorem

$$\left\langle \langle \delta\lambda_i \delta\lambda_j \rangle \right\rangle \geq G^{-1}_{ij}$$

[Heldstrom 1976]

↑

Errors in estimation of values of λi
based on quantum (POVM) measurements of ρ(λ)

Warmup example: Pure States in CFTs

Consider the locally excited state $|\Psi(w)\rangle = O(w,\bar{w})|0\rangle$.

Primary with conformal dim. h

The Bures distance is computed as

$$D_B(|\Psi(w)\rangle\langle\Psi(w)|, |\Psi(w')\rangle\langle\Psi(w')|)^2 = 2(1 - |\langle\Psi(w)|\Psi(w')\rangle|).$$

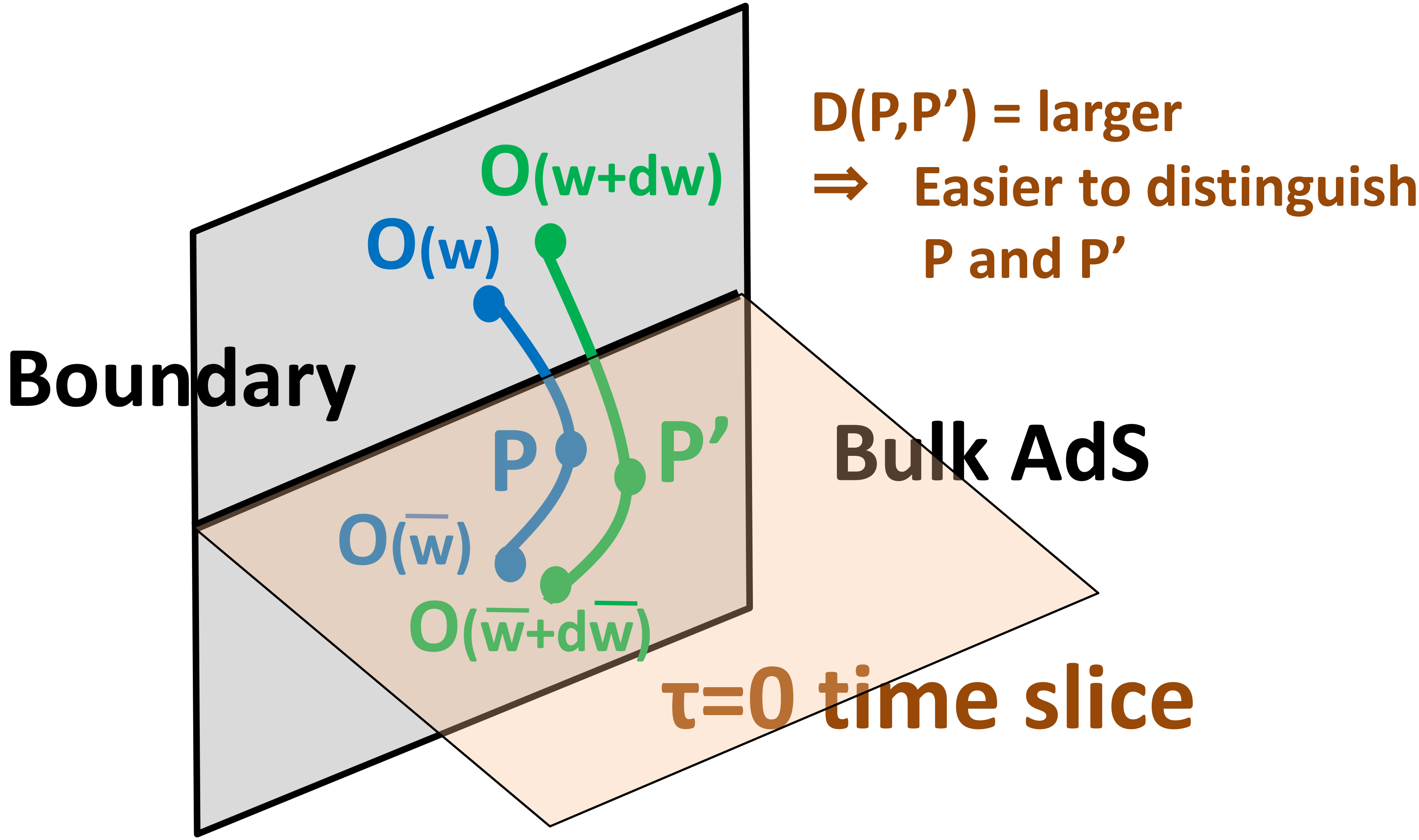
$$|\langle\Psi(w)|\Psi(w')\rangle| = |w - \bar{w}|^{2h} |w' - \overline{w'}|^{2h} |w - \overline{w'}|^{-4h}.$$

The Bures metric reads (w=x+iτ)

$$ds_B^2 = \frac{h}{\tau^2}(dx^2 + d\tau^2)$$

⇒ Coincides with the metric on the time slice of AdS up to normalization

Information Metric ∝ Geometric Metric in Gravity Dual



However, the info. metric is universal for any CFTs.

⇒ The situation largely changes for ρ_A as we will see.

[7-3] Single Interval Case

Primary with conformal dim. h

Consider the reduced density matrix

$$\rho_A(w,\bar{w}) = \mathrm{Tr}_B\left[O(w,\bar{w})|0\rangle\langle 0|O^\dagger(\bar{w},w)\right].$$

We choose A to be an interval A=[0,L] on R^2.

We would like to calculate its Bures metric.

As a first step, let us start with the computation of `Renyi version' of Bures distance:

[Introduced by Cardy 2014]

$$I(\rho,\rho') = \frac{\mathrm{Tr}[\rho\rho']}{\sqrt{\mathrm{Tr}[\rho^2]\mathrm{Tr}[\rho'^2]}} \leq 1.$$

Cf. $2 - D_B(\rho,\rho')^2$
$= 2\mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$

$$I(\rho,\rho') = 1 \;\leftrightarrow\; \rho = \rho'.$$

We can calculate **Tr[ρρ']** by pasting the w-plane for ρ and that for ρ' via the conformal map: $z^2 = \frac{w}{w-L}$.

