RENYI ENTROPY FOR LOCAL QUENCHES
FROM NUMERICAL CONFORMAL BLOCK IN 2D CFTS

Yuya Kusuki (YITP)

Based on a paper to appear,
in a collaboration with
Tadashi Takayanagi

My talk

Motivation

Excited Renyi entropy in holographic CFTs

Our Method

Expression for General Conformal blocks

Result

Completely understanding excited RE in large c CFTs

Motivation – Renyi entropy

- Renyi entanglement entropy after a local quench

$$\Delta S_A^{(n)}(t) = S_A^{(n)}(\,|\psi(t)\rangle) - S_A^{(n)}(|0\rangle)$$

Where

$$|\psi(t)\rangle = N e^{-iHt} e^{-\epsilon H} O(x)|0\rangle$$

→understanding…

· the properties of QFTs (ex. chaotic or not)

· the mechanism of AdS/CFT

Motivation – known results

Known results

Time dependence of EE in various CFTs	
RCFT	$const.$
Holographic CFT	$\log t$
Orbifold irrational CFT	$\log\log t$

Motivation – known results

Known results

Time dependence of RE in various CFTs	
RCFT	$const.$
Holographic CFT	$\log t$
Orbifold irrational CFT	$\log\log t$

Moreover

Renyi entropy in holographic CFTs	
RE ($n \geq 2$) for light operators	$\frac{2nh_O}{n-1}\log\frac{t}{\epsilon}$
EE($n \to 1$) for heavy operators	$\frac{c}{6}\log\frac{t}{\epsilon}$

Motivation – known results

Known results

Time depend…
RCFT
Holographic C…
Orbifold irrati…

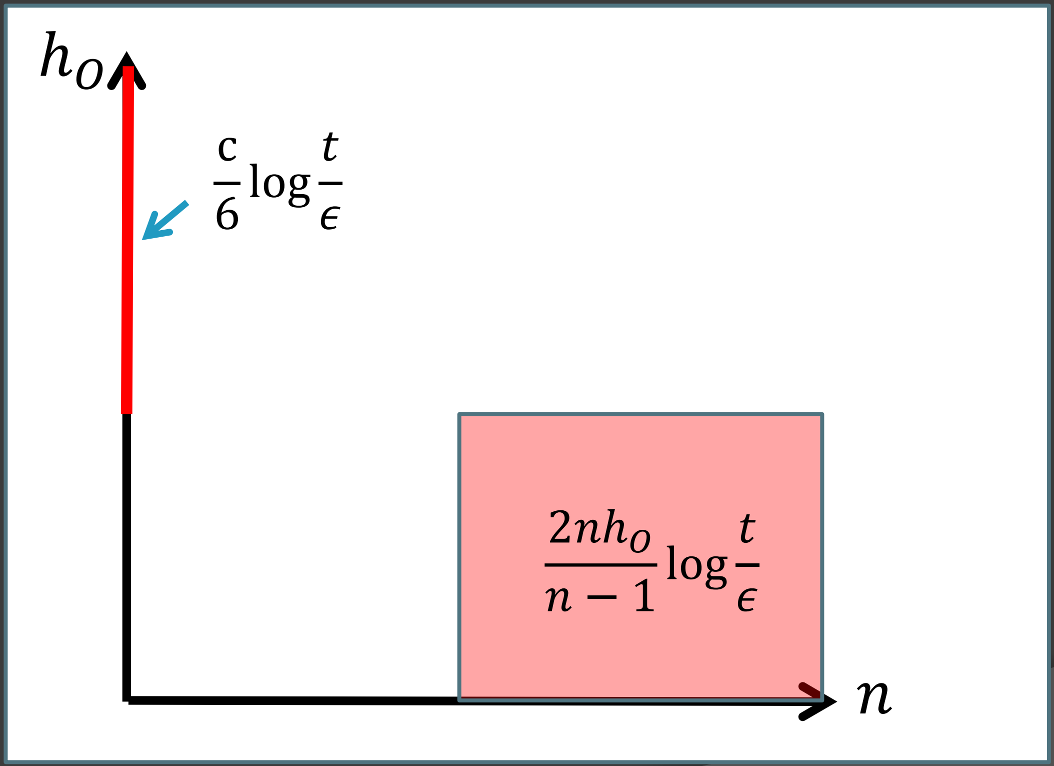
We have a very limited knowledge of REE in holographic CFTs!!

Moreover

Renyi entropy in holographic CFTs	
RE ($n \geq 2$) for light operators	$\frac{2nh_O}{n-1}\log\frac{t}{\epsilon}$
EE($n \to 1$) for heavy operators	$\frac{c}{6}\log\frac{t}{\epsilon}$

