Euclidean Path Integral Optimization in CFT's: holographic complexity and an attempt to a derivation of AdS/CFT

Nilay Kundu
Yukawa Institute for Theoretical Physics, Kyoto University

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Based on the following works ..

1) Phys. Rev. Lett. 119 (2017) no.7, 071602

2) arXiv : 1706.07056 [hep-th] — (Accepted in JHEP)

With ... Pawel Caputa, Masamichi Miyaji, Tadashi Takayanagi and Kento Watanabe

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Holographic Complexity

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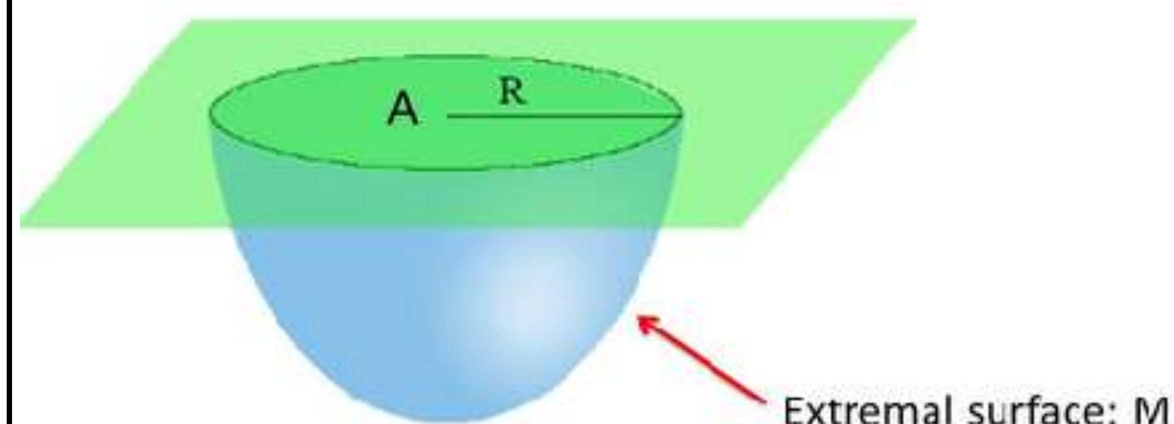
- One such quantitative suitable measure happens to be : the entanglement entropy or von-Neumann entropy ...

Holographic Entanglement Entropy



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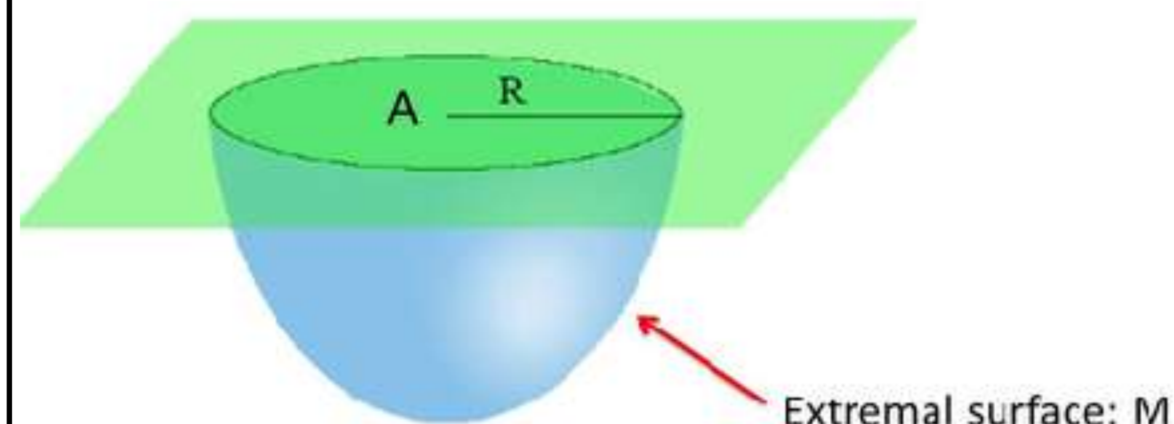

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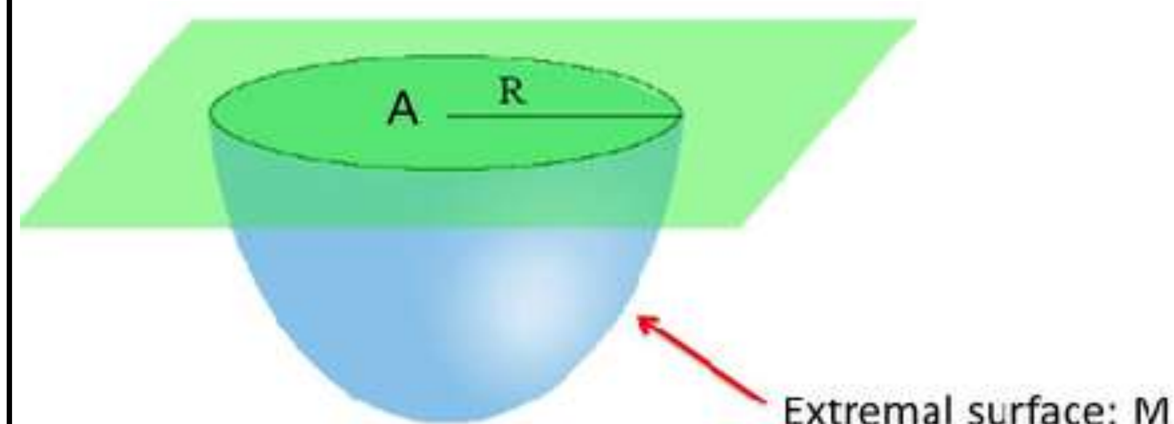

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Tensor Networks are an useful and explicit framework to realize this interpretation

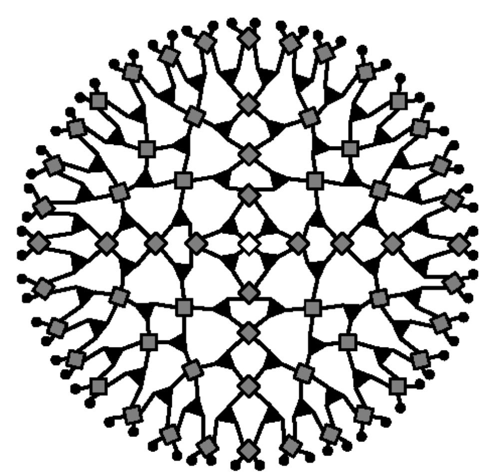
AdS/CFT or more generally holography explicitly realizes **the geometrization of dynamics in CFT’s**

$\updownarrow$

Tensor networks uses quantum entanglement to represents **the geometrization of quantum states in many body system**

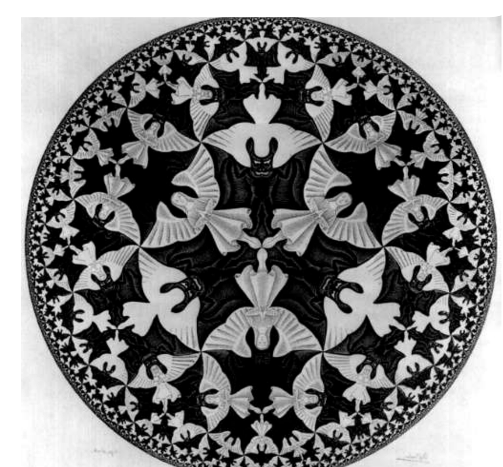
Quantum States

$$\left|\Psi(t)\right\rangle = \sum_{\{i_k\}} c_{\{i_k\}}(t)\ \left|i_1\right\rangle \otimes \left|i_2\right\rangle \ldots \otimes \left|i_N\right\rangle \quad \longleftrightarrow$$



Tensor Network

=



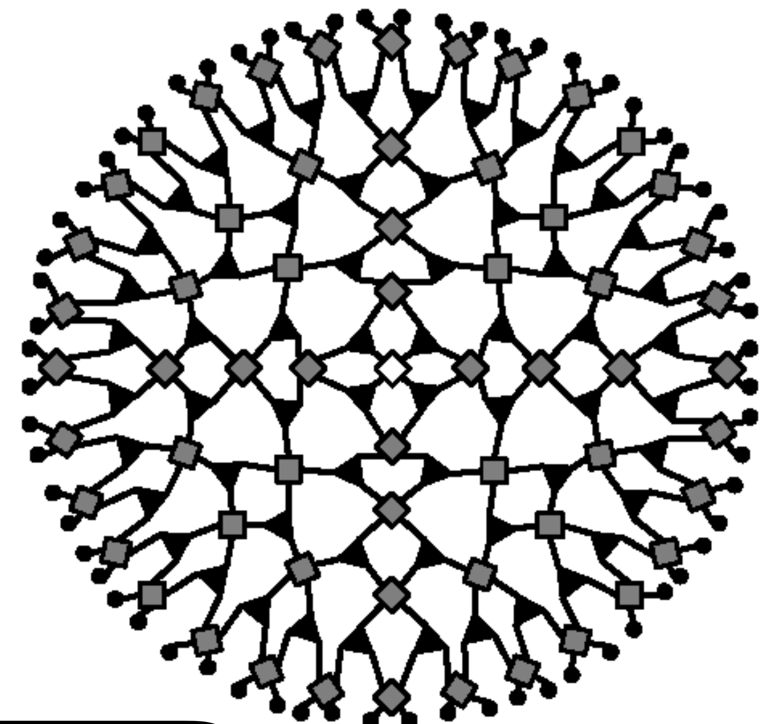
AdS

Tensor Network : An efficient variational ansatz for the ground state wave functions in quantum many body systems

A Tensor Network diagram = A many-body wave-fn

↕

A network of tensors that geometrizes quantum entanglement

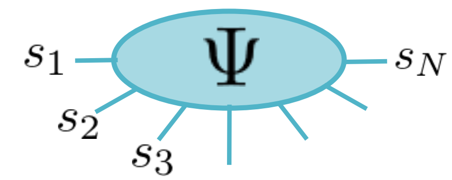


Requirement : A good ansatz should reproduce the correct quantum entanglement of the ground state wave-fn

Generic States :

$\boxed{\text{Exponential parameters ==> inefficient}}$

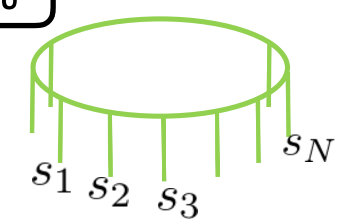
$$|\Psi\rangle = \sum_{\{s_N\}} \Psi_{s_1,\cdots,s_N} |s_1,\cdots,s_N\rangle$$



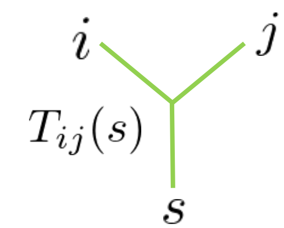
MPS :

$\boxed{\text{Polynomial parameters ==> efficient}}$

$$|\Psi\rangle = \sum_{\{s_i\}} \mathrm{Tr}[T(s_1)T(s_2)\cdots]|s_1,s_2,\cdots\rangle$$



$i,j = 1,\cdots,\chi$

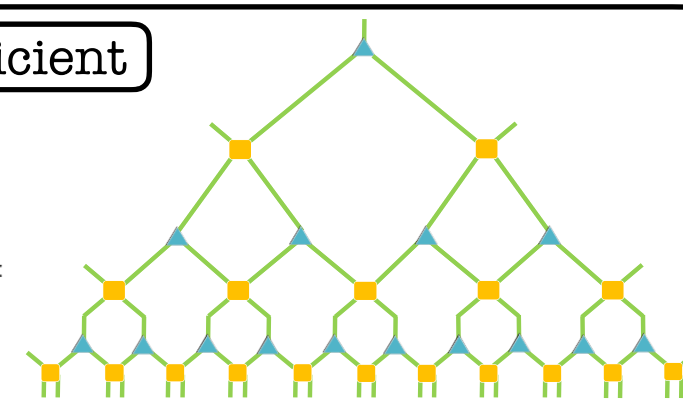


[DMRG: White 92,...,
Rommer-Ostlund 95,..]