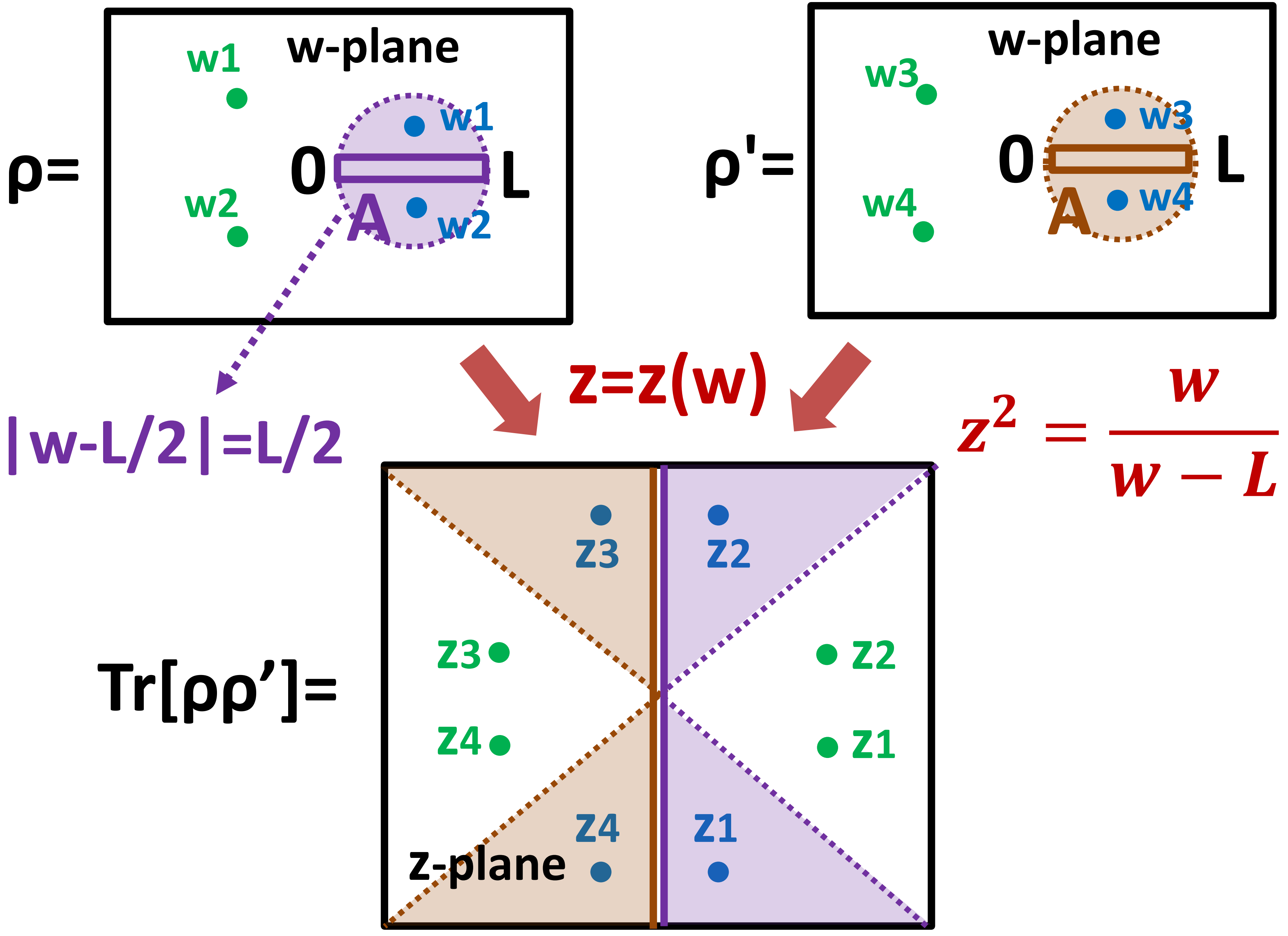
$$\mathrm{Tr}[\rho\rho'] = \left[\left|\frac{dz_1}{dw_1}\right|\left|\frac{dz_2}{dw_2}\right|\left|\frac{dz'_3}{dw'_3}\right|\left|\frac{dz'_4}{dw'_4}\right|\right]^{2h} \cdot \frac{H(z_1, z_2, z'_3, z'_4)\cdot Z^{(2)}}{(Z^{(1)})^2},$$

$$H(z_1, z_2, z'_3, z'_4) \equiv \frac{\langle O^\dagger(z_1,\bar{z}_1)O(z_2,\bar{z}_2)O^\dagger(z'_3,\bar{z}'_3)O(z'_4,\bar{z}'_4)\rangle}{\langle O^\dagger(w_1,\bar{w}_1)O(w_2,\bar{w}_2)\rangle\langle O^\dagger(w'_3,\bar{w}'_3)O(w'_4,\bar{w}'_4)\rangle},$$

where $\langle\cdots\rangle$ denotes the normalized correlation function such that $\langle 1\rangle = 1$ and we also write the vacuum partition function on a n-sheeted complex plane by $Z^{(n)}$. Finally we obtain

$$I(\rho,\rho') = \frac{F(z_1, z_2, z'_3, z'_4)}{\sqrt{F(z_1, z_2, z_3, z_4)F(z'_1, z'_2, z'_3, z'_4)}},$$

$$F(z_1, z_2, z'_3, z'_4) \equiv \langle O^\dagger(z_1,\bar{z}_1)O(z_2,\bar{z}_2)O^\dagger(z'_3,\bar{z}'_3)O(z'_4,\bar{z}'_4)\rangle.$$



ρ=
w-plane
w1
w2
w1
w2
0
L
A
ρ'=
w-plane
w3
w4
w3
w4
0
L
A
|w-L/2|=L/2
z=z(w)
$z^2 = \frac{w}{w-L}$
Tr[ρρ']=
Z3
Z2
Z3
Z2
Z4
Z1
Z4
Z1
z-plane

I(ρ,ρ') in 2d Holographic CFTs

The 4-pt function $F(z_1, z_2, z_3, z_4) = \langle O^\dagger O O^\dagger O \rangle$ is given by applying the generalized free field prescription:

$$F \approx |z_1 - z_2|^{-4h}|z_3 - z_4|^{-4h} + |z_2 - z_3|^{-4h}|z_1 - z_4|^{-4h}.$$

Trivial Wick Contraction

$\Rightarrow$ **Always $I(\rho, \rho') = 1$**

$\Rightarrow$ $\rho = \rho'$.

(i.e. Indistinguishable)

Non-Trivial Wick Contraction

$\Rightarrow$ **In general, $I(\rho, \rho') < 1$**

(i.e. distinguishable)

Having in mind the information metric, we take w ≈ w'.

[i] Case 1: the trivial Wick contraction is dominant:

$$|z_2 - z_3| < |z_1 - z_4| \quad \Rightarrow |w - L/2| > L/2$$

w ∉ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are indistinguishable !
Always we find I(ρ,ρ')=1.

[ii] Case 2: the non-trivial Wick contraction is domiant:

$$|z_2 - z_3| > |z_1 - z_4| \quad \Rightarrow |w - L/2| < L/2$$

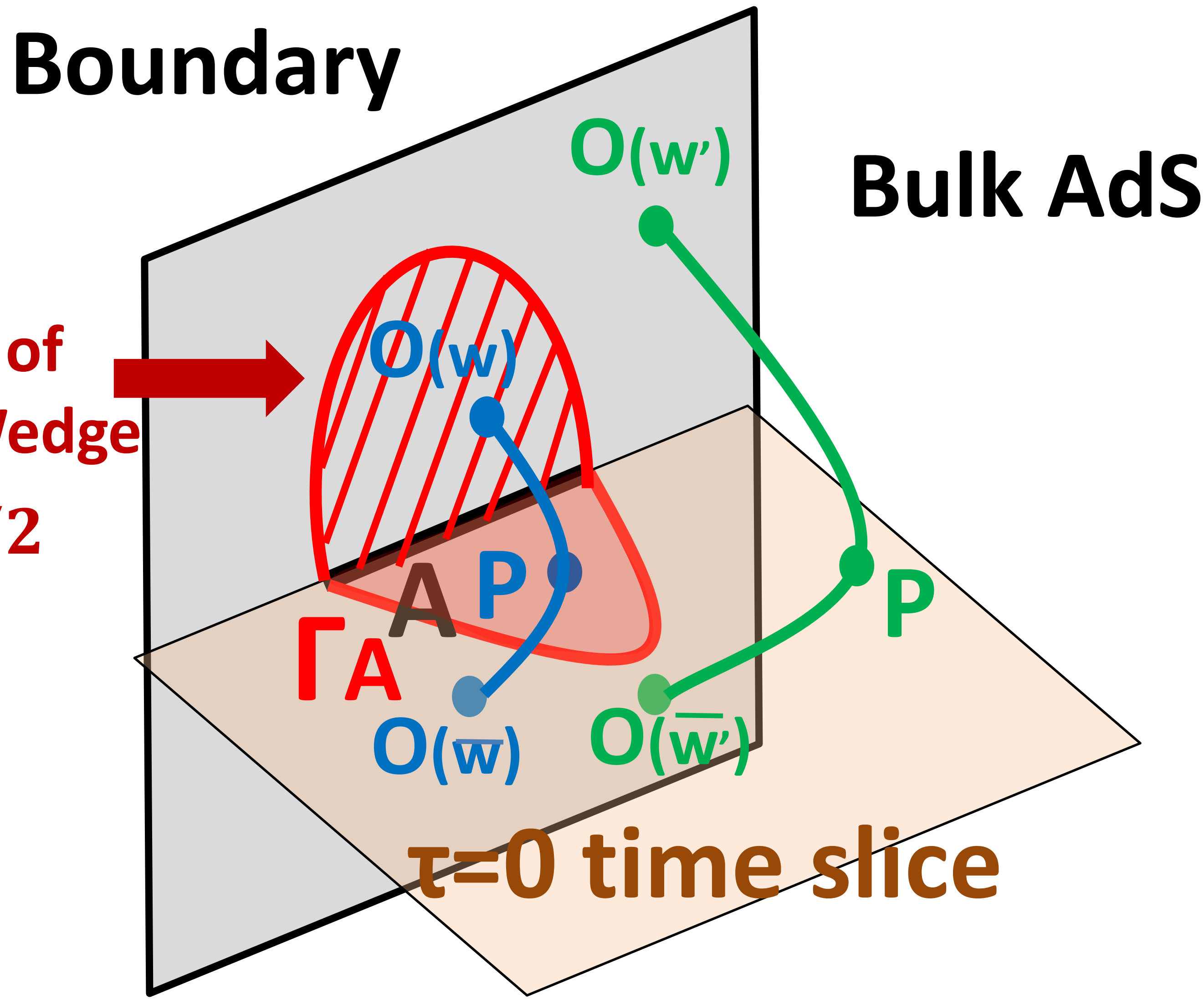
w ∈ Entanglement Wedge

$\rho_A(w)$ and $\rho_A(w')$ are distinguishable !
I(ρ,ρ')=1 only when w=w'.