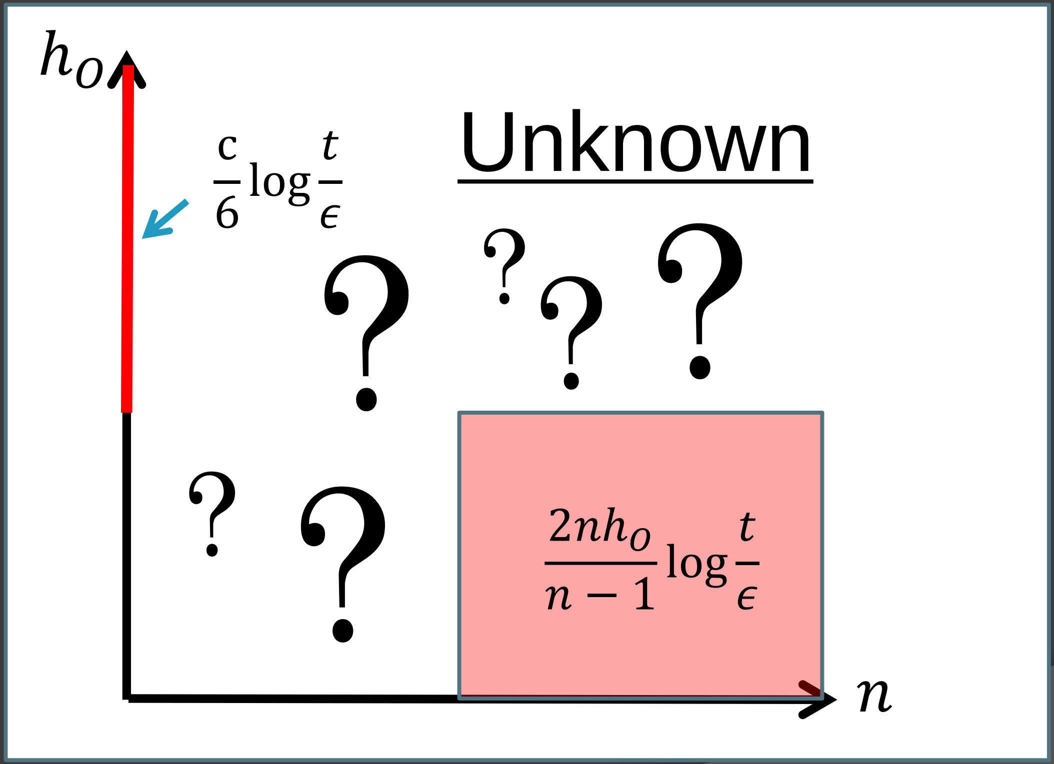
Motivation – known results

Known results



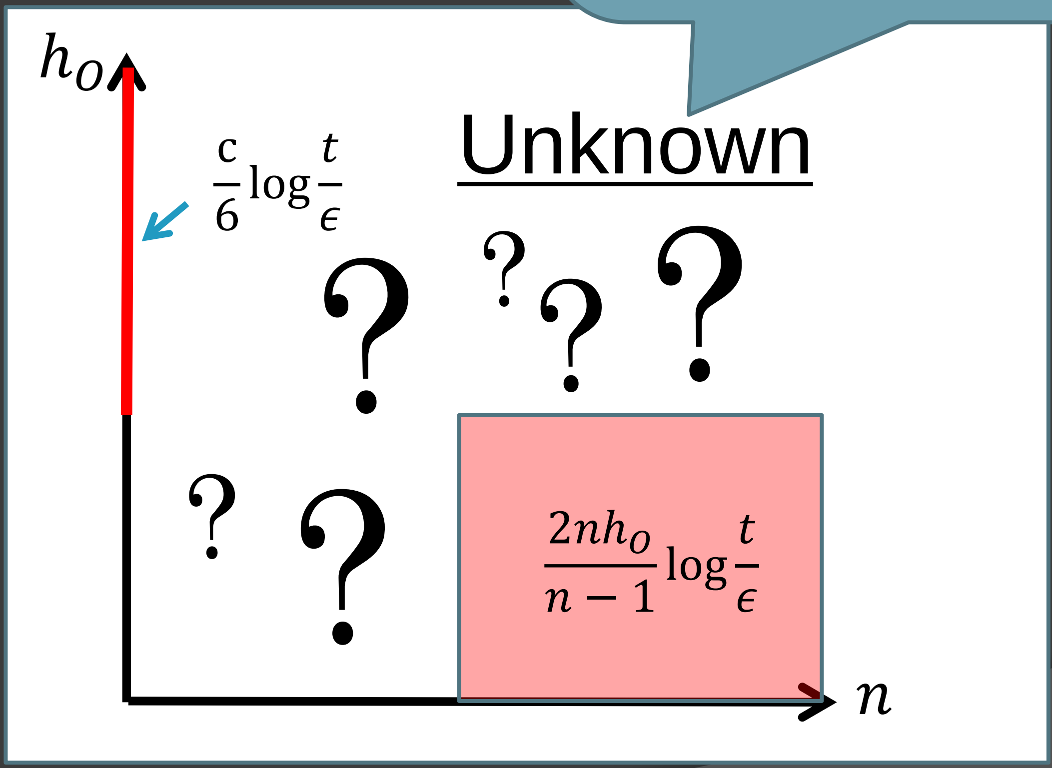
Motivation – known results

Known results



Motivation – know

Known results



We want to <u>completely</u> understand the REE in holographic CFTs.

Method – how to calculate REE

- Roughly, excited Renyi entropy can be given by

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left(|z(t)|^{4\,h_{\sigma_n}}\left|F_{h_{O^n}\,h_{O^n}}^{h_{\sigma_n}h_{\sigma_n}}(0|z(t))\right|^2\right)$$

Method – how to calculate REE

- Roughly, excited Renyi entropy can be given by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log\left(|z(t)|^{4 h_{\sigma_n}} \left| F^{h_{\sigma_n} h_{\sigma_n}}_{h_{O^n} h_{O^n}}(0|z(t)) \right|^2 \right)$$

All we need is the Vacuum Conformal Block

in holographic CFTs

Method – how to calculate REE

- Roughly, excited Renyi entropy can be given by

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\left(|z(t)|^{4h_{\sigma_n}}\left|F^{h_{\sigma_n}h_{\sigma_n}}_{h_{O^n}\,h_{O^n}}(0|z(t))\right|^2\right)$$