MERA :

$\boxed{\text{Polynomial parameters ==> efficient}}$

2 type of tensors ==> $|\Psi\rangle =$



[Vidal 05]

<u>Examples</u>

- Gapped systems

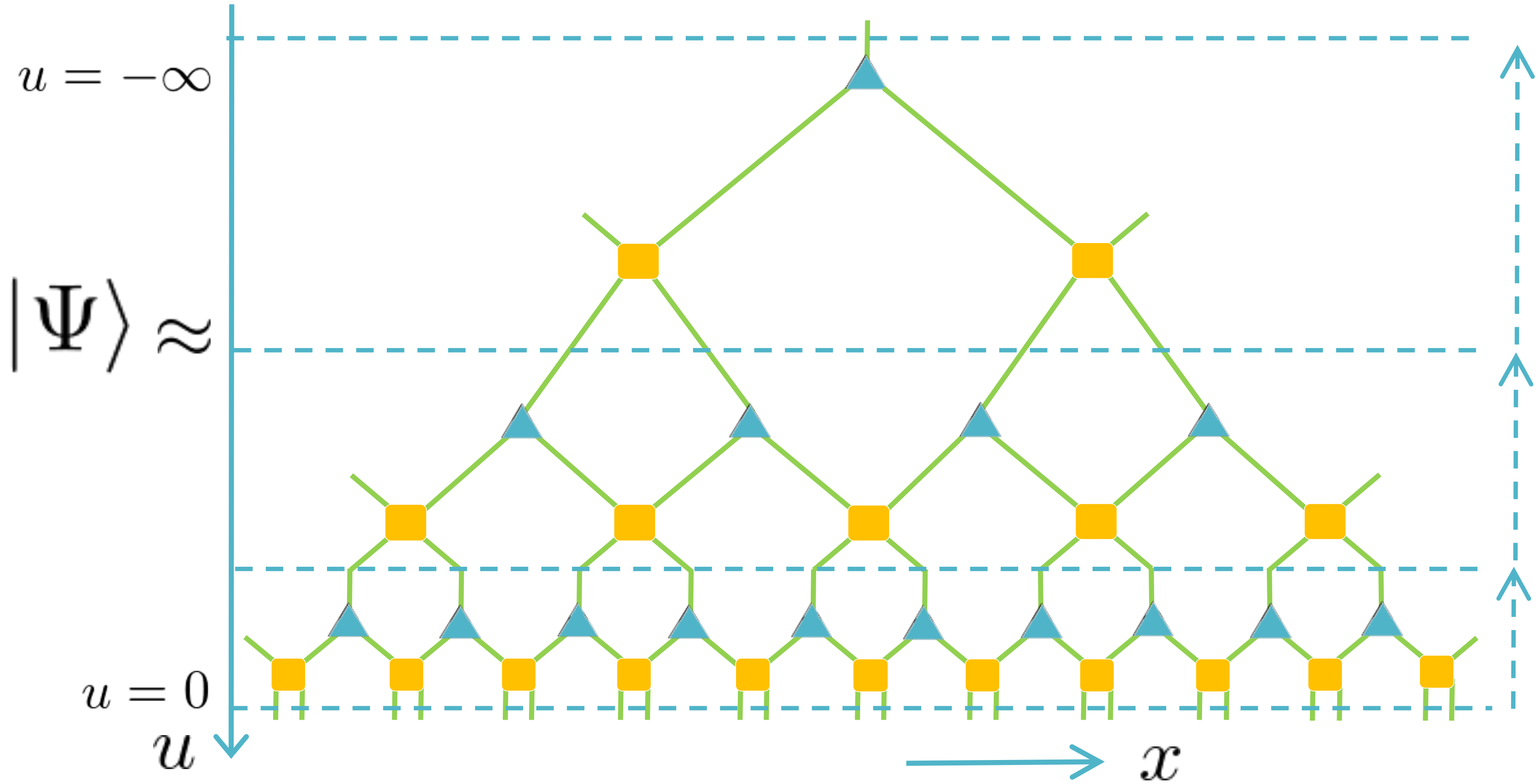
  Matrix Product State(MPS), Density, Matrix Renormalization Group(DMRG)

- Critical or gapped systems

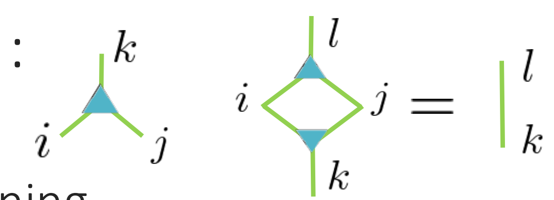
  Multi-scale Entanglement Renormalization Anzatz(MERA)

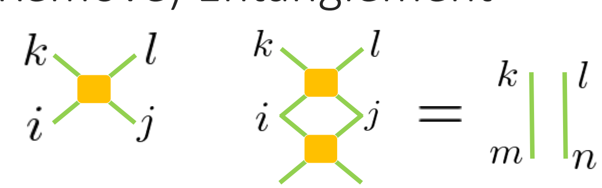
Effective variational ansatz for ground state wave-fn :
Tensor Networks that Optimizes the energy $\langle\Psi|H|\Psi\rangle$

Multiscale Entanglement Renormalization Ansatz (MERA)



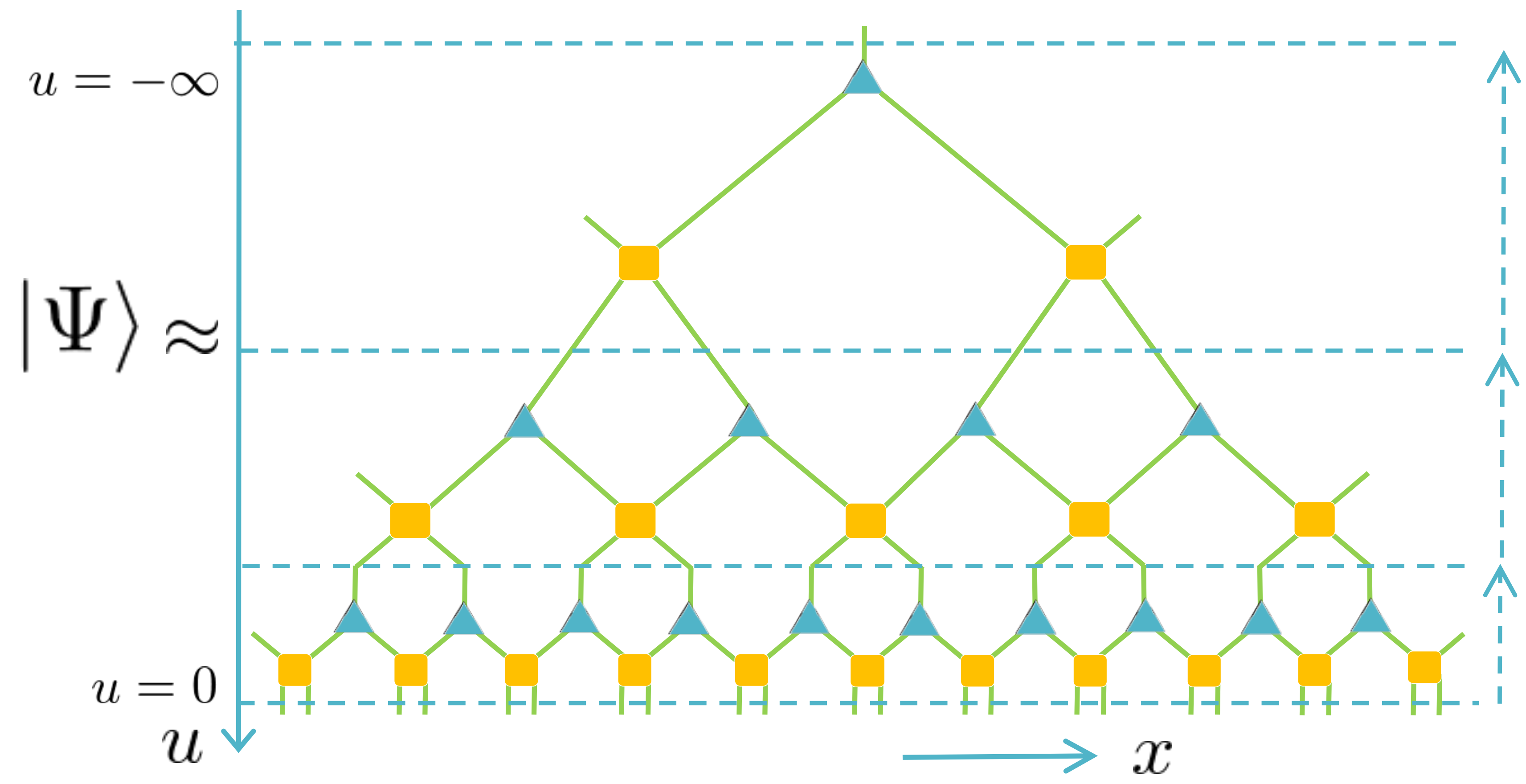
- 2 kinds of Tensors :
  - Isometry
    Coarse-graining
  - (Dis-) Entangler
    Add (Remove) Entanglement





Multiscale Entanglement Renormalization Ansatz (MERA)