A Sketch of Entanglement Wedge from CFTs

CFT counterpart of Entanglement Wedge

$|w - L/2| < L/2$



I(ρ,ρ’) in 2d Free Scalar CFTs

As a comparison, consider the massless free scalar CFT in two dimensions. We simply choose O to be

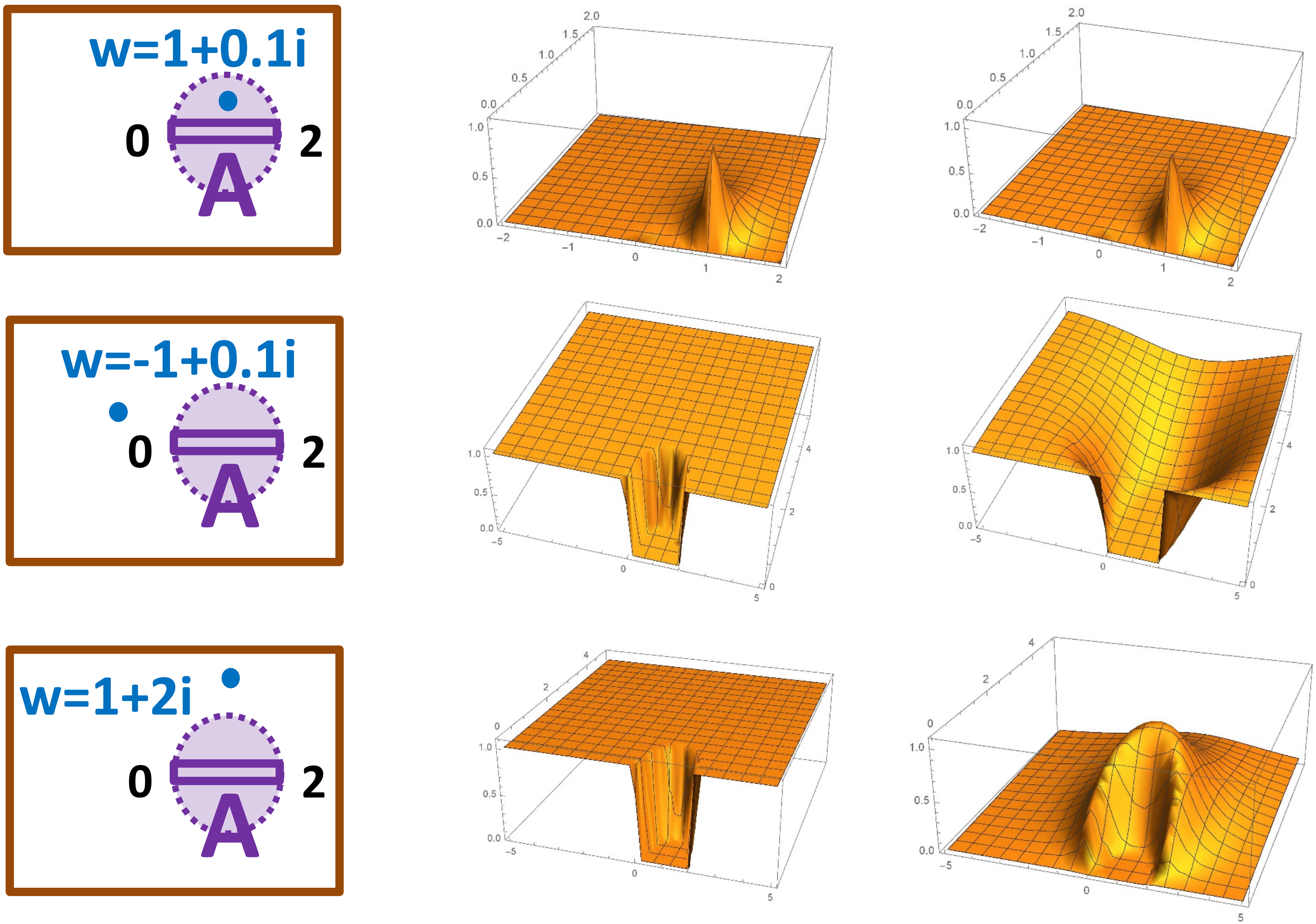
$$O(w,\bar{w}) = e^{i\alpha\varphi(w,\bar{w})}. \qquad h = \alpha^2/2$$

In this case we find (z1=z, z’3=-z’)

$$I(\rho,\rho') = \left[\frac{|z+z'|^2|z+\bar{z}||z'+\overline{z'}|}{4|z||z'||z+\bar{z}'|^2}\right]^{4h}$$

Plot of I(ρ,ρ')

Holographic CFT | **Free Scalar CFT**



Bures Metric in Holographic CFTs

To be exact, we need to calculate the Bures metric.

We introduce $A_{n,m} = \mathrm{Tr}[(\rho^m \rho' \rho^m)^n]$.

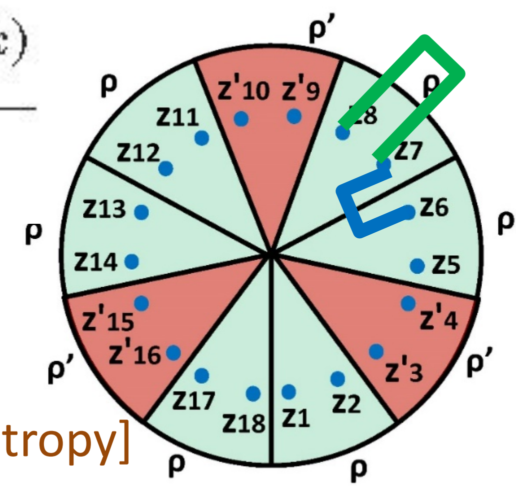
We compute the Bures metric via an analytical continuation:

$$A_{n=\frac{1}{2},m=\frac{1}{2}} = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right].$$

To evaluate Anm, consider the map: $z^k = \frac{w}{w-L}$, k=(2m+1)n

$$A_{n,m} = \frac{\langle O^\dagger(w_1)O(w_2)\cdots O^\dagger(w_{2k-1})O(w_{2k})\rangle \cdot Z^{(k)}}{\prod_{i=1}^{k}\langle O^\dagger(w_{2i-1})O(w_{2i})\rangle \cdot (Z^{(1)})^k}$$

$A_{n=3,m=1} =$



[See also Lashkari 2014-15, Sarosi-Ugajin 2016-18 for relative entropy]

Trivial Wick Contraction (Outside EW: |w-L/2|>L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle \simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-1})O(z_{2j})\rangle \simeq \prod_{j=1}^{k}|z_{2j-1}-z_{2j}|^{-4h}$$

➡ $A_{1/2,1/2}=1$, $D_B(\rho,\rho')=0$

Trivial metric

Non-Trivial Wick Contraction (Inside EW: |w-L/2|<L/2)

$$\langle O^\dagger(z_1)O(z_2)\cdots O^\dagger(z_{2k-1})O(z_{2k})\rangle \simeq \prod_{j=1}^{k}\langle O^\dagger(z_{2j-2})O(z_{2j-1})\rangle \simeq \prod_{j=1}^{k}|z_{2j-2}-z_{2j-1}|^{-4h}.$$