All we need is the Vacuum Conformal Block in holographic CFTs

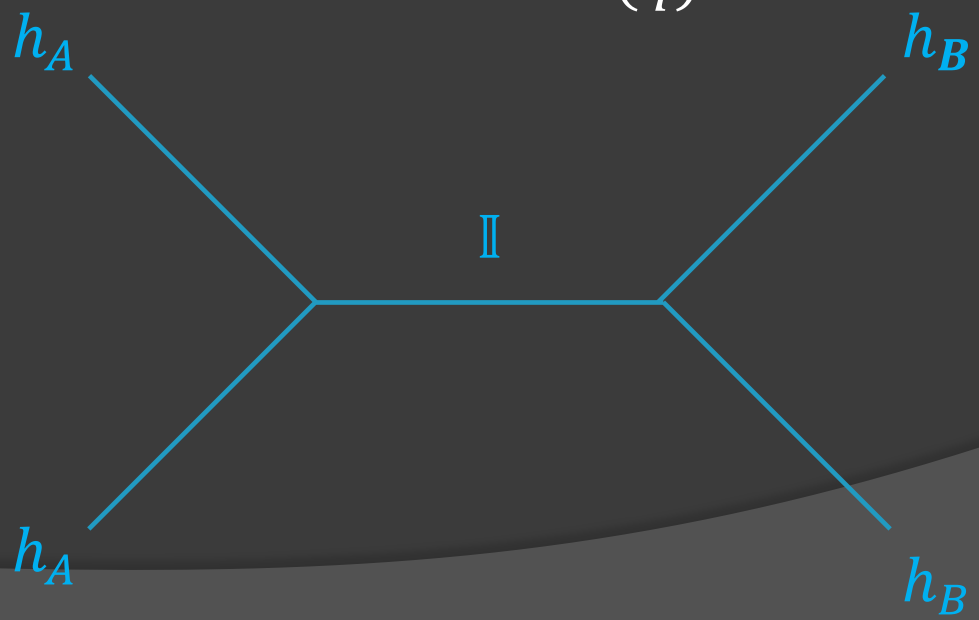
- The main tool is

Zamolodchikov recursion relation

Method – recursion formula

- The conformal block for $\langle O_A(0) O_A(z) O_B(1) O_B(\infty) \rangle$

$$F^{h_A h_A}_{h_B\, h_B}(0|z)$$
$$= (16q)^{-\frac{c-1}{24}} z^{\frac{c-1}{24}} (1-z)^{\frac{c-1}{24}-h_A-h_B} \theta_3(q)^{\frac{c-1}{2}-8(h_A+h_B)}$$
$$\times H^{h_A,h_B}(q)$$



Method – recursion formula

- The conformal block for $\langle O_A(0) O_A(z) O_B(1) O_B(\infty)\rangle$

$$F^{h_A h_A}_{h_B\, h_B}(0|z)$$
$$= (16q)^{-\frac{c-1}{24}} z^{\frac{c-1}{24}} (1-z)^{\frac{c-1}{24}-h_A-h_B} \theta_3(q)^{\frac{c-1}{2}-8(h_A+h_B)}$$
$$\underline{\times H^{h_A,h_B}(q)}$$

The function of $H(q)$ can be calculated recursively by

$$H(h_A, h_B, h_p|q) = 1 + \sum_{n=1}^{\infty} \frac{q^{mn} R_{m,n}}{h_p - h_{m,n}} H(h_A, h_B, h_{m,n} + mn|q)$$

where $h_{m,n}$ is a zero of the Kac determinant.

Method – recursion formula

- The conformal block for $\langle O_A(0) O_A(z) O_B(1) O_B(\infty) \rangle$

$$F^{h_A h_A}_{h_B\, h_B}(0|z)$$
$$= (16q)^{-\frac{c-1}{24}} z^{\frac{c-1}{24}} (1-z)^{\frac{c-1}{24}-h_A-h_B} \theta_3(q)^{\frac{c-1}{2}-8(h_A+h_B)}$$
$$\underline{\times H^{h_A,h_B}(q)}$$

$$\big\downarrow \text{Series Expansion}$$

$$H^{h_A,h_B}(q) = 1 + \sum_{n=1}^{\infty} c_n\, q^{2n}$$

Method – recursion formula

$$H^{h_A, h_B}(q) = 1 + \sum_{n=1}^{\infty} c_n \, q^{2n}$$

- c_n can be calculated recursively in the same way as $H(q)$.

Method – recursion formula

$$H^{h_A,h_B}(q) = 1 + \sum_{n=1}^{\infty} c_n\, q^{2n}$$

- c_n can be calculated recursively in the same way as $H(q)$.
- In our case $h_1 = h_2, h_3 = h_4$, we can show that odd powers of q don't appear.

Method – recursion formula

$$H^{h_A,h_B}(q) = 1 + \sum_{n=1}^{\infty} c_n\, q^{2n}$$

- c_n can be calculated recursively in the same way as $H(q)$.
- In our case $h_1 = h_2, h_3 = h_4$, we can show that odd powers of q don't appear.
- For any dynamical configuration, the corresponding q satisfies $|q| < 1$, so this expansion is a good approximation.