For CFT ground states, a good TN



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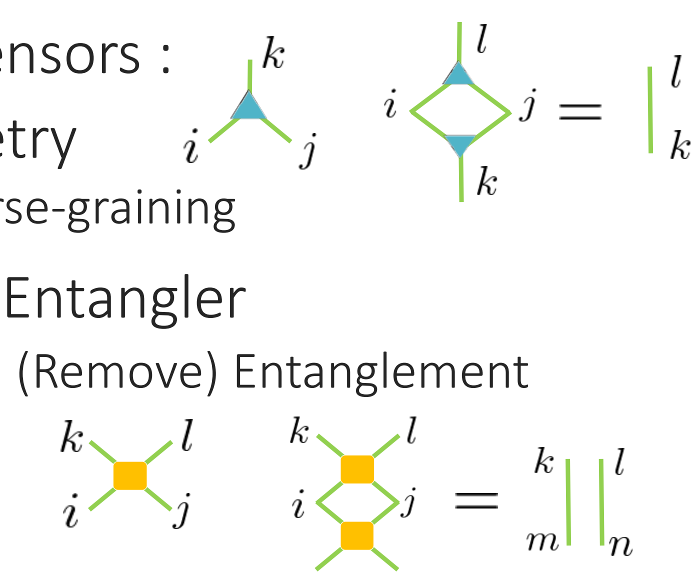
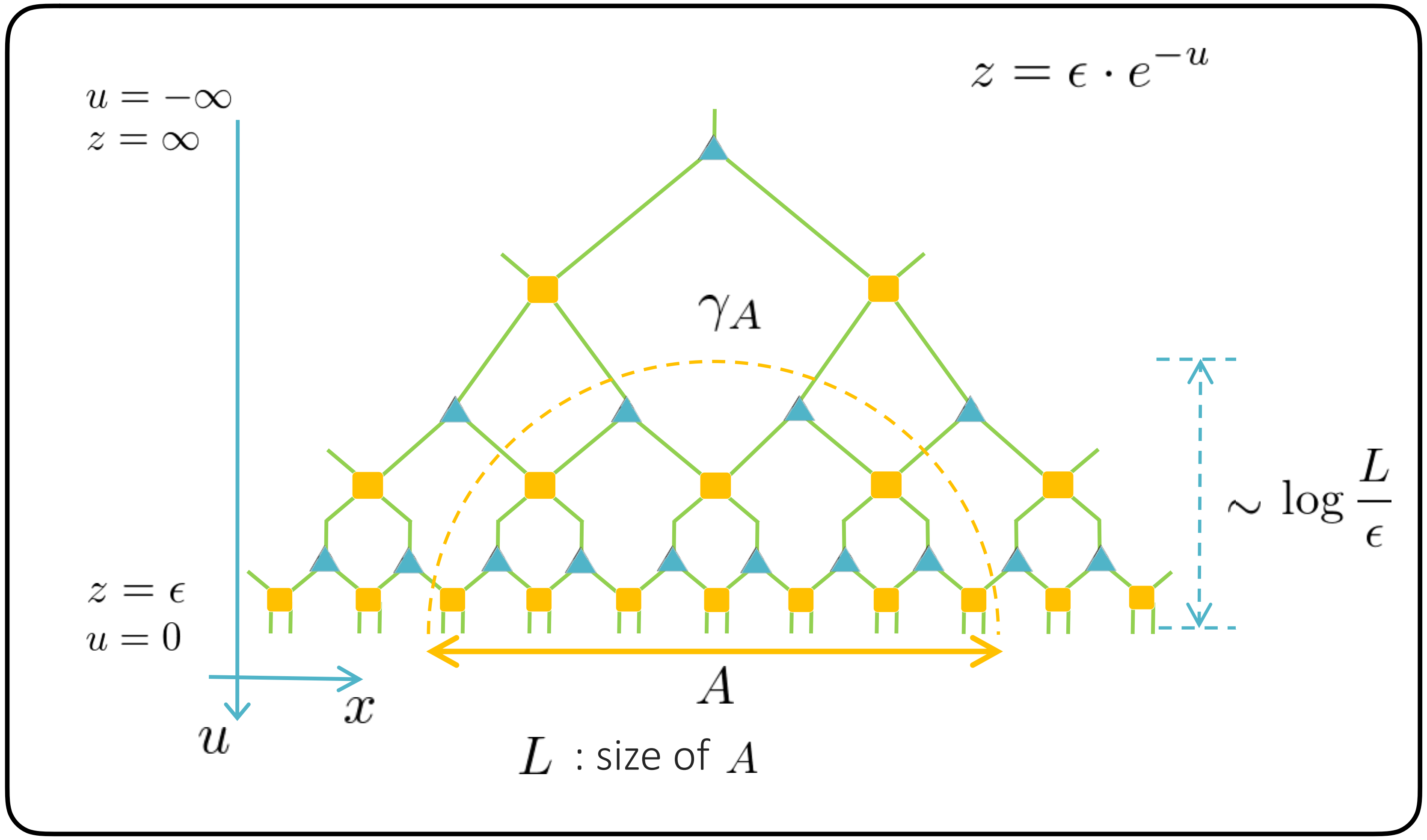


MERA and time slice of AdS and Entanglement entropy



$$ds^2 = du^2 + \frac{e^{2u}}{\epsilon^2} dx^2$$
$$= \frac{dz^2 + dx^2}{z^2}$$

Hyperbolic space H_2

(a time slice of AdS_3)

$S_A \le \min_{\gamma_A} [\#\text{cuts}] \cdot \log \chi$ $\propto \log \frac{L}{\epsilon}$ AdS_3/CFT_2

Holographic EE saturates this bound

AdS/CFT and tensor network

Significant similarity between geometric expression of entanglement entropy in AdS/CFT and MERA.

What we learn from tensor networks :
MERA geometry is equal to time slice of AdS

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Evidences :

1) A real-space RG structure of MERA :: The extra radial direction and evolution in bulk AdS
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1) Is it possible to explore locality beyond the AdS scale ..
2) Why do we need to have saturation of entanglement entropy ..
3) Is the full conformal symmetry structure realizable ?
4) Ultimately we want to go beyond the lattice formulation and see the AdS in true continuum sense ..

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We propose an alternative method using OPTIMIZATION of EUCLIDEAN PATH-INTEGRALS for ground state wave-fn in CFT's

An "OPTIMIZATION" of the Euclidean Path Integral

$$\Psi(\tilde{\varphi}(x)) = \int \left(\prod_x \prod_{\epsilon < z < \infty} D\varphi(x,z) \right) e^{-S_{CFT}(\varphi)} \prod_x \delta(\varphi(\epsilon, x) - \tilde{\varphi}(x))$$

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The ground state wave-fn as a functional of the UV boundary value of the fields in the CFT

The measure of the path-integration over the fields in the CFT

The CFT action

The delta-fn ensuring the boundary value for the field at UV (z=epsilon) bdy.

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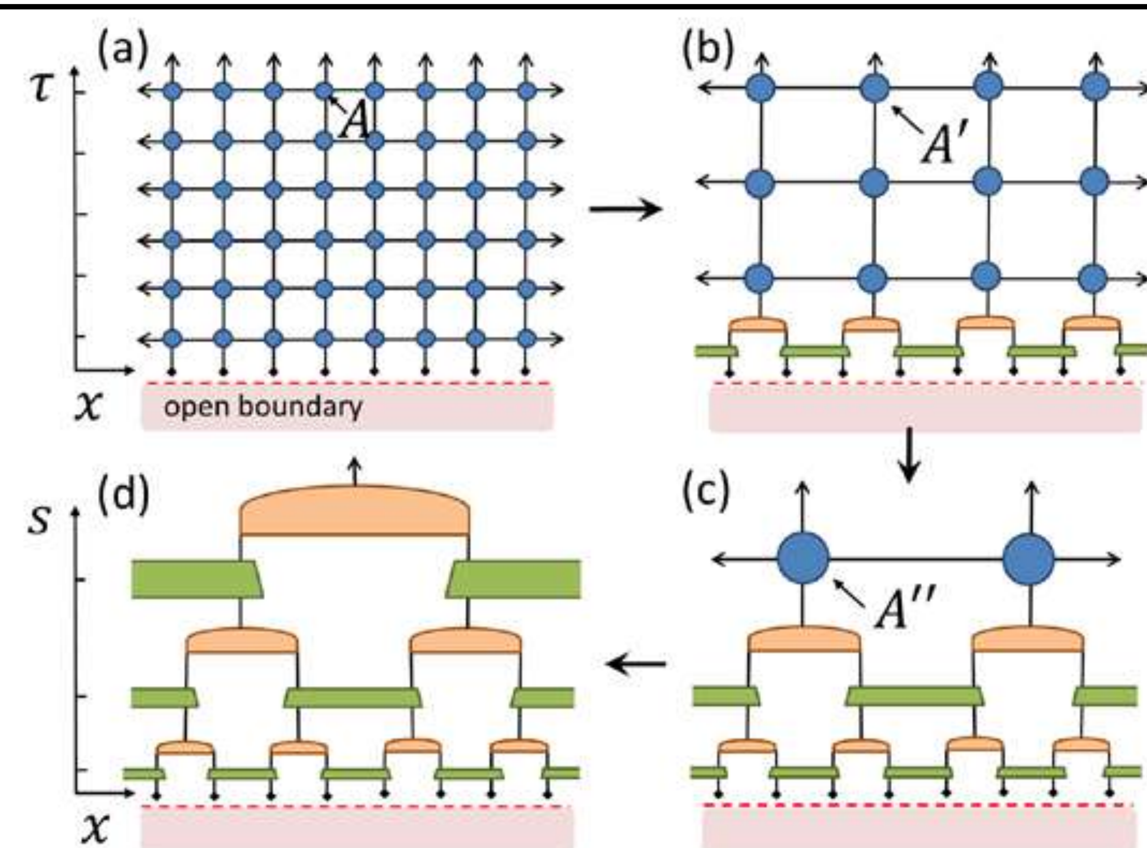
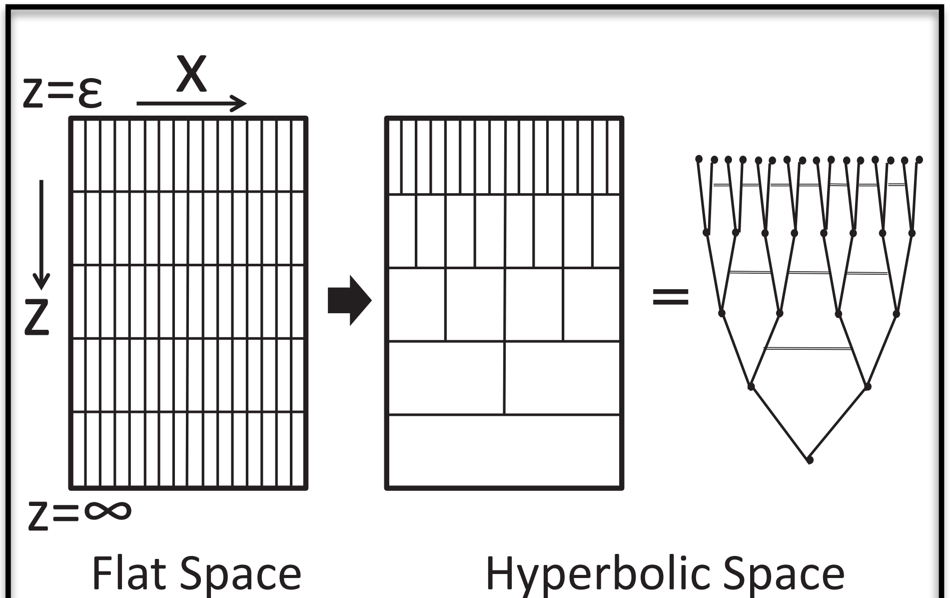
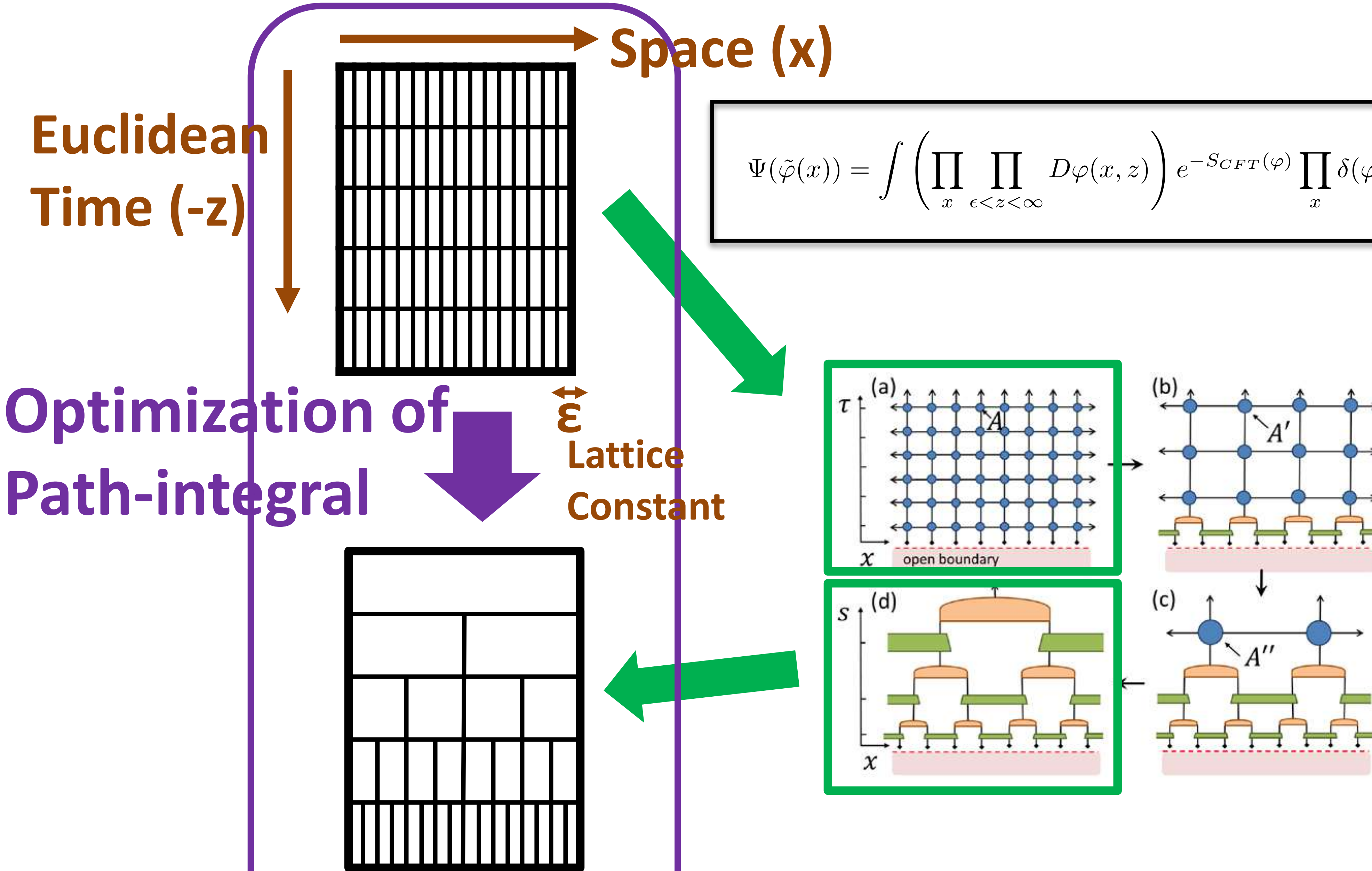