In the limit $n \to 1/2$ and $m \to 1/2$, this leads to

$$A_{1/2,1/2} \simeq |w-\bar{w}|^{2h}|w'-\bar{w}'|^{2h}|w'-\bar{w}|^{-4h},$$

$$dD_B^2 \simeq \frac{h}{\tau^2}(dx^2+d\tau^2).$$

➡ Reproduce the time slice of AdS

Other examples of Bure Metrics in Hol. CFTs

[1] Hol. CFT on Cylinder

$$ds_B^2 = \frac{h}{(\sinh\tau)^2}\left(d\tau^2 + dx^2\right)$$ ➡ **Agrees with Global AdS**

[2] Hol. CFT at Finite temp.

$$ds_B^2 = \frac{h(2\pi/\beta)^2}{\left(\sin\frac{2\pi}{\beta}\tau\right)^2}\left(d\tau^2 + dx^2\right)$$ ➡ **Agrees with Global BTZ**

Bures Distance/Metric in free scalar CFT

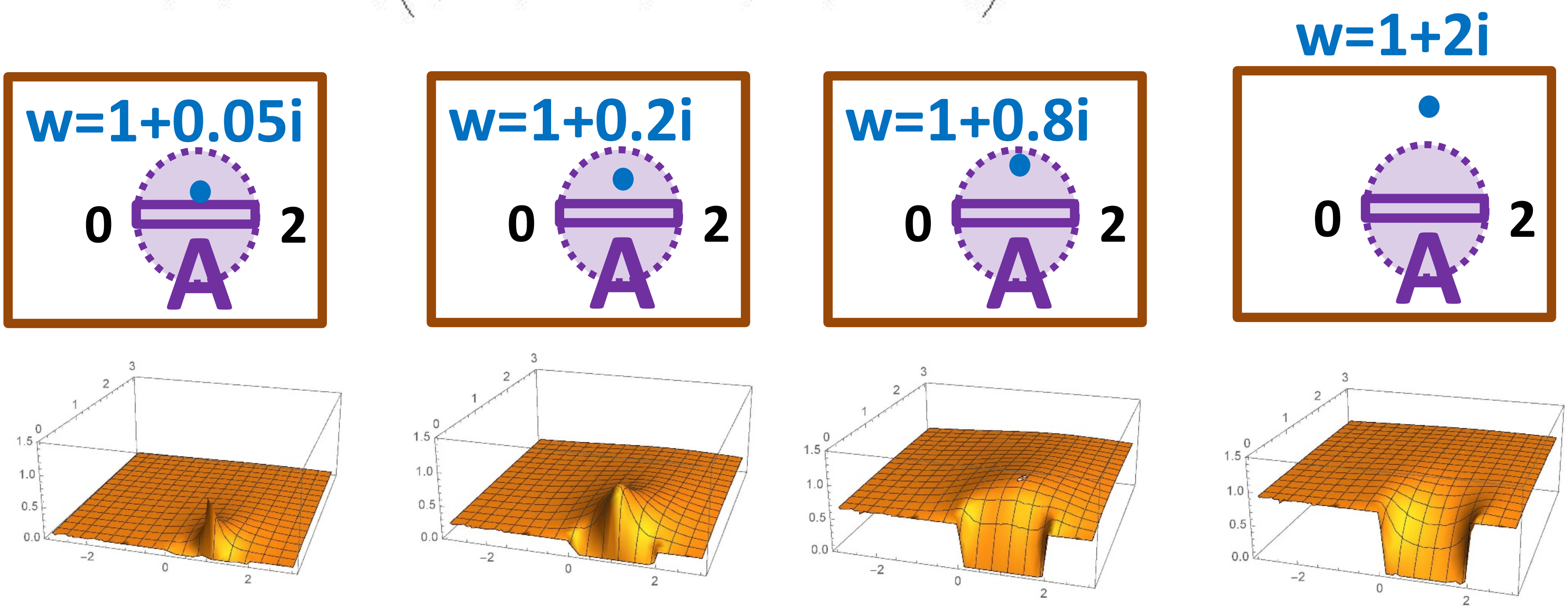
$$A_{\frac{1}{2},\frac{1}{2}} = \frac{(\sqrt{z}+\sqrt{z'})(\sqrt{\bar{z}}+\sqrt{\bar{z}'})(\sqrt{z}+\sqrt{\bar{z}})(\sqrt{z'}+\sqrt{\bar{z}'})}{4\sqrt{|z||z'|}(\sqrt{z}+\sqrt{\bar{z}'})(\sqrt{\bar{z}}+\sqrt{z'})},$$

$$dD_B^2 = -\frac{L^2(dw)^2}{16w^2(L-w)^2} - \frac{L^2(d\bar{w})^2}{16\bar{w}^2(L-\bar{w})^2} + \frac{L^2(dw)(d\bar{w})}{2|w||w-L|\left(\sqrt{\bar{w}(w-L)}+\sqrt{w(\bar{w}-L)}\right)^2}.$$

We choose

$$\boldsymbol{O} = \boldsymbol{e^{i\varphi}}$$

$$(\boldsymbol{\alpha} = \mathbf{1})$$



[7-4] Double Intervals Case

Let us move on to a more complicated case where the subsystem A consists of double intervals in 2d CFTs.

We choose A as A=A1 ∪ A2, A1=[0,s] and A2=[l+s,l+2s].

Comformal Map:
from a cpx plane with two slits to a cylinder

$$z = f(w) = \frac{J(\kappa^2)}{2} - \frac{J(\kappa^2)}{2K(\kappa^2)}\,\mathrm{sn}^{-1}(\widetilde{w}, \kappa^2),$$

where we introduced

$$\tilde{w} = \frac{2}{l}\left(w - s - \frac{l}{2}\right), \quad \mathbf{J}\left(\kappa^2\right) = 2\pi\frac{K(\kappa^2)}{K(1-\kappa^2)},$$

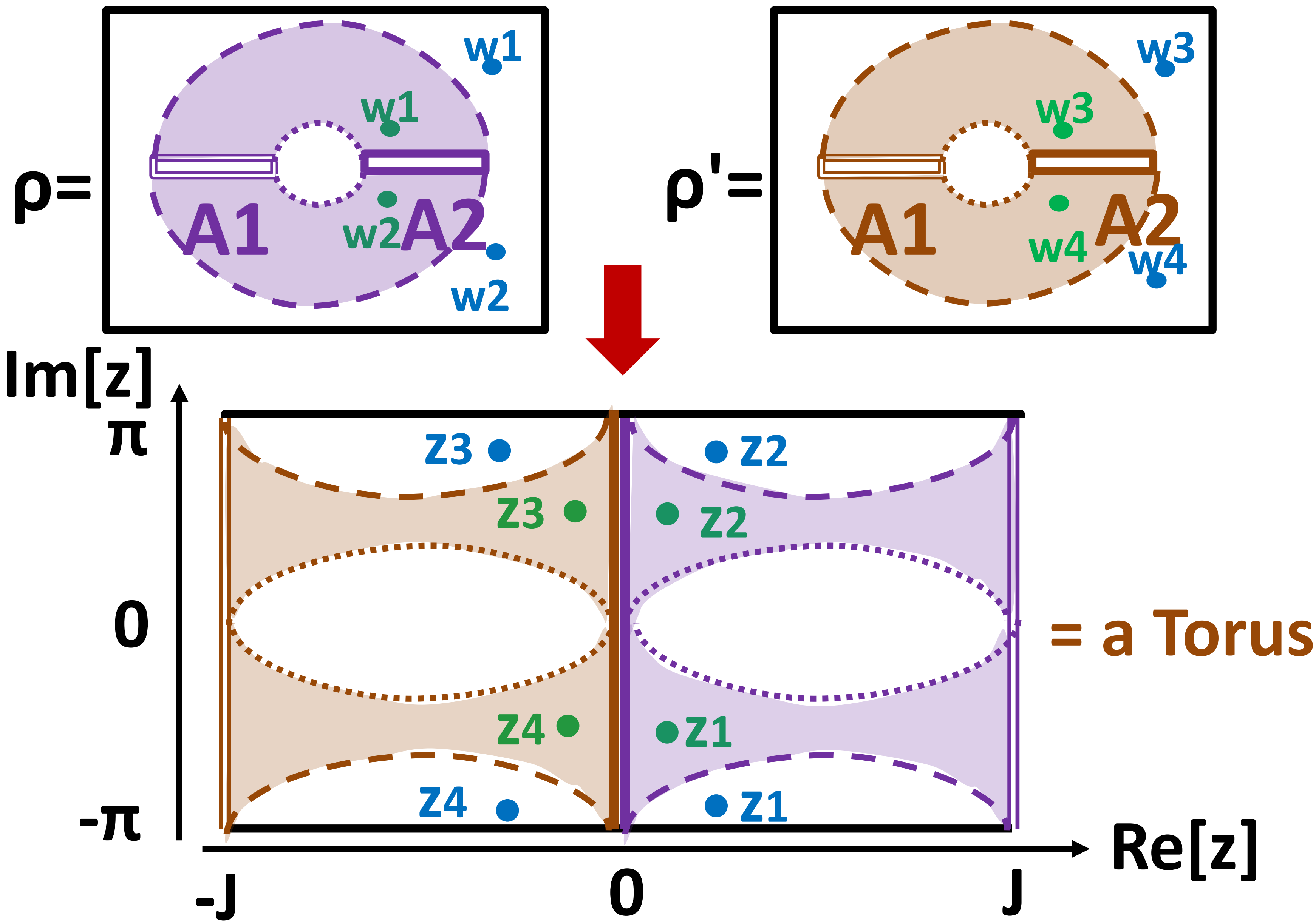
$$K(\kappa^2) = \int_0^1 \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}, \quad \kappa = \frac{l}{l+2s}.$$