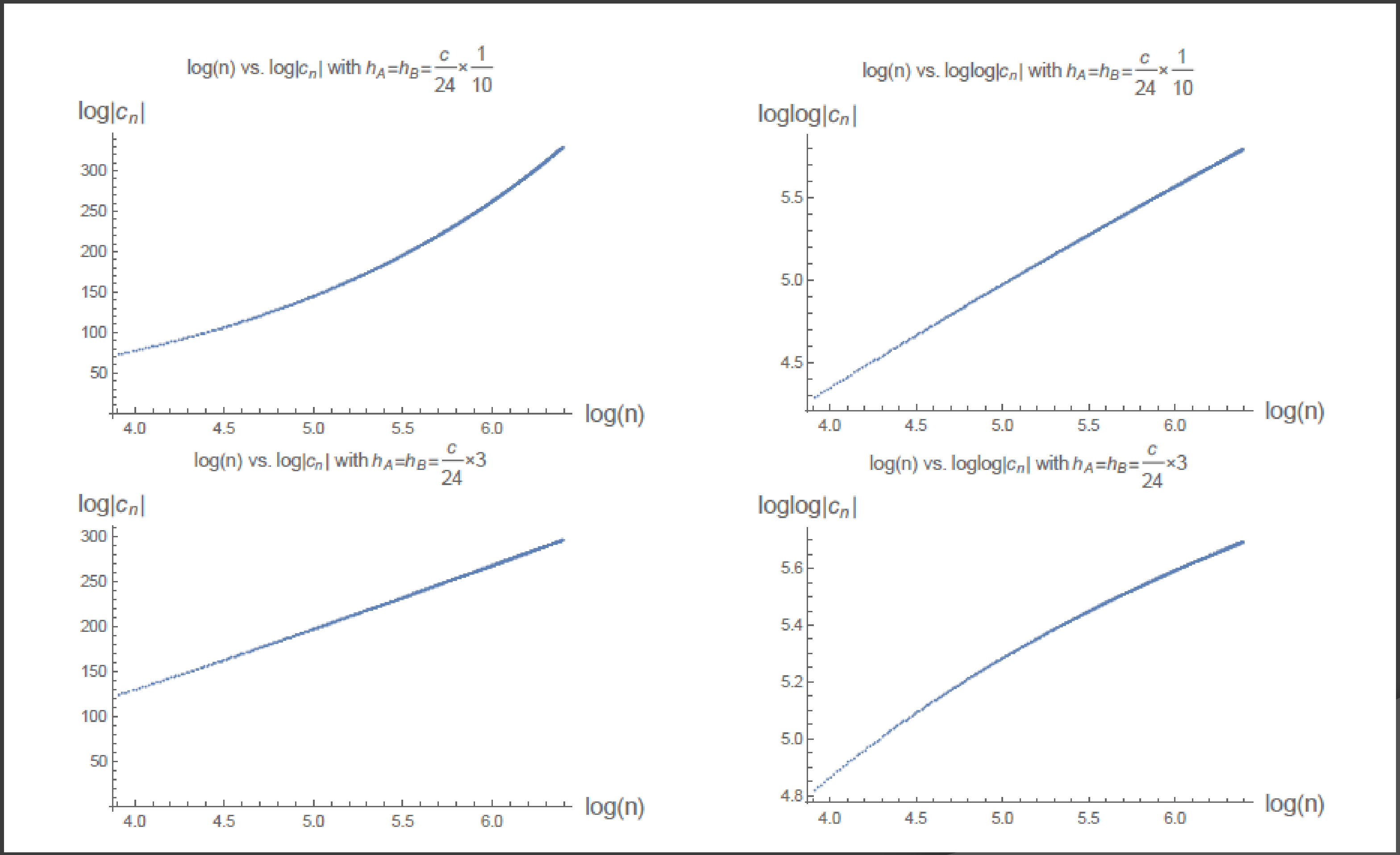
Method – recursion formula

$$H^{h_A,h_B}(q) = 1 + \sum_{n=1}^{\infty} c_n\, q^{2n}$$

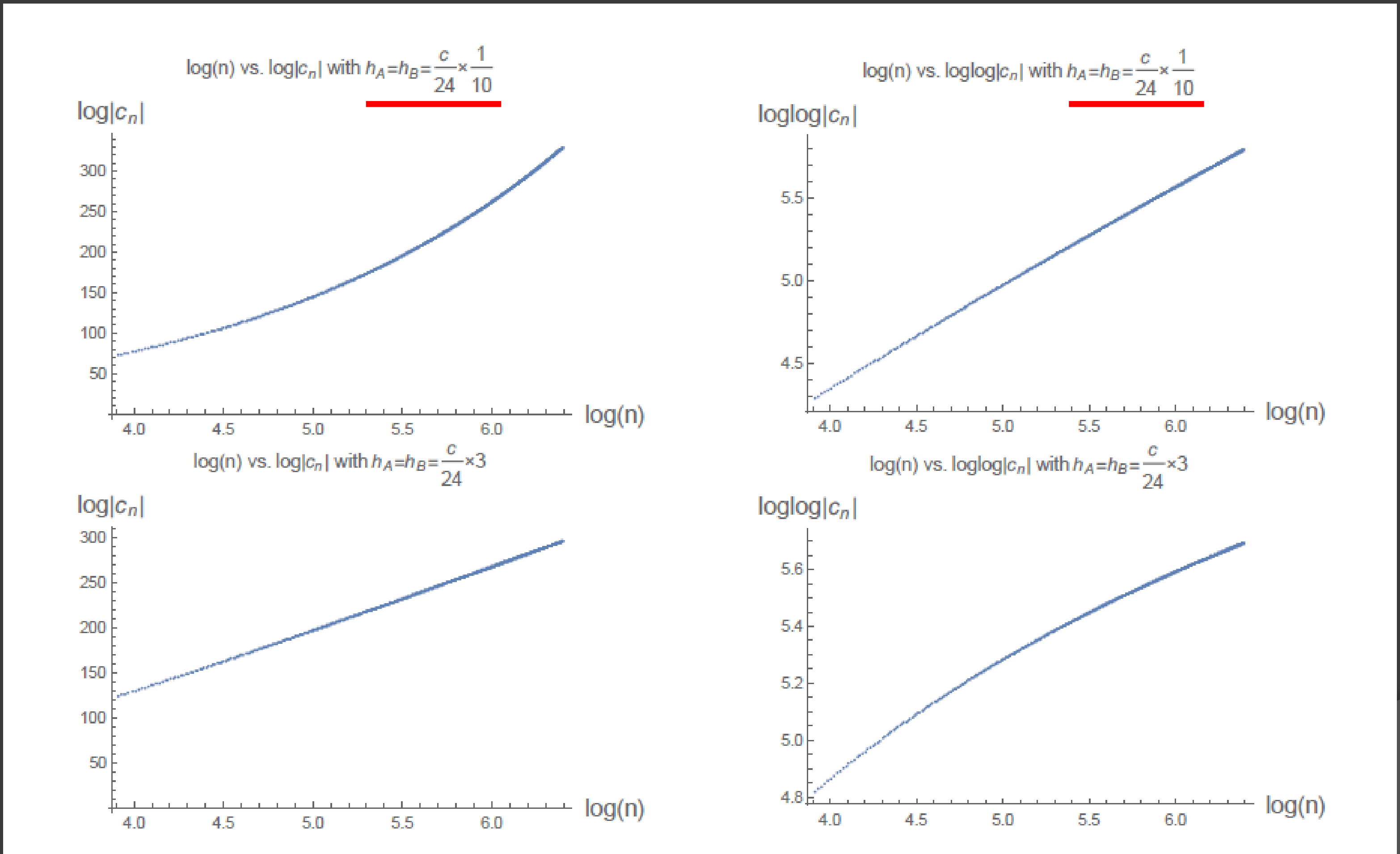
We studied the n dependence of $|c_n|$ by numerical calculation and found the fact:

- There are only two patterns of the n dependence of $|c_n|$ for large n.