Picture taken from Evenbly-Vidal arXiv:1502.05385



Euclidean path Integral can be rewritten as MERA network but we would like to get rid of the possible lattice artifact

Tensor Network Renormalization (TNR) : OPTIMIZATION of Tensor Networks



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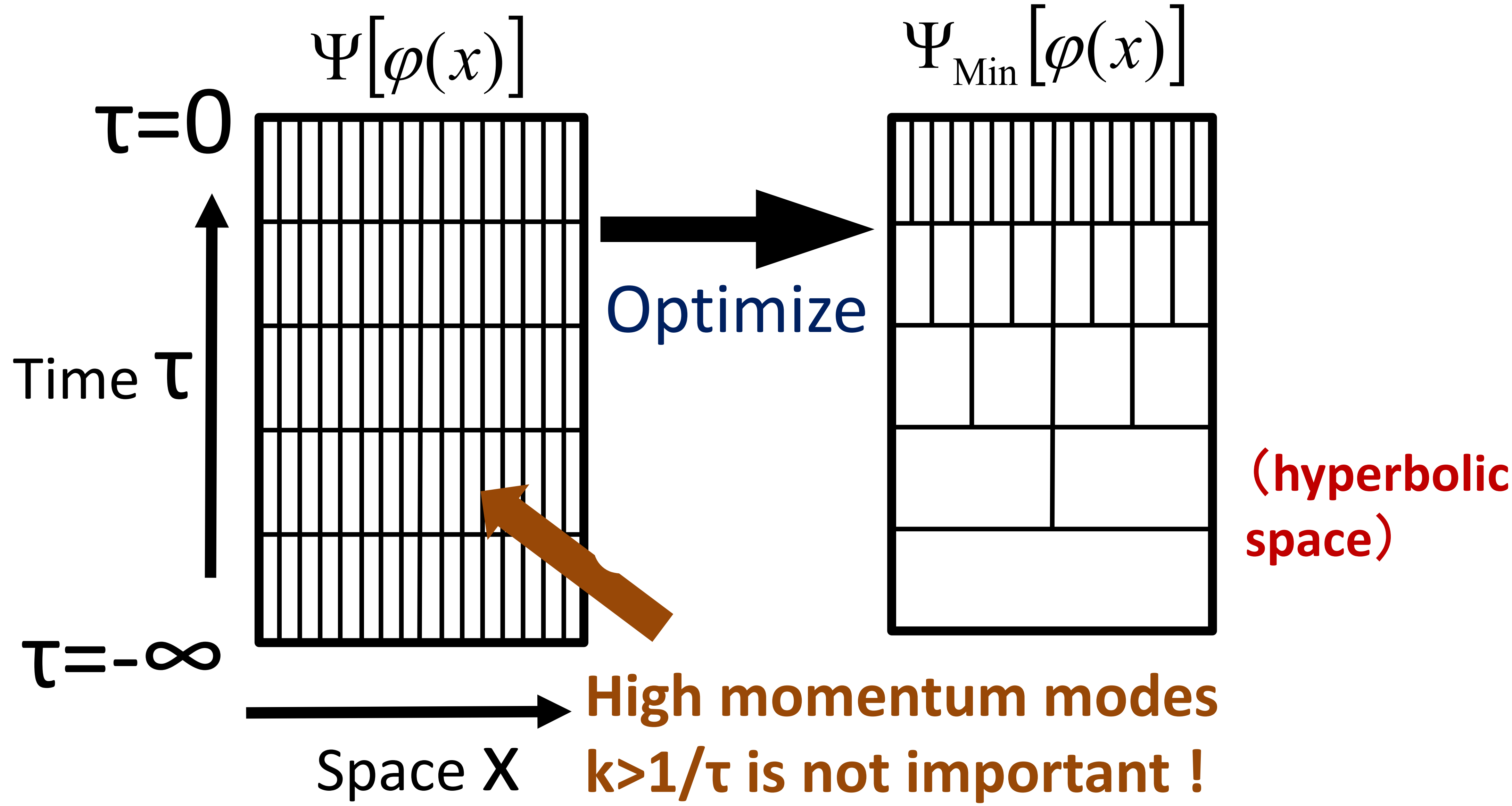
Hyperbolic Space = Time slice of AdS3

Path-Integral Wave-fn : it's Discretization and Optimization

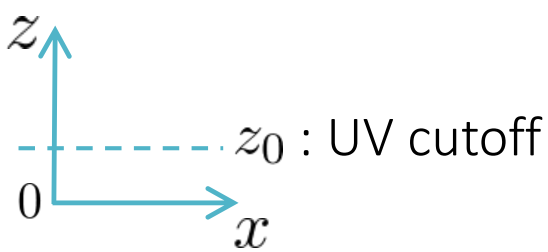
$$\Psi[\varphi(x)] = \int \prod_{\substack{-\infty<\tau<0 \\ -\infty<x<\infty}} D\phi(x,\tau)\, e^{-S_{CFT}(\phi)} \cdot \delta\big(\varphi(x) - \phi(x,\tau=0)\big)$$



(Miyaji, Watanabe, Takayanagi-2016)

Another way of looking at it : Free scalar Field

CFT$_2$ on $\mathbb{C}$



Free Scalar $S_{CFT} = \int dxdz \left[(\partial_z\phi)^2 + (\partial_x\phi)^2\right]$

EOM & regular at $z = \infty$

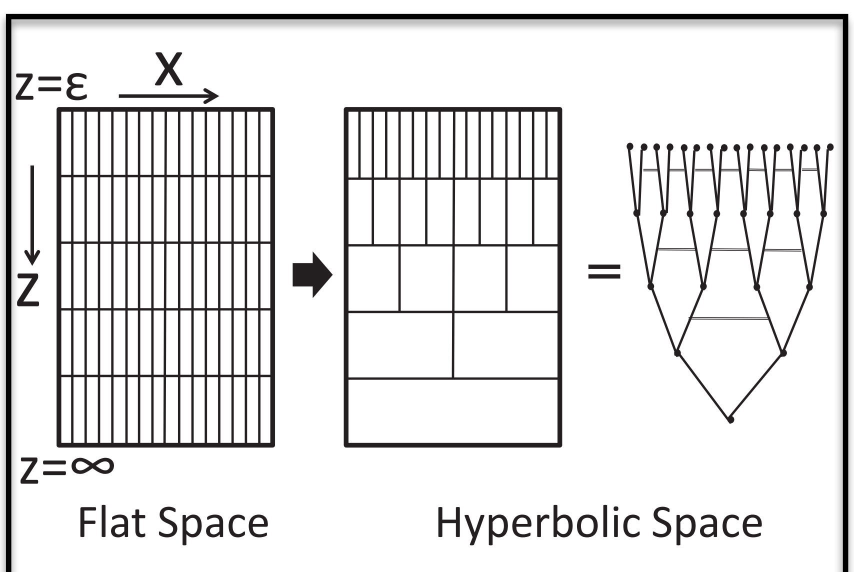
$$\rightarrow \phi(z,x) = \int_{-\infty}^{\infty} dk\phi(k)e^{ikx-|k|z}$$

$$\rightarrow \Psi[\phi(x)] = \int \prod_{z_0<z<\infty} D\phi(z,x)\cdot\delta(\phi(z_0,x)=\phi(x))\cdot e^{-S_{CFT}(\phi)} \boxed{\propto e^{-S_{on-shell}}}$$

$$\rightarrow S_{on-shell} = 4\pi\int_0^{\infty} dz \int_{-\infty}^{\infty} dk|k|^2 \boxed{e^{-2|k|z}}\phi(k)\phi(-k)$$