[e.g. Rajabpour 2015]

The function $\mathrm{sn}^{-1}(\tilde{w}, \kappa^2)$ is the Jacobi elliptic function:

$$\mathrm{sn}^{-1}(\tilde{w}, \kappa^2) = \int_0^{\tilde{w}} \frac{dx}{\sqrt{(1-x^2)(1-\kappa^2 x^2)}}.$$

Computation of Tr[ρρ’]



There are two phases depending on the torus moduli.

[As in mutual information for double intervals: Headrick 2010, Hartman 2013,…]

(i) Connected Phase

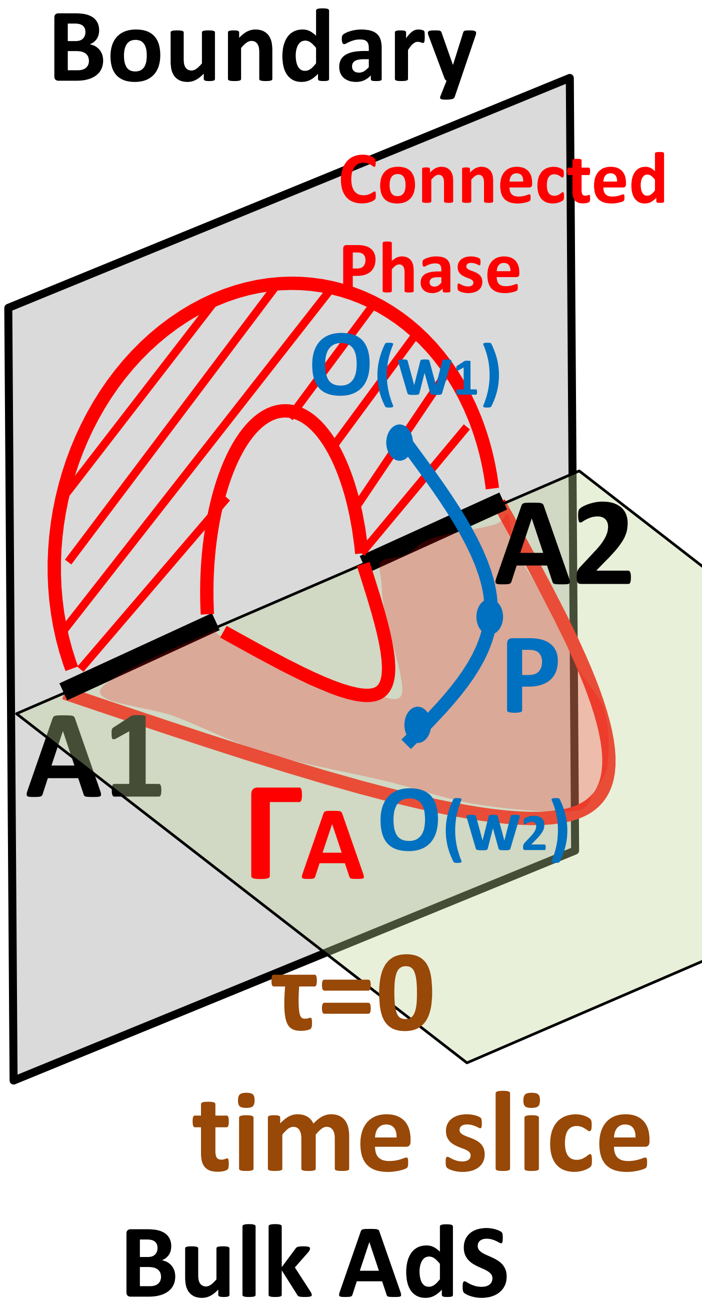
$J < \pi \ \ (or\ \kappa < 3\text{-}2\sqrt{2})$

$$\langle O^\dagger(z) O(z') \rangle \approx \left| \mathrm{Sin}\left(\frac{\pi(z-z')}{2J} \right) \right|^{-4h}$$

(ii) Disconnected Phase

$J > \pi \ \ (or\ \kappa > 3\text{-}2\sqrt{2})$

$$\langle O^\dagger(z) O(z') \rangle \approx \left| \mathrm{Sinh}\left(\frac{z-z'}{2} \right) \right|^{-4h}$$



Again, there are two possibilities:
Trivial Wick contraction and Non-trivial Wick contraction

(i) Connected Phase

$J < \pi \ \ (or\ \kappa < 3\text{-}2\sqrt{2})$

$$\left|\mathrm{Sin}\left(\frac{\pi(z2-z1)}{2J}\right)\right| > \left|\mathrm{Sin}\left(\frac{\pi(z3-z2)}{2J}\right)\right|$$

(ii) Disconnected Phase

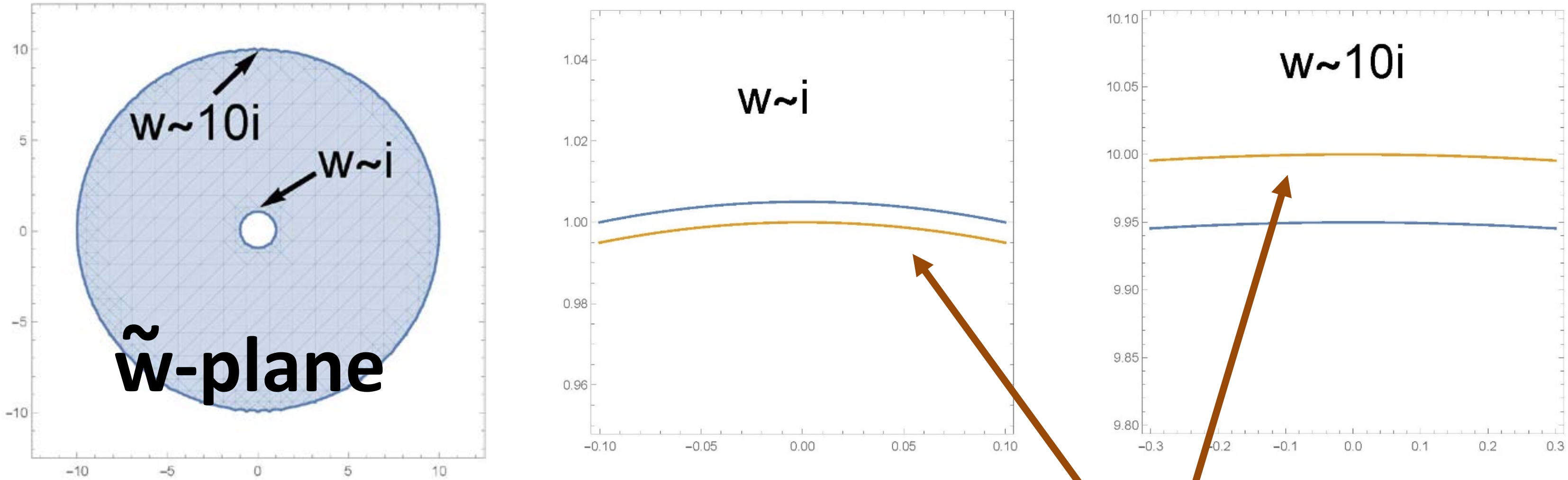
$J > \pi \ \ (or\ \kappa > 3\text{-}2\sqrt{2})$

$$\left|\mathrm{Sinh}\left(\frac{z2-z1}{2}\right)\right| > \left|\mathrm{Sinh}\left(\frac{\pi(z3-z2)}{2J}\right)\right|$$