Method – numerical results

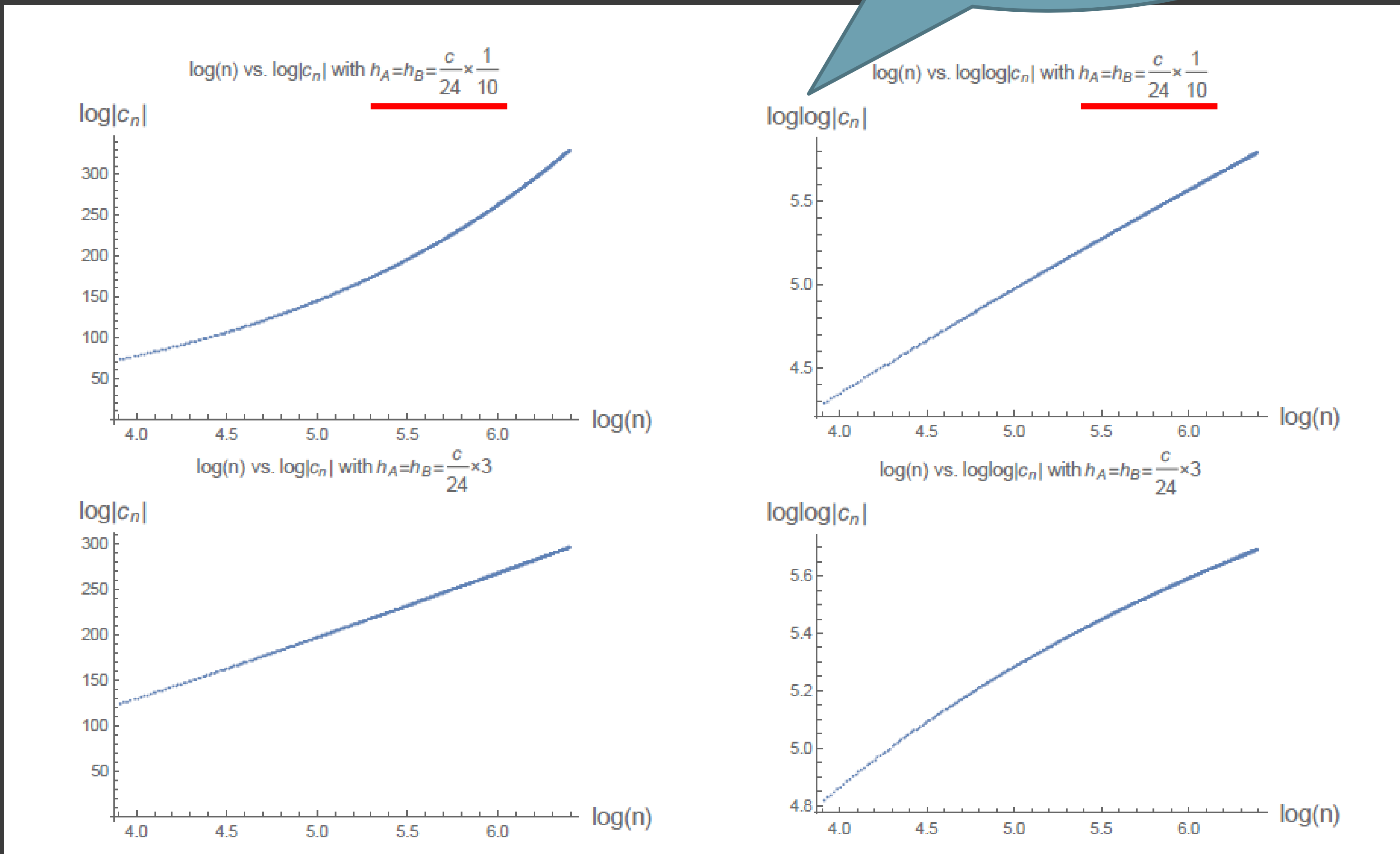


Method – numerical results

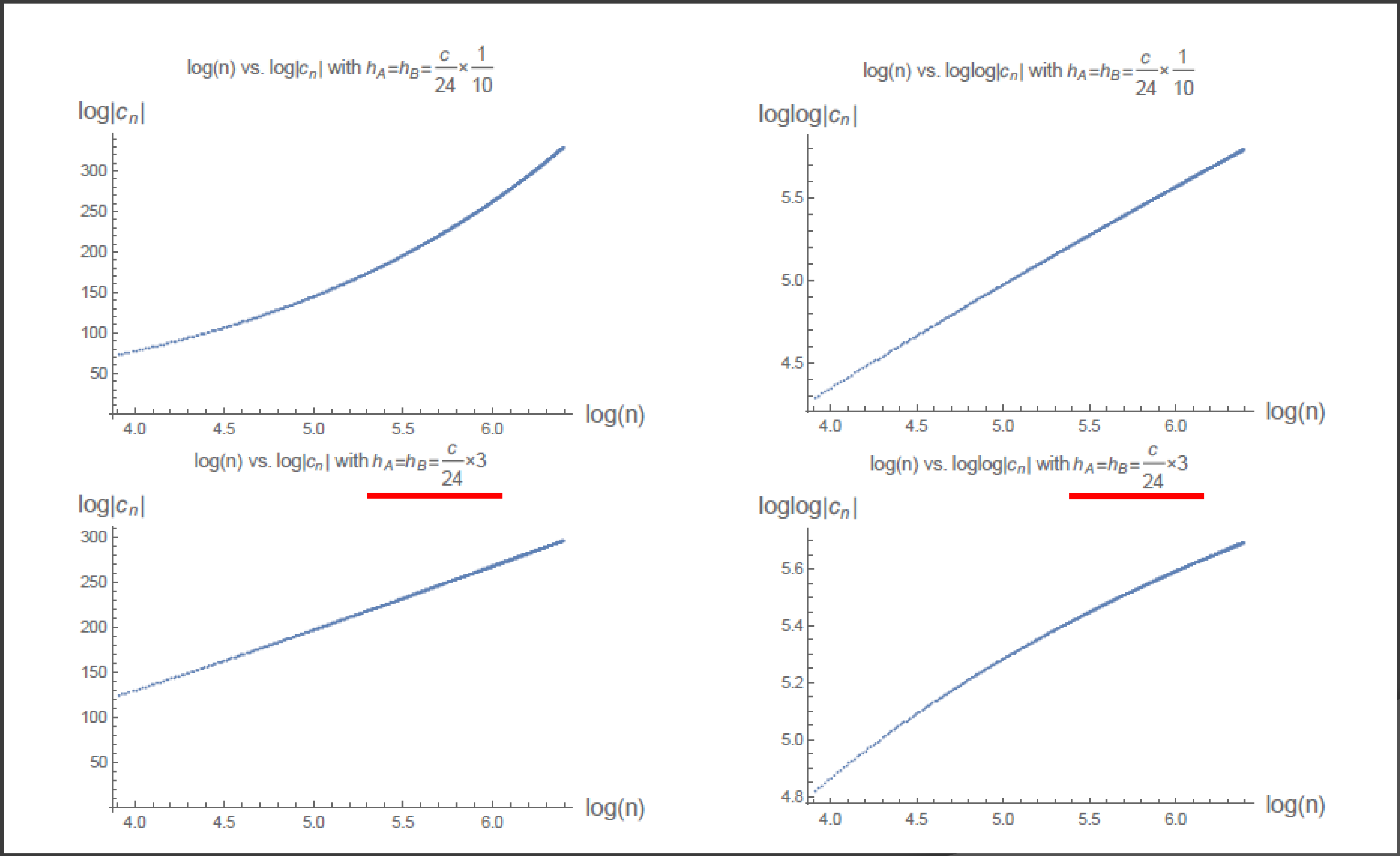


Method – numerical r

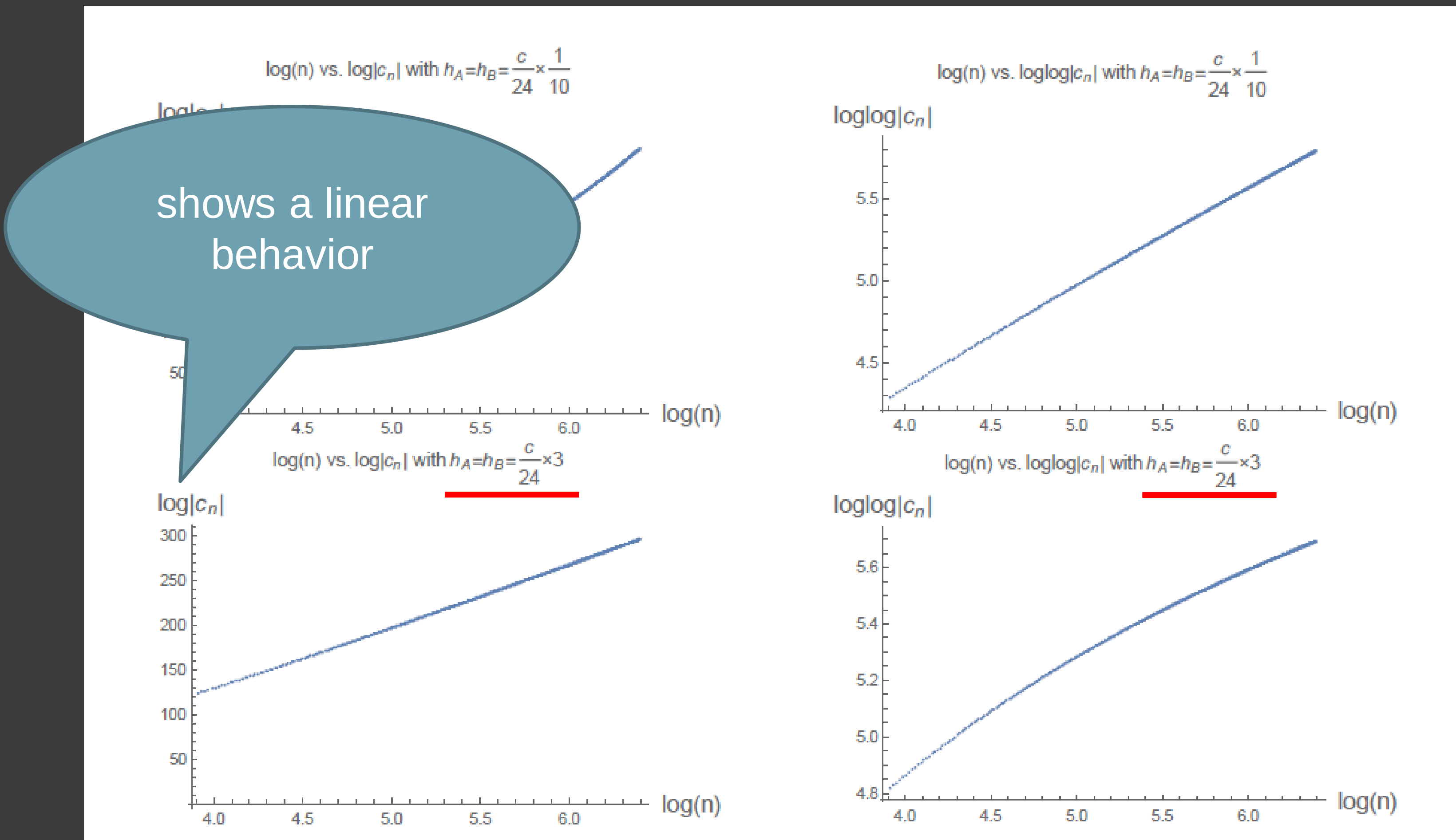