$$= 2\pi\int_{-\infty}^{\infty} dk|k|\phi(k)\phi(-k)$$

At fixed z ,only modes with $|k| \lesssim 1/z$ contribute in the k-integral



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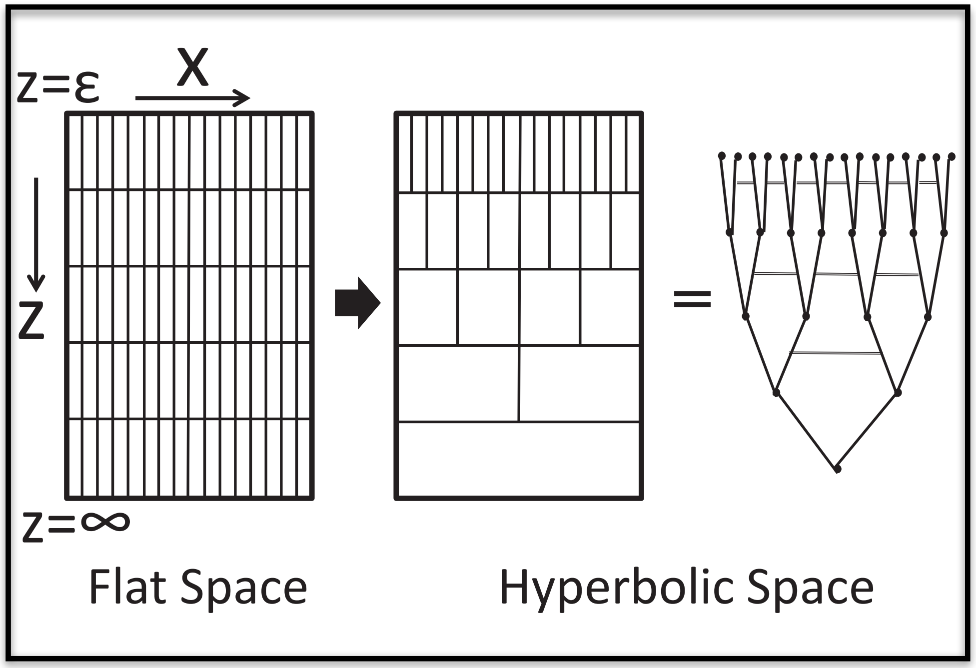
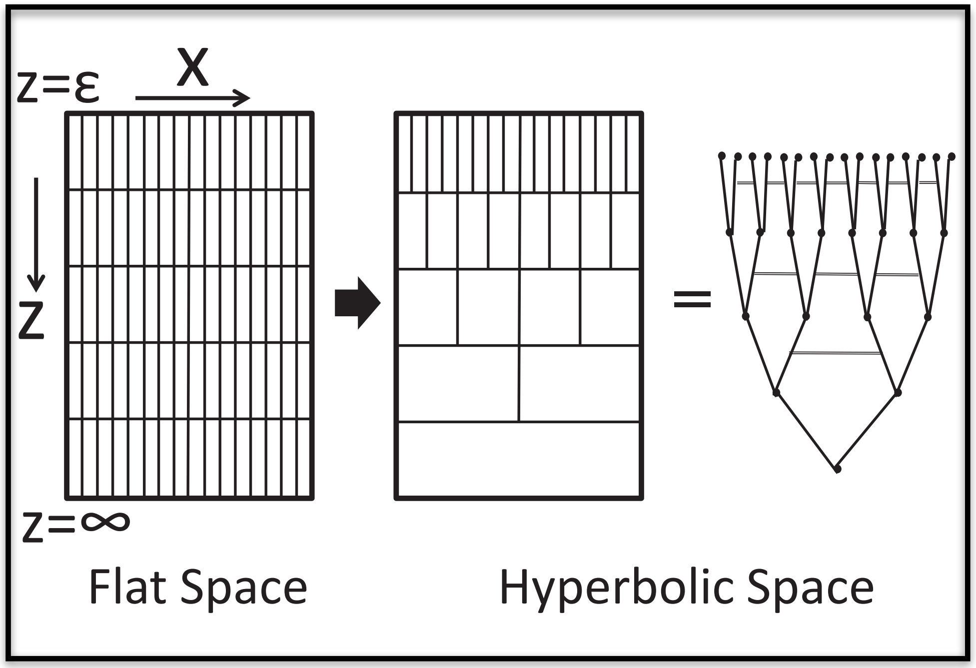
Position dependent cut-off



What we learn so far:

In the language of Tensor networks
==> Eliminating extra tensors in TN is creating most efficient TN
==> The algorithm at work for this, is the "OPTIMIZATION" of TN for a given state.

For free fields ==> Introducing momentum dependent cut-off.



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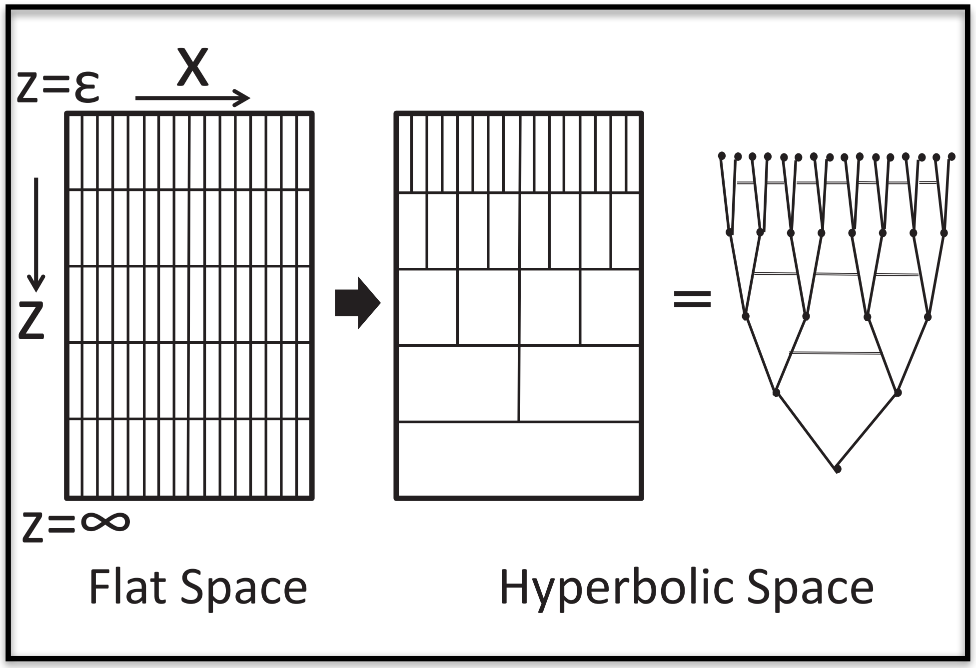
Rearrangement of the lattice-structure for the tensors in TN, can be engineered by changing the background metric in the Euclidean Path Integral, keeping the boundary condition for the fields at UV unchanged.



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What we are still lacking :

What is the counterpart of the OPTIMIZATION algorithm in TN, that will be the guiding principle to determine the changed (optimized) background metric ..

This takes us to the “COMPLEXITY” part of our story ..

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How to define complexity of a state in CFT’s ..

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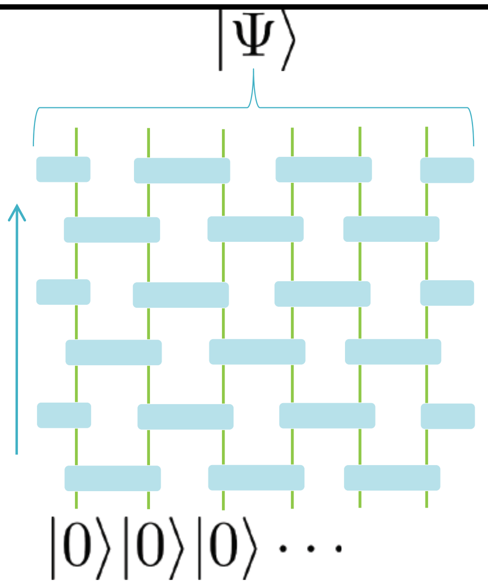
Computational complexity of a quantum state
= Min. no. of quantum gates required to prepare it from a given "simple" reference state
= Min. no. of tensors in the TN description .



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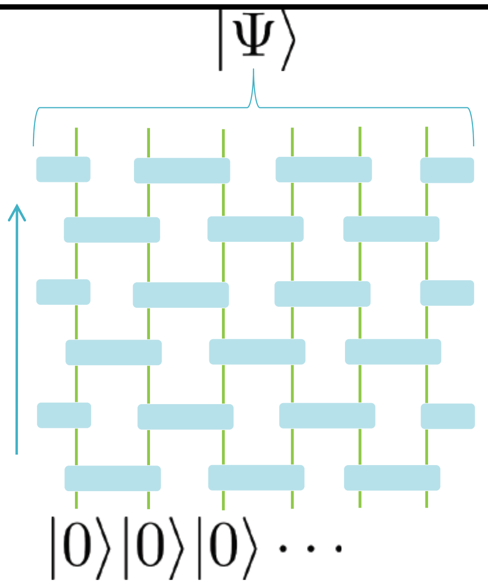

In the literature holographic formulas for computing complexity is already proposed :

1) Complexity = Max. Volume in AdS (Standford-Susskind ..)
2) Complexity = Gravity action in WDW patch of AdS (Brown, Roberts, Susskind, Myers ...)

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= We define the CFT analogue of the Complexity = $I_{\Psi}[g_{ab}]$

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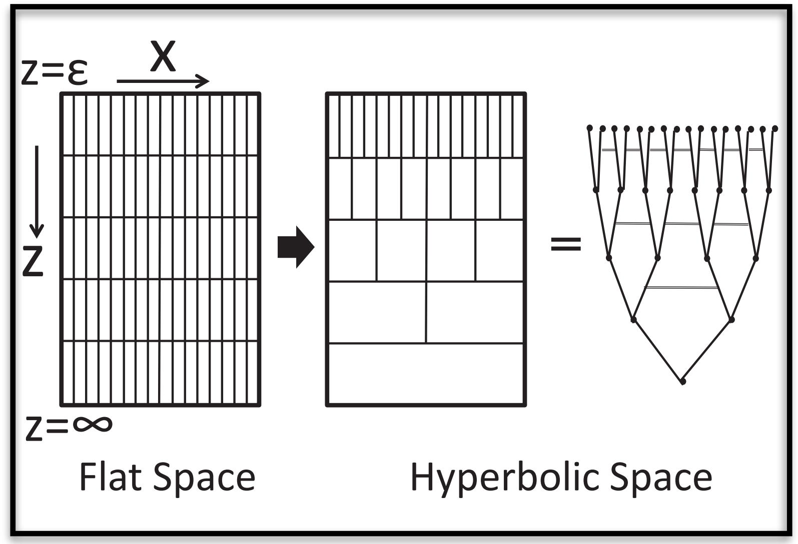

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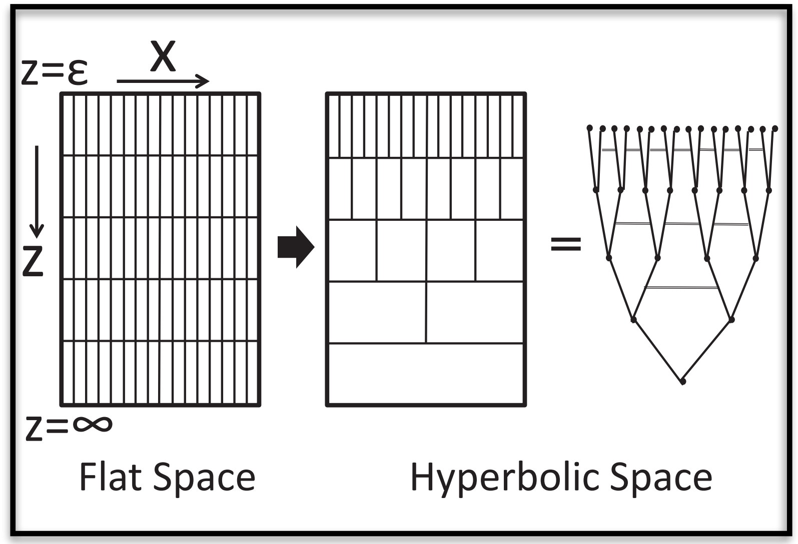

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Optimization of TN for a given state is equivalent to Minimizing the path-integral complexity with respect to the back-ground metric.

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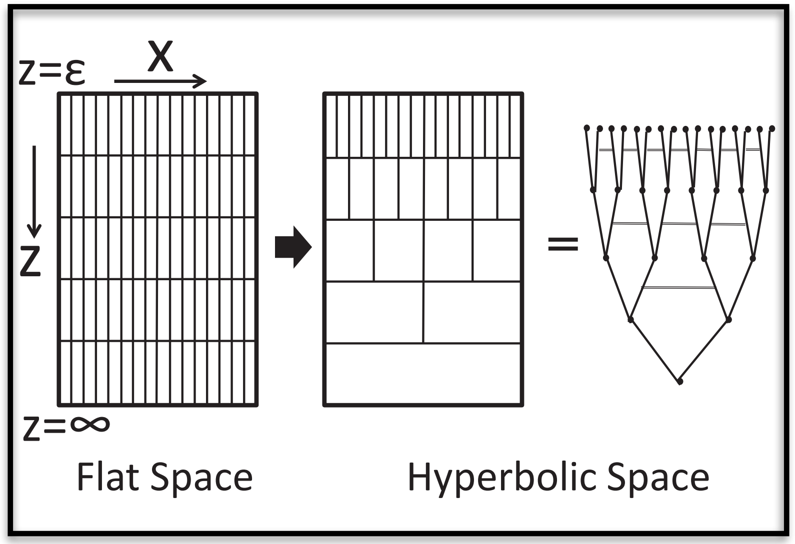

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The final crucial element :
Do we have a good answer for how to define the path-integral complexity given one CFT ..