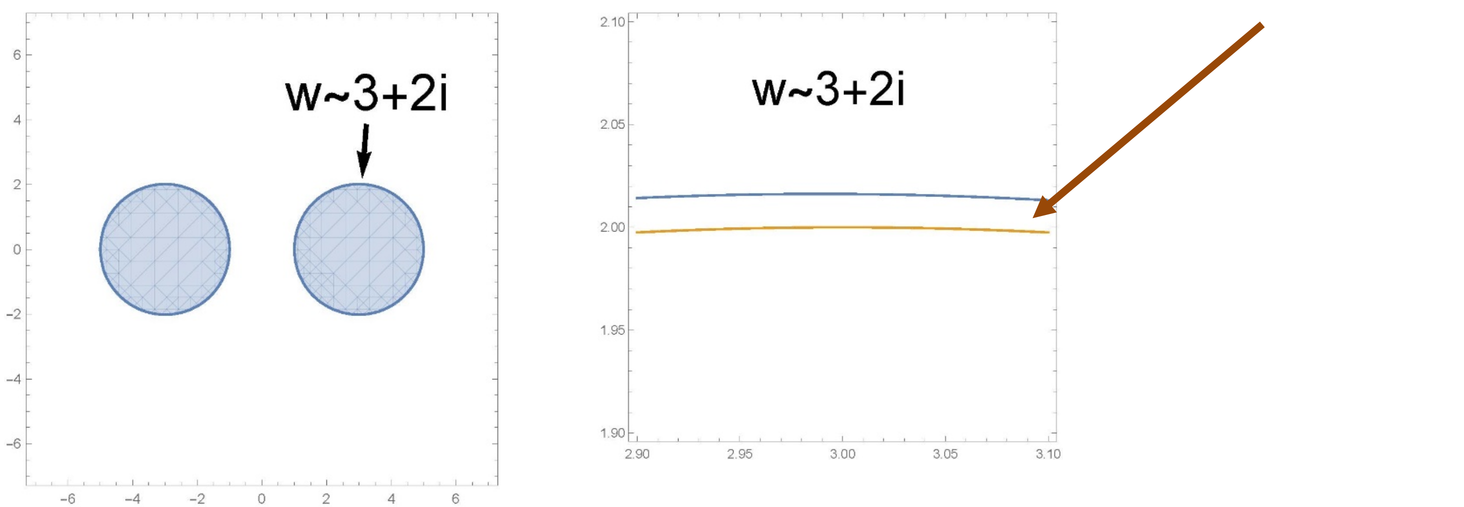
Non-trivial Contraction is favored

Numerical Plots of Regions of Non-trivial Contraction

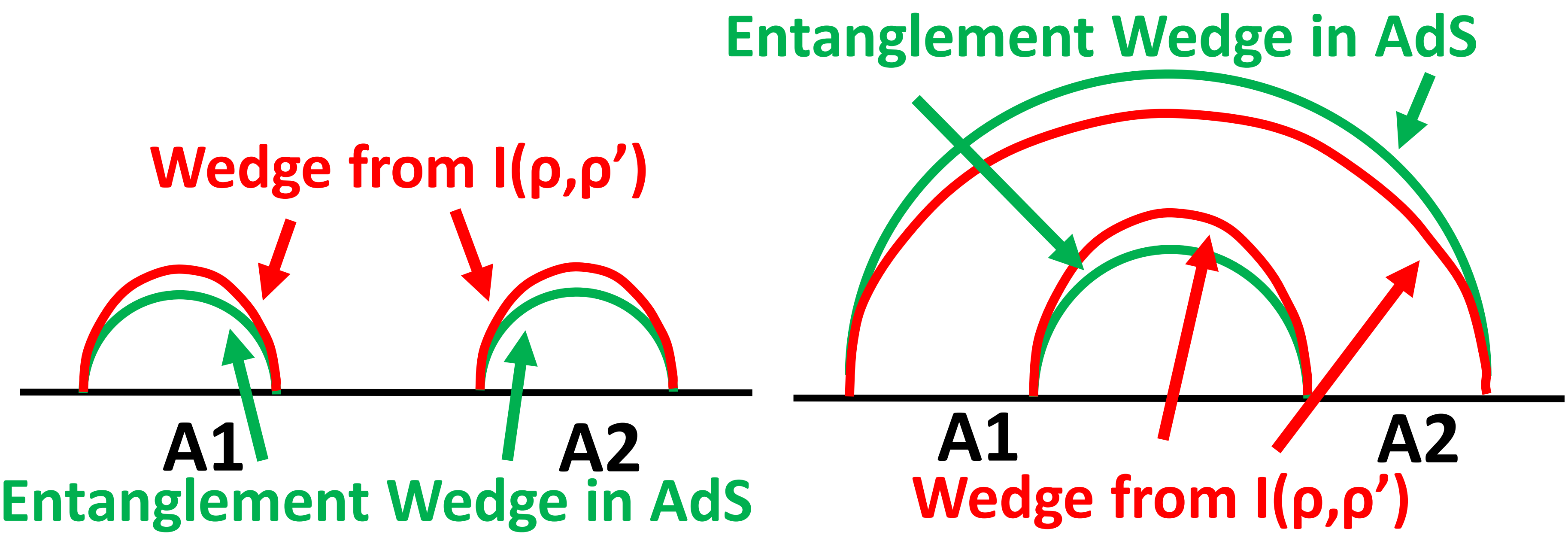
(i) Connected Phase (κ=0.1)



(ii) Disconnected Phase (κ=0.2)



The distinguishability analysis for I(ρ,ρ’) reproduces correct entanglement wedges up to a few percent errors.