shows a linear behavior



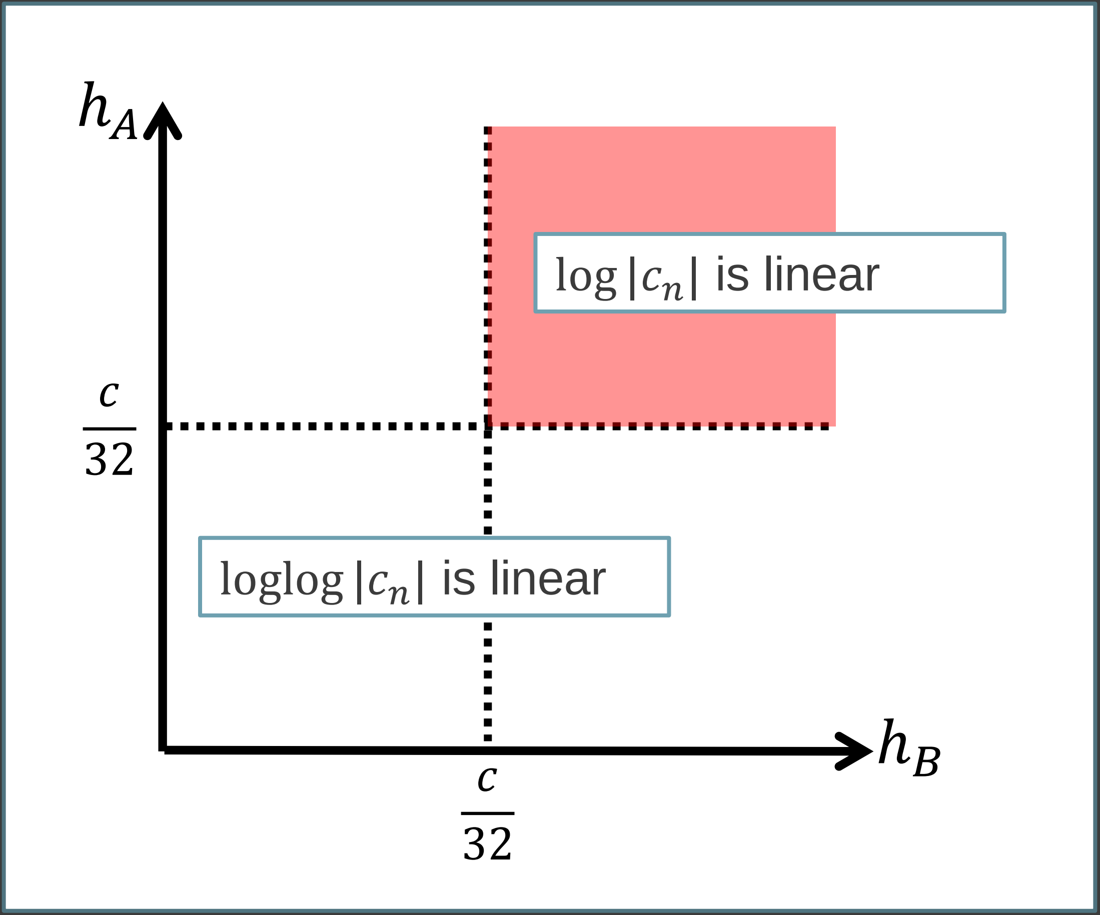
Method – numerical results



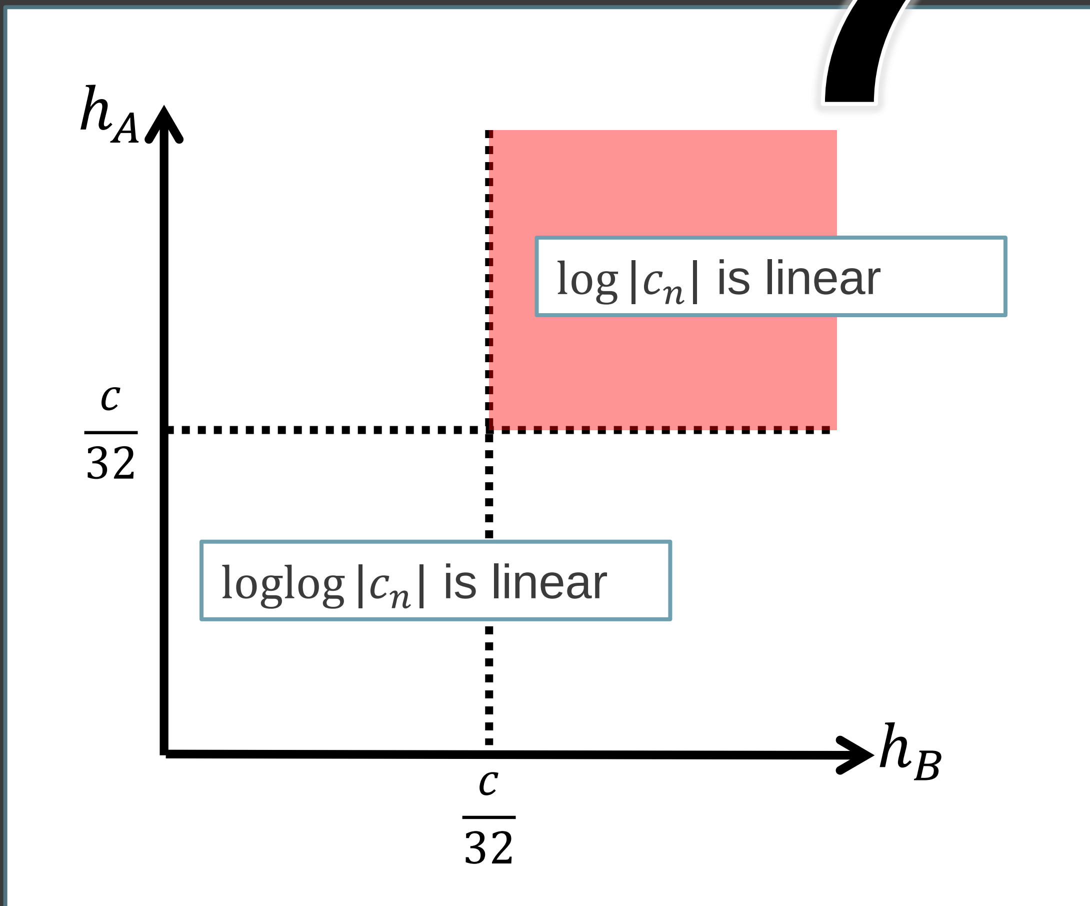
Method – numerical results