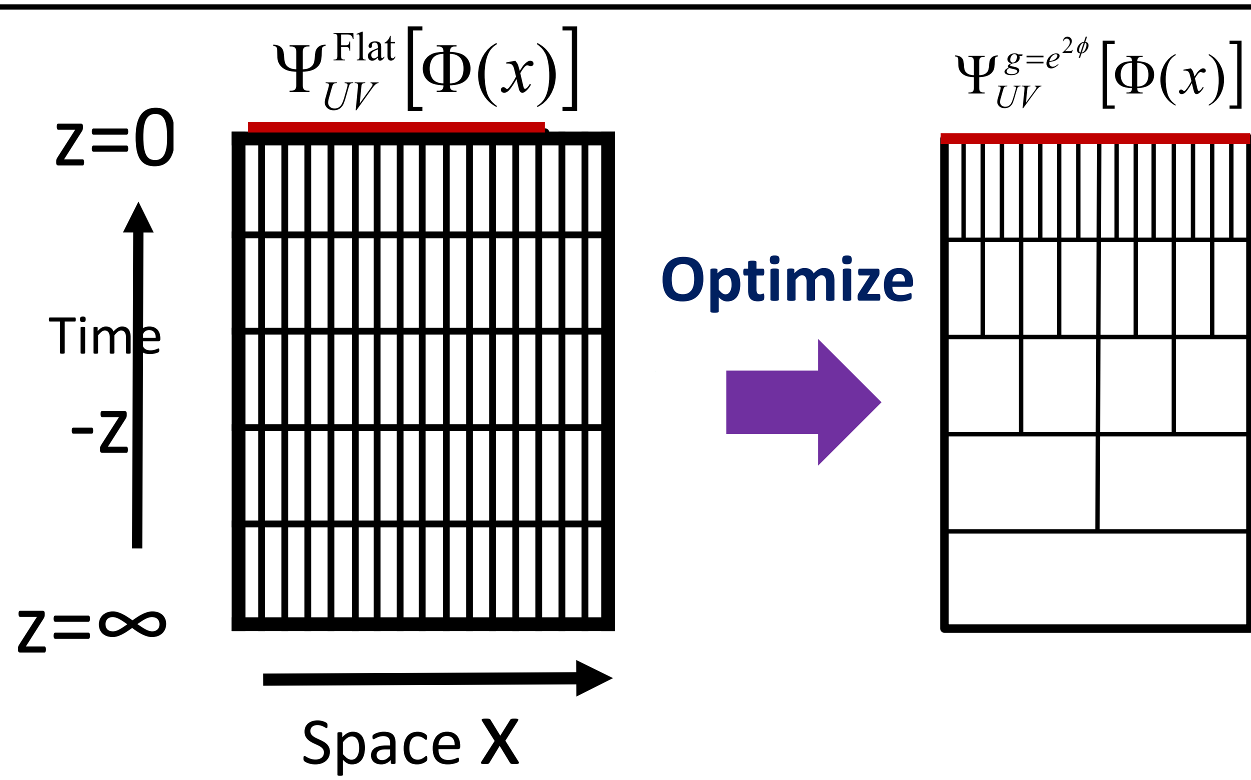
FOR a 2D Euclidean CFT We have a very good answer !!

In a 2D CFT with coordinates z (Euclidean time), x (space) we can always write any metric in the conformally flat form

$$ds^2 = e^{2\phi(x,z)}(dx^2 + dz^2)$$

Remember, the unoptimized metric is flat space (with UV cutoff given by "epsilon")...

$$ds^2 = \varepsilon^{-2} \cdot (dx^2 + dz^2)$$



The OPTIMIZATION should maintain the boundary condition

$$e^{2\phi}\big|_{z=\varepsilon} = \varepsilon^{-2}$$

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In a 2D CFT path-integral for the wave-fn of vacuum ==

background metric

$$\boxed{\Psi_{UV}^{g}\left[\Phi(x)\right]} = \int \prod_{\substack{0<z<\infty \\ -\infty<x<\infty}} D\Phi(x,z)\, e^{-S_{CFT}(\Phi)} \cdot \delta\left(\Phi(x) - \Phi(x,z=0)\right)$$

Due to Weyl invariance in CFT

$$\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}\left[\Phi(x)\right] = \exp\left(I[\phi(x,z)]\right) \cdot \Psi_{UV}^{\mathrm{Flat}}\left[\Phi(x)\right] .$$

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$$\boxed{\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}\left[\Phi(x)\right]} = \exp\left(I[\phi(x,z)]\right) \cdot \boxed{\Psi_{UV}^{\mathrm{Flat}}\left[\Phi(x)\right]}$$

OPTIMIZED wave-fn

UN-OPTIMIZED wave-fn

FOR a 2D Euclidean CFT We have a very good answer !!

In a 2D CFT path-integral for the wave-fn of vacuum ==

background metric

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Our Proposal for the optimization of the path-integral in CFT's

MINIMIZE $\exp\big(I[\phi(x,z)]\big)$ w.r.t. $\phi(x,z)$ with $e^{2\phi}\big|_{z=\varepsilon} = \varepsilon^{-2}$

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OPTIMIZED wave-fn

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$I[\phi(x,z)] =$ **LIOUVILLE ACTION**

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Minimize the Liouville action "= path-integral complexity"

$$\Psi_{UV}^{g_{ab}=e^{2\phi}\delta_{ab}}\left[\Phi(x)\right]=\exp\left(I[\phi(x,z)]\right)\cdot\Psi_{UV}^{\mathrm{Flat}}\left[\Phi(x)\right]$$

$$I[\phi]=\mathrm{Log}\left[\frac{\Psi_{g=e^{2\phi}\delta_{ab}}}{\Psi_{g=\delta_{ab}}}\right]=S_L[\phi]$$

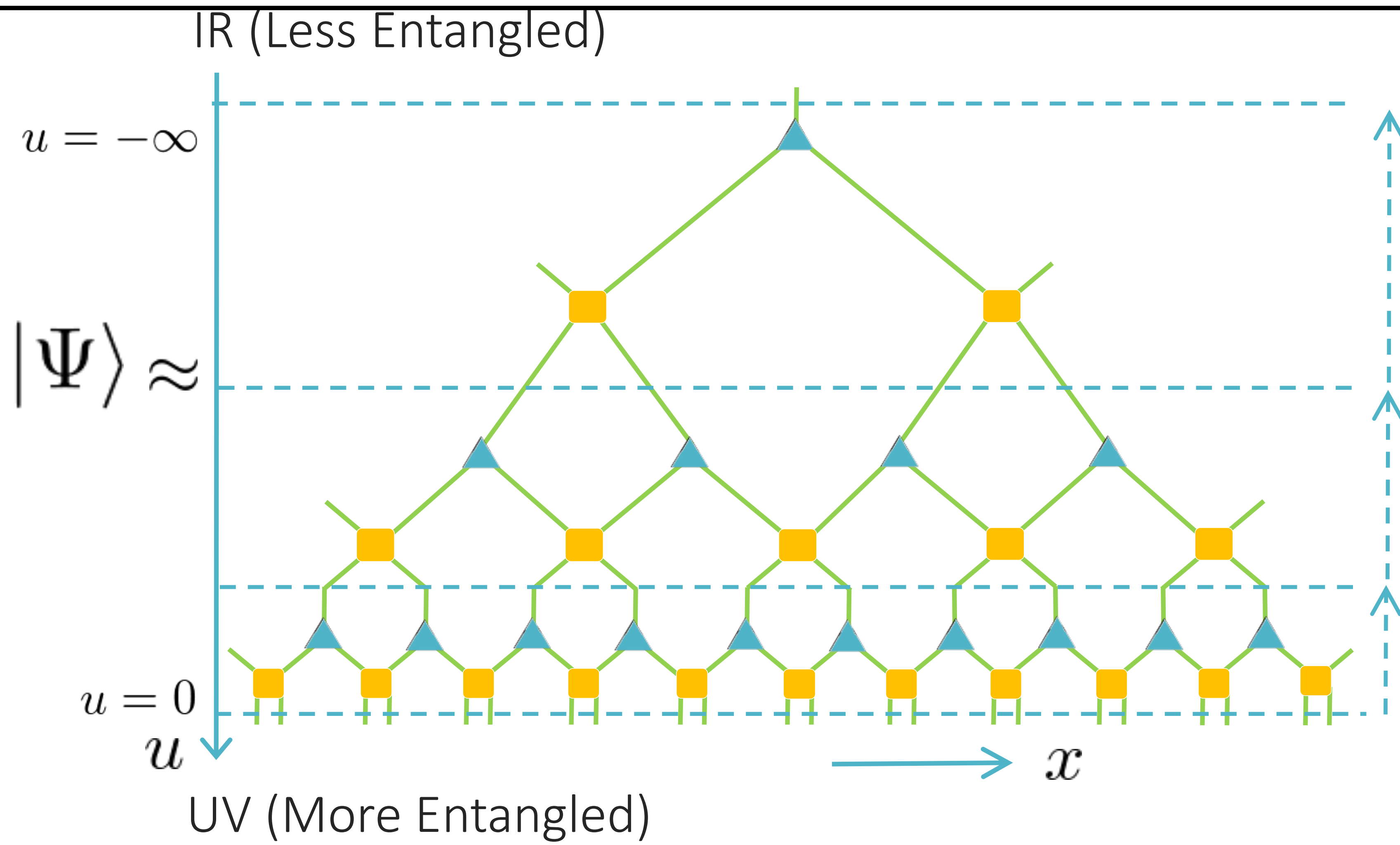
$$S_L=\frac{c}{24\pi}\int_{-\infty}^{\infty}dx\int_{\epsilon}^{\infty}dz\left[(\partial_x\phi)^2+(\partial_z\phi)^2+\mu e^{2\phi}\right]$$

$$S_L=\frac{c}{24\pi}\int dxdz\left[(\partial_x\phi)^2+(\partial_z\phi+\sqrt{\mu}e^{\phi})^2\right]-\frac{c}{12\pi}\int dx[\sqrt{\mu}e^{\phi}]_{z=\epsilon}^{z=\infty}\geq\frac{c\sqrt{\mu}L}{12\pi\epsilon}$$

$$\Rightarrow \mathrm{Minimum:}\ e^{2\phi}=\frac{1}{z^2}\ \Rightarrow\qquad ds^2=\frac{dx^2+dz^2}{z^2}$$

Hyperbolic plane = Time slice of AdS_3

Minimize the Liouville action "= path-integral complexity"



$$S_L = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz \left[\boxed{(\partial_x\phi)^2 + (\partial_z\phi)^2} + \boxed{\mu e^{2\phi}}\right]$$

$\overset{k}{\underset{i \quad j}{\triangle}}$ no. of isometries !!