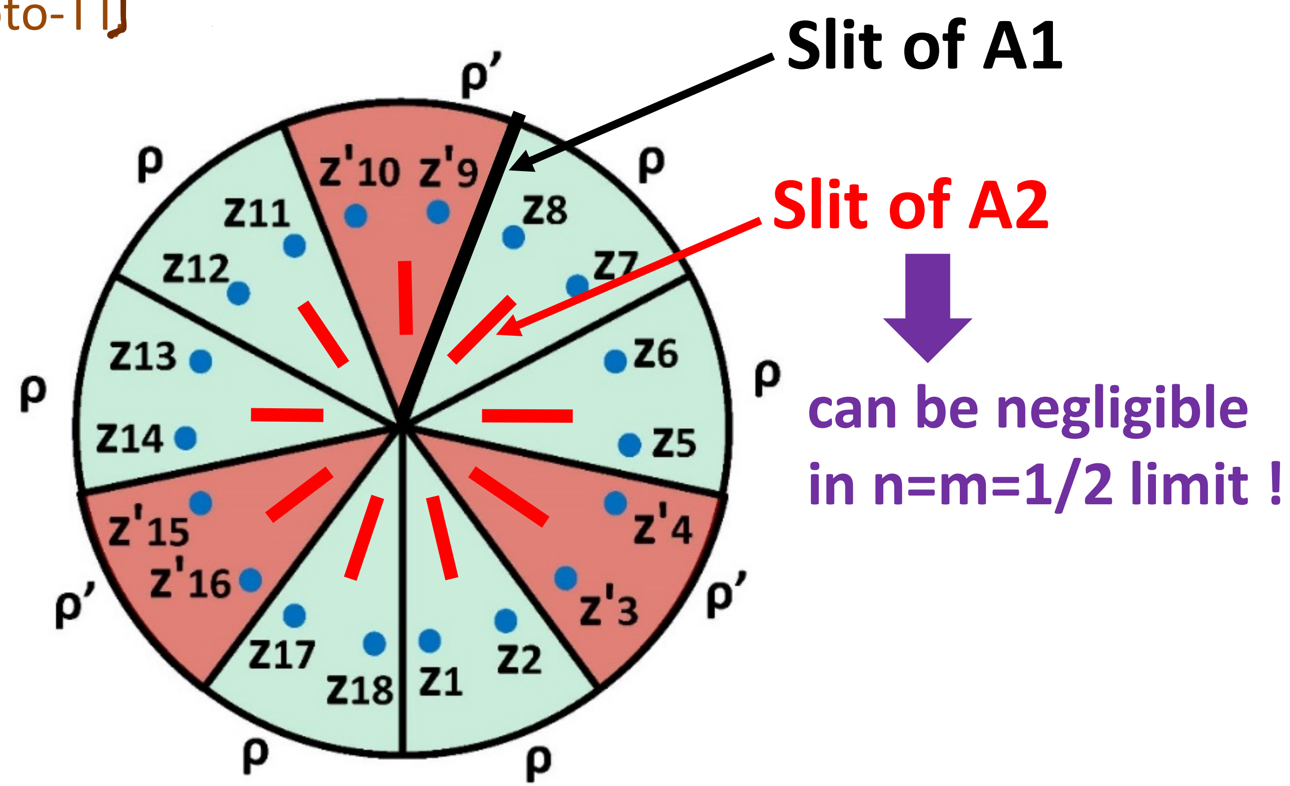
The wedge from I(ρ,ρ’) satisfies

(a) the wedge for A1 ∪ A2 ⊃ the wedge for A1 (or A2)
(b) the wedge for A is the complement of that for A^c

We expect these small deviations are because we employ I(ρ,ρ’) instead of the Bures distance D_B(ρ,ρ’).

Indeed, the following heuristic argument shows the Bures metric reproduces the genuine Entanglement Wedge:
[Kusuki-Suzuki-Umemoto-TT]

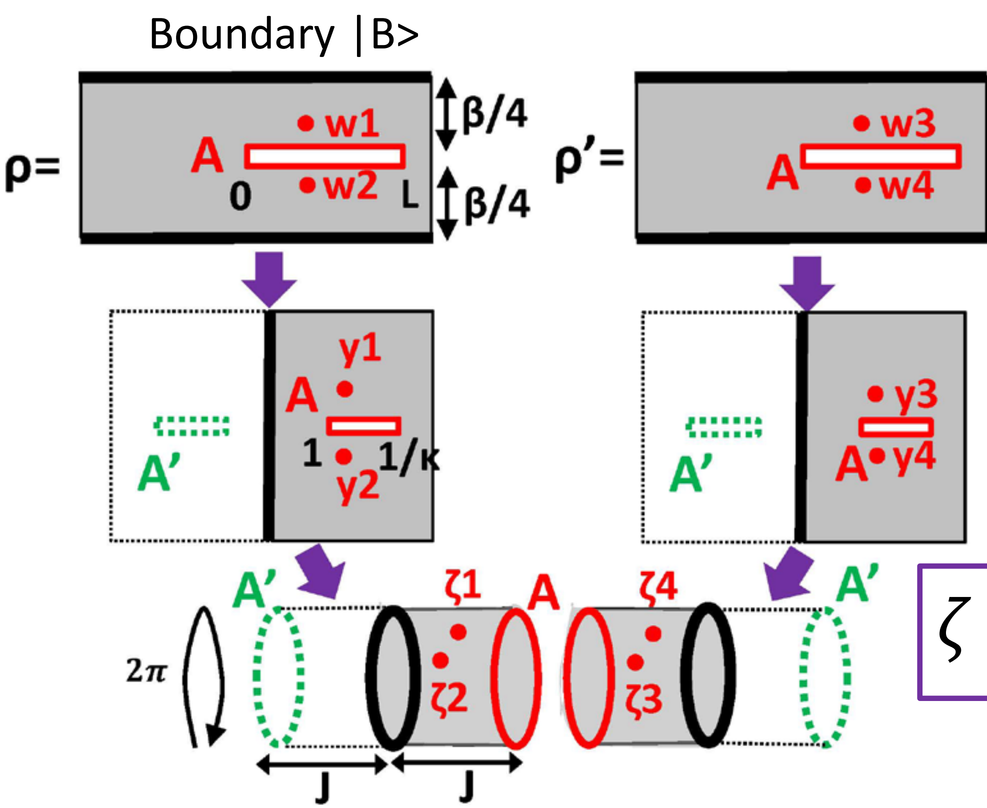
$$A_{n=3,m=1} =$$



[7-5] AdS/BCFT Case

Consider a quantum state in BCFT: $|\Psi\rangle = e^{-\beta H/2}|B\rangle$, which describes a global quantum quench.

The calculation of **Tr[ρρ']** can be done by the mapping:



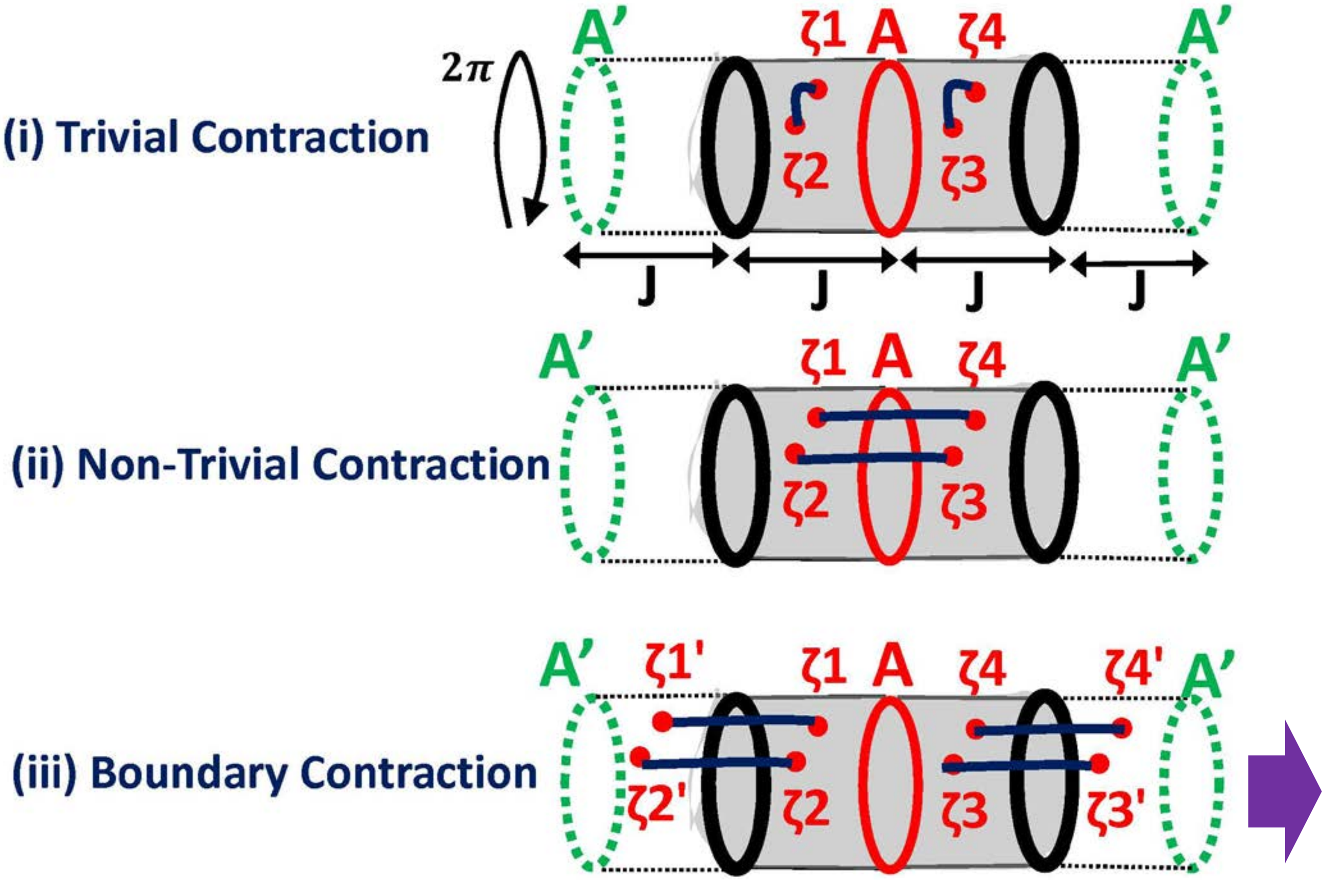
A=[0,L]

$$\kappa = e^{-\frac{2\pi}{\beta}L}$$

$$y = e^{\frac{2\pi}{\beta}w}$$

$$\zeta = \frac{\pi}{K(1-\kappa^2)}\mathrm{sn}^{-1}(y,\kappa^2)$$

Wick Contractions of Tr[ρρ']$\sim \langle O_1 O_2 O_3 O_4 \rangle$ i.e. 4pt function



This vanishes if Bdy 1pt function is zero.