Method – numerical results

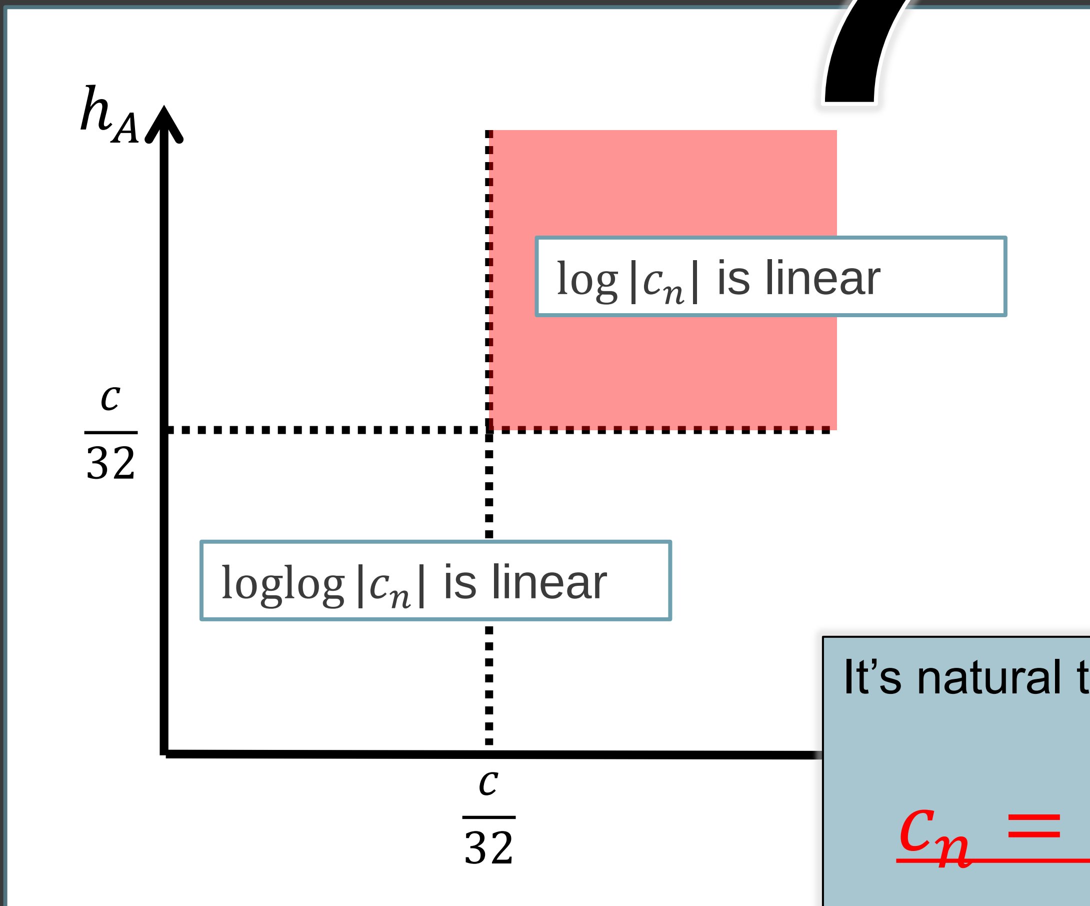


Method – numerical results



h_A, h_B	$\lvert c_n \rvert \quad (n \gg c)$
$h_A, h_B > \frac{c}{32}$	$n^{\#}$