Czech '17

no. of unitaries ! $\overset{k \quad l}{\underset{i \quad j}{\blacksquare}}$

The Thermo-field double state in 2D CFT

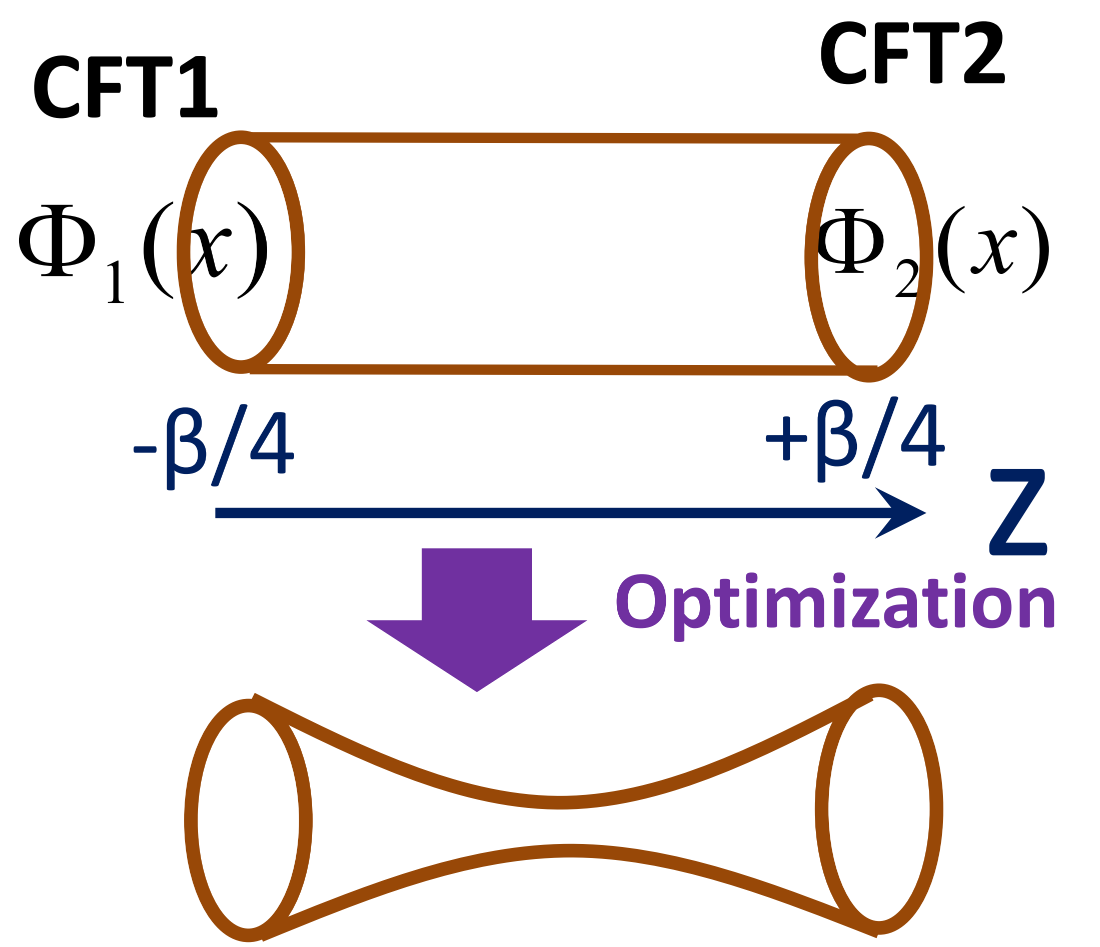
The path-integral for TFD state $-\frac{\beta}{4}(\equiv z_1) < z < \frac{\beta}{4}(\equiv z_2)$

$$\Psi[\tilde{\varphi}_1, \tilde{\varphi}_2] = \int \left(\prod_x \prod_{-\frac{\beta}{4}<z<\frac{\beta}{4}} D\varphi(x,z) \right) e^{-S_{CFT}(\varphi)} \prod_{-\infty<x<\infty} \delta\big(\varphi(z_1,x) - \tilde{\varphi}_1(x)\big)\delta\big(\varphi(z_2,x) - \tilde{\varphi}_2(x)\big)$$

$$\propto e^{S_L}$$

Minimization of S_L

$$\Rightarrow \quad e^{2\phi(z)} = \frac{4\pi^2}{\beta^2}\sec^2\left(\frac{2\pi z}{\beta}\right)$$



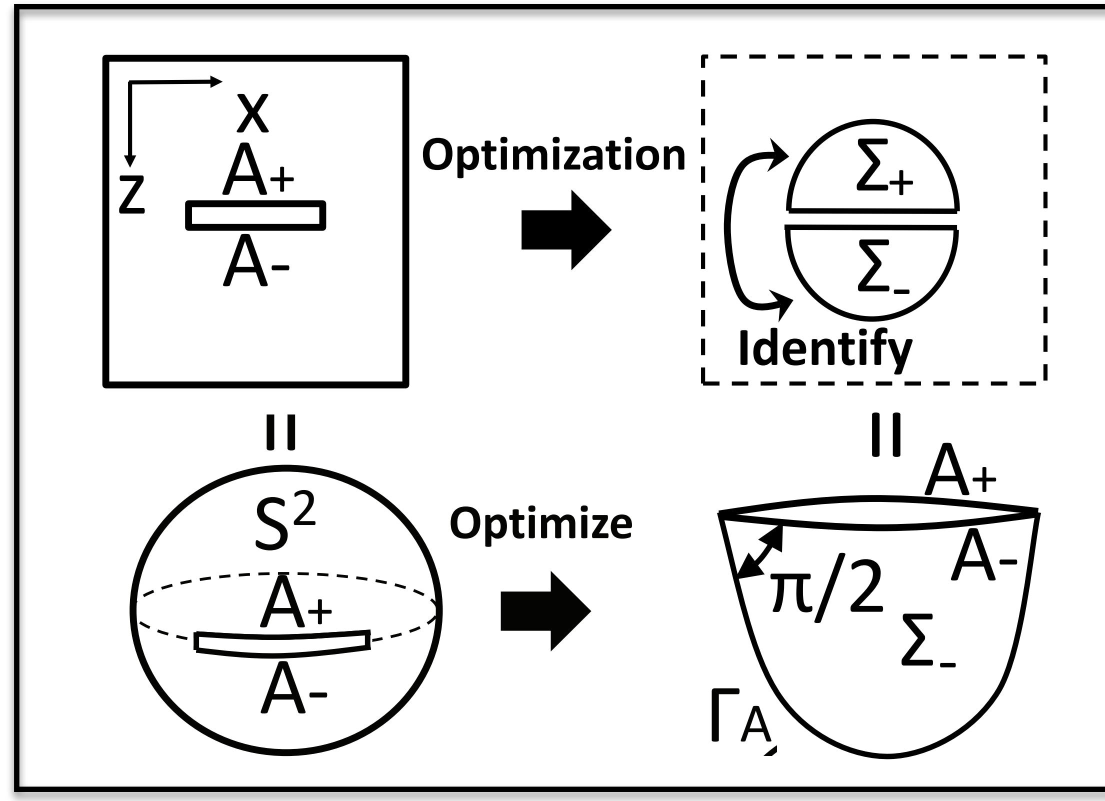
Time slice of BTZ blackhole == Einstein-Rosen bridge

Optimization of density matrices ?

- The optimization is done by introducing the back-ground metric

$$ds^2 = e^{2\phi}(dx^2 + dz^2) \qquad e^{\phi}\big|_{A_\pm} = 1/\epsilon$$

- Optimization squeezes the infinite background to finite size.



- Finally one identifies $\Sigma_\pm$ along the boundaries $\partial\Sigma_\pm$

- The shape of the boundary is given by extremizing the boundary Liouville action

$$S_L = 2\times\frac{c}{24\pi}\int_\Sigma dxdz\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + e^{2\phi}\right] + 2\times\frac{c}{12\pi}\int_{\partial\Sigma} ds\left[K_0\cdot\phi\right].$$

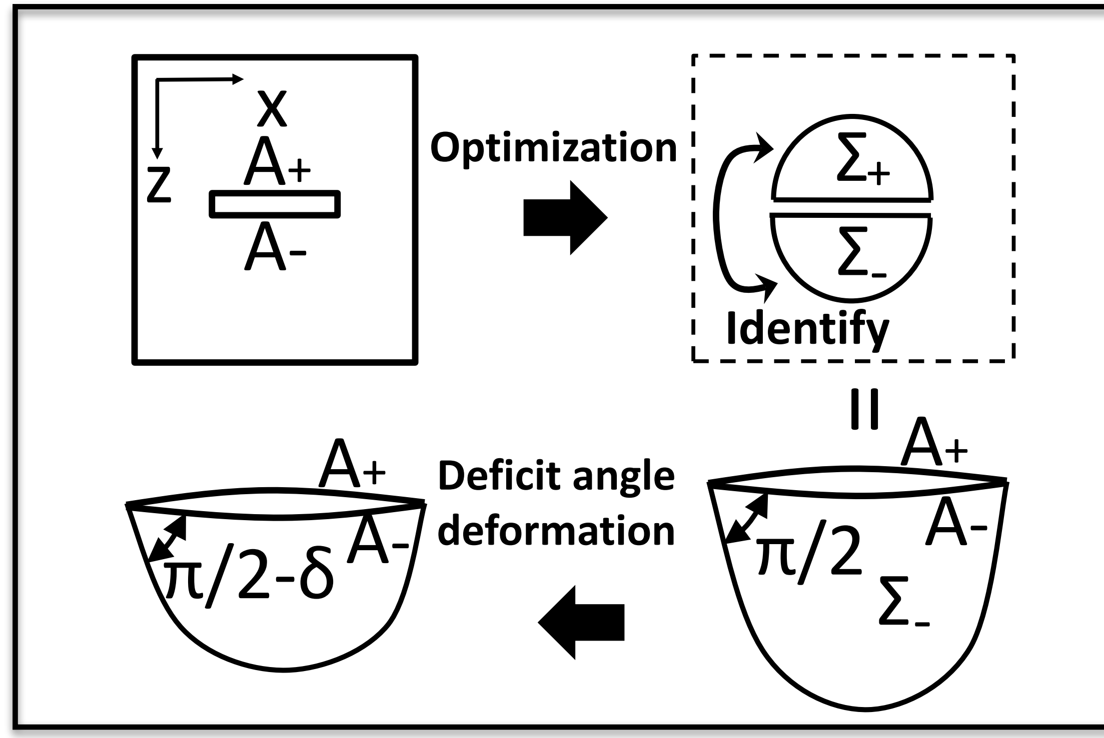
- Impose Neumann boundary condition => The extrinsic curvature K = 0

$$e^{\phi}K = n^a\partial_a\phi + K_0 = 0 \qquad\qquad x^2 + z^2 = l^2$$

- The solution is :=> The half circle. => The Entanglement Wedge for gravity dual of density matrix

Entanglement Entropy from the optimization?

- We should consider the standard n-sheeted geometry with the two end points of the slit having conical deficit angle : $2\pi(n-1)$
  - After the optimization this is equivalent to changing the boundary condition



$$S_L^{(n)}[\phi] = \frac{c}{6\pi}\int_\Sigma dxdz\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + e^{2\phi}\right] + \frac{c}{3\pi}\int_{\partial\Sigma} ds\left[K\phi + \mu_B e^{\phi}\right]$$

$$K + \mu_B = e^{-\phi}(n^a\partial_a\phi + K_0) + \mu_B = 0$$

$$K = \pi(n-1), \quad \mu_B = \pi(1-n)$$

- The Entanglement Entropy is evaluated as

$$S_A = -\partial_n\left[\frac{c\mu_B}{12\pi}\int_{\partial\Sigma_+} ds\ e^{\phi} + \frac{c\mu_B}{12\pi}\int_{\partial\Sigma_-} ds\ e^{\phi}\right]_{n=1} = \frac{c}{6}\int_{\partial\Sigma_+} ds\ e^{\phi} = \frac{c}{3}\log\frac{l}{\epsilon}$$

One-dimensional system ...