To correctly probe the bulk by local operator, we assume the bdy 1pt function in BCFT is zero !

[Even for pure states, we cannot probe unless <O> =0.]

CFT wedges for I(ρ,ρ’)

Entanglement Wedge in AdS/CFT

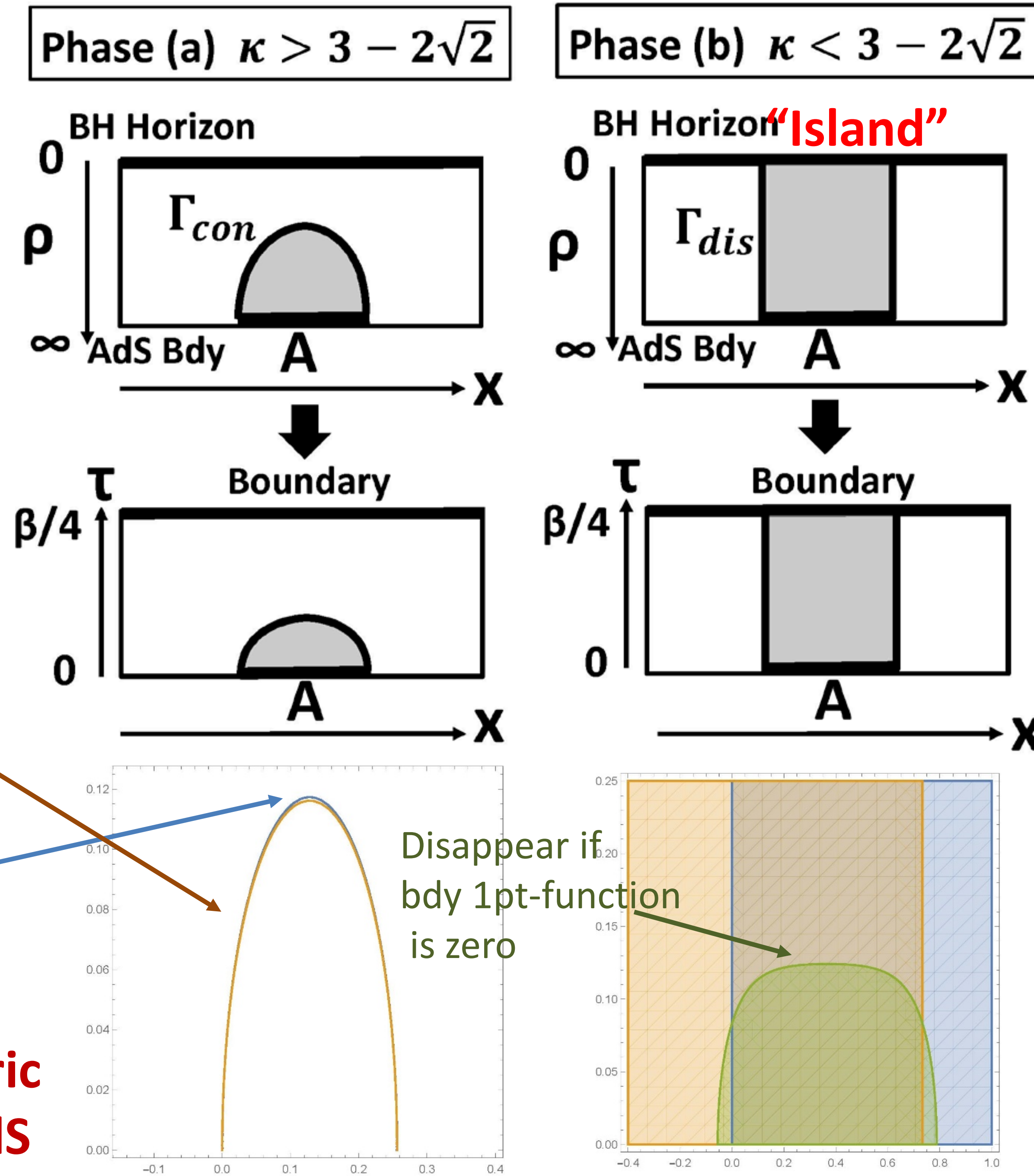
Shadow of EW

↕ **Very small deviation !**

CFT wedge of I(ρ,ρ’)

↓

CFT wedges of Bures metric agree with true EWs in AdS



[7-6] HKLL States (Speculative Argument)

The quantum Cramer-Rao predicts

$$\left\langle\left\langle \delta x_i \delta x_j \right\rangle\right\rangle \geq G_{ij}^{-1} \sim O\left(\frac{1}{h}\right).$$

However, in the actual AdS/CFT predicts
a sub-AdS scale resolution of coordinate values:

$$\left\langle\left\langle \delta x_i \delta x_j \right\rangle\right\rangle \sim O\left(\frac{1}{c^2}\right).$$

This is because the operator $O(w, \overline{w})$ is not precisely dual to a fully localized bulk excitation.

The actual bulk local excitation is described by the HKLL states (2d CFT on a cylinder):

$$|\Psi_\alpha(\rho,\theta)\rangle = U(\rho,\theta)\cdot e^{-\delta\cdot H_{CFT}}\cdot e^{\frac{\pi i}{2}\cdot H_{CFT}}|J_\alpha\rangle$$

Unitary by SL(2,R) which changes localized points

UV regularization (for descendants)

Global Ishibashi State [SL(2,R) boundary state]

[Miyaji-Numasawa-Shiba-Watanbe-TT 2015]

The Bures metric reads

$$ds_B^2 = \frac{1}{4\delta^2}\left(d\rho^2 + \sinh^2\rho\, d\theta^2\right).$$

(i) From our previous analysis, we expect Bures metric for the reduced density matrix ρ_A is the same as that for the pure state $|\Psi_\alpha(\rho,\theta)\rangle$ if the point is inside the wedge.

(ii) Moreover, since we consider large c CFTs (Hol. CFTs), it is natural to take $\delta \sim 1/c$.

If we assume these speculations, we find the Bures metric for reduced matrices of HKLL state:

$$ds_B^2 \approx c^2\left(d\rho^2 + \sinh^2\rho\, d\theta^2\right).$$

This agrees with the correct metric of global AdS and the expected resolution: $\left\langle\langle \delta x_i \delta x_j \rangle\right\rangle \sim O(c^{-2})$.

[7-7] Conclusions

- In this talk, we presented an approach using the Bures metric DB(ρ,ρ') to detect entanglement wedges directly from CFTs.

- We found that the large c property of holographic CFTs plays a crucial role for the emergence of EW. [Generalizations to higher dim. CFTs, AdS/BCFT, and TFD etc. are also possible. See forthcoming paper]

- We can perfectly reproduce the EW for a single interval. For double intervals, the Renyi-like measure I(ρ,ρ') reproduces the correct EW up to a few percent errors. On the other hand, the Bures metric in CFTs correctly reproduces the actual EW.

$I(\rho,\rho')\sim \mathrm{Tr}[(\rho\rho')]$ → Probe both low and high energy

$F(\rho,\rho') = \mathrm{Tr}\left[\sqrt{\sqrt{\rho}\rho'\sqrt{\rho}}\right]$ → Probe low energy (Code Subspace)