- Another interesting application is the one dimensional case with the metric

$$\boxed{\begin{gathered} ds^2 = e^{2\phi} d\tau^2 \\ \Psi_{g_{\tau\tau}=e^{2\phi}}(\tilde{\varphi}(x)) = e^{S_1[\phi]} \cdot \Psi_{g_{\tau\tau}=1}(\tilde{\varphi}(x)), \quad S_1[\phi] = N \int d\tau \left[(\partial_\tau \phi)^2 + \mu e^{\phi} \right] \end{gathered}}$$

- To stabilize the optimization procedure for AdS-2/CFT-1, one needs to add the conformal symmetry breaking Schwarzian derivative term, which is explicitly realized in the SYK model.
- The end result is again time slice of 2-dim AdS

$$ds^2 = e^{2\phi} d\tau^2 = \frac{d\tau^2}{\tau^2}$$

Excited state in 2D - CFT's

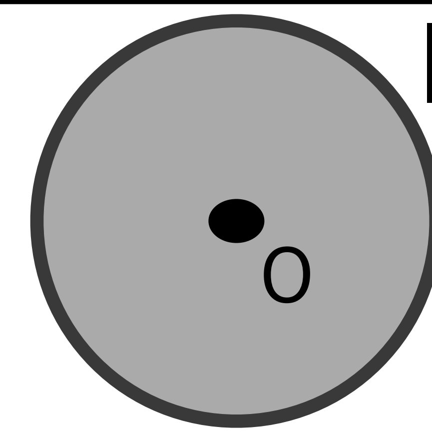
- a primary operator O, with conformal dimension = h, is inserted at the origin of a disk ..

$$O(w,\bar{w}) \propto e^{-2h\phi}. \quad \Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}) \simeq e^{S_L} \cdot e^{-2h\phi(0)} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}).$$

- We are considering a 2-D CFT defined on the disk, and the EOM looks like

$$|\omega| < 1 \qquad \partial_w \partial_{\bar{w}} \phi - \frac{\mu}{4} e^{2\phi} + \frac{\pi}{2}(1-a)\delta^2(w) = 0 \qquad a = 1 - \frac{12h}{c}$$

- The solution is => $e^{2\phi} = \frac{4}{\mu} \cdot \frac{a^2}{|w|^{2(1-a)}(1-|w|^{2a})^2}$



- Compare with time slice of 3D - AdS and matches when the back-reaction is small

$$a = \sqrt{1 - \frac{24h}{c}}$$

$$h \ll c$$

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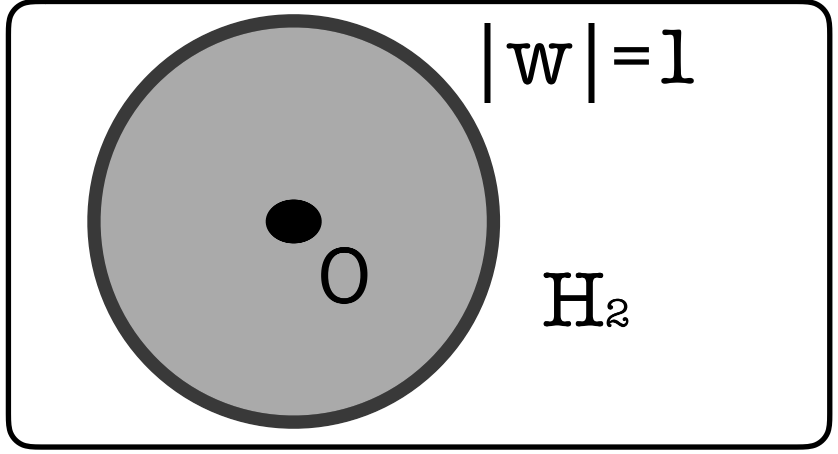

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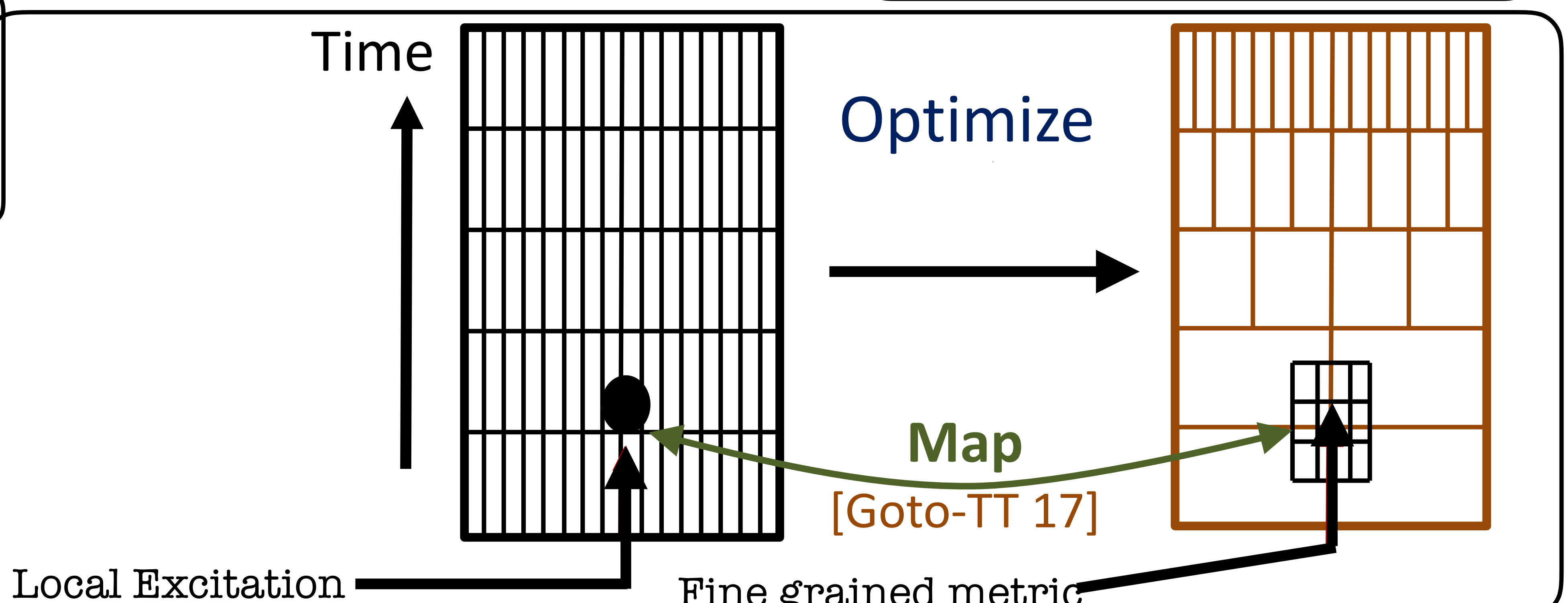


Higher dimensional CFT's ? We don't have a "very" good answer ! !

We can not write the metric with just a Weyl mode !

For convenience we just focus on the Weyl form

$$ds^2 = e^{2\phi(x,z)}\left(dz^2 + d\vec{x}^2\right)$$

We argue that the path-integral complexity should be given by

$$I_d[\phi] = N\int dz dx^{d-1}\left[e^{d\phi} + e^{(d-2)\phi}(\partial_z\phi)^2 + e^{(d-2)\phi}(\partial_x\phi)^2 + \frac{e^{(d-2)\phi}\cdot R_0}{(d-1)(d-2)}\right]$$
$$+2N\int_{bdy} dx^{d-1}\left[\frac{e^{(d-2)\phi}\cdot K_0}{(d-1)(d-2)} + \frac{\mu_B}{d-1}e^{(d-1)\phi}\right]. \qquad \left(N \equiv \frac{(d-1)R_{AdS}^{d-1}}{16\pi G_N}\right).$$

Such that $\lim_{d\to 2}\left[I_d[\phi] - I_d[0]\right] = S_L[\phi] - S_L[0]$

We then show that

1) time slice of AdS_(d+1) == vacua of CFT
2) For a spherical round ball ==> HEE is reproduced ..
3) AdS BH deformation is also produced

Higher dimensional CFT's ?
Evaluation of path-integral complexity

2d CFT (1) Poincare AdS3: $C = \frac{c}{12\pi} \cdot \frac{L}{\varepsilon}.$

(2) global AdS3: $C = \frac{c}{6} \cdot \left[\frac{1}{\varepsilon} - 1 \right].$

(3) BTZ(TFD): $C = \frac{c}{3} \left[\frac{1}{\varepsilon} - \frac{\pi^2}{2\beta} \right].$

3d CFT global AdS4: $C = 4\pi N \left[\frac{1}{\varepsilon^2} + \frac{1}{2} + \log\left(\frac{2}{\varepsilon} \right) \right].$

4d CFT global AdS5: $C = 2\pi^2 N \left[\frac{2}{3\varepsilon^3} + \frac{1}{\varepsilon} - \frac{5}{12} \right].$

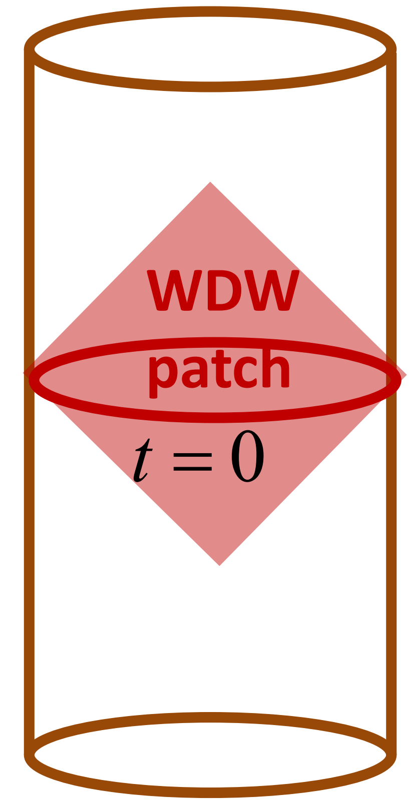
An Universal divergence structure :
complexity = Volume law divergence + sub-leading terms

Holographic results : Myers et.al., Reynolds-Ross

A connection to “Complexity = Action” Proposal

Consider the following patch of pure AdS_(d+1) :

$$ds^2 = R_{AdS}^2\left(-dt^2 + \cos^2 t \cdot e^{2\phi(x)} \cdot h_{ab}dx^a dx^b\right).$$
$$-\pi/2 \leq t \leq \pi/2$$



This patch precisely covers the WdW patch in the literature

When $e^{2\phi(x)} \cdot h_{ab}dx^a dx^b$ is the hyperbolic plane

We obtain ..

$$I_{WDW} = \frac{1}{16\pi G_N}\int_{WDW} dx^d dt\sqrt{-g}\left[R - 2\Lambda\right] + (\text{bdy. term})$$
$$= (d-2)\cdot n_d \cdot I_d[\phi] + (\text{IR surface term}).$$

Summary

1) We introduce the optimization of path integral in CFTs and obtain the time slice of AdS
2) Provide a proposal for computational complexity for CFTs : path-integral complexity
3) The minimization of path-integral complexity : time slice of AdS
4) We provided generalizations to higher dim cases ..

In future :
1) Time - component of metric ..
2) Time dependent cases ..
3) Non-conformal CFTs ..
4) More concrete higher-dim analysis ..

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Thank you for your Kind Attention