otherwise	$\exp(n^{\#})$

Method – numerical results

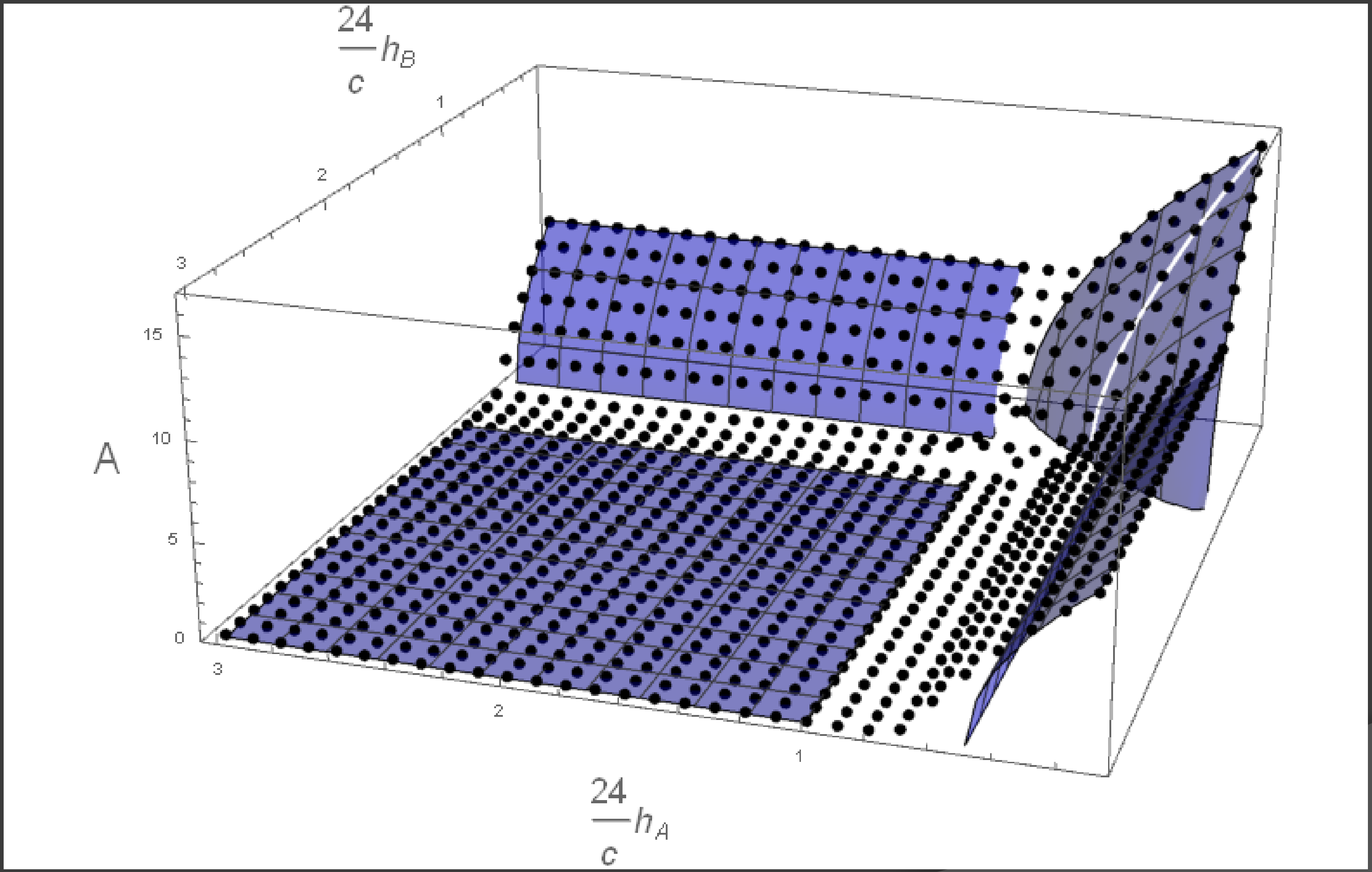


h_A, h_B	$\|c_n\| \quad (n \gg c)$
$h_A, h_B > \frac{c}{32}$	$n^{\#}$
otherwise	$\exp(n^{\#})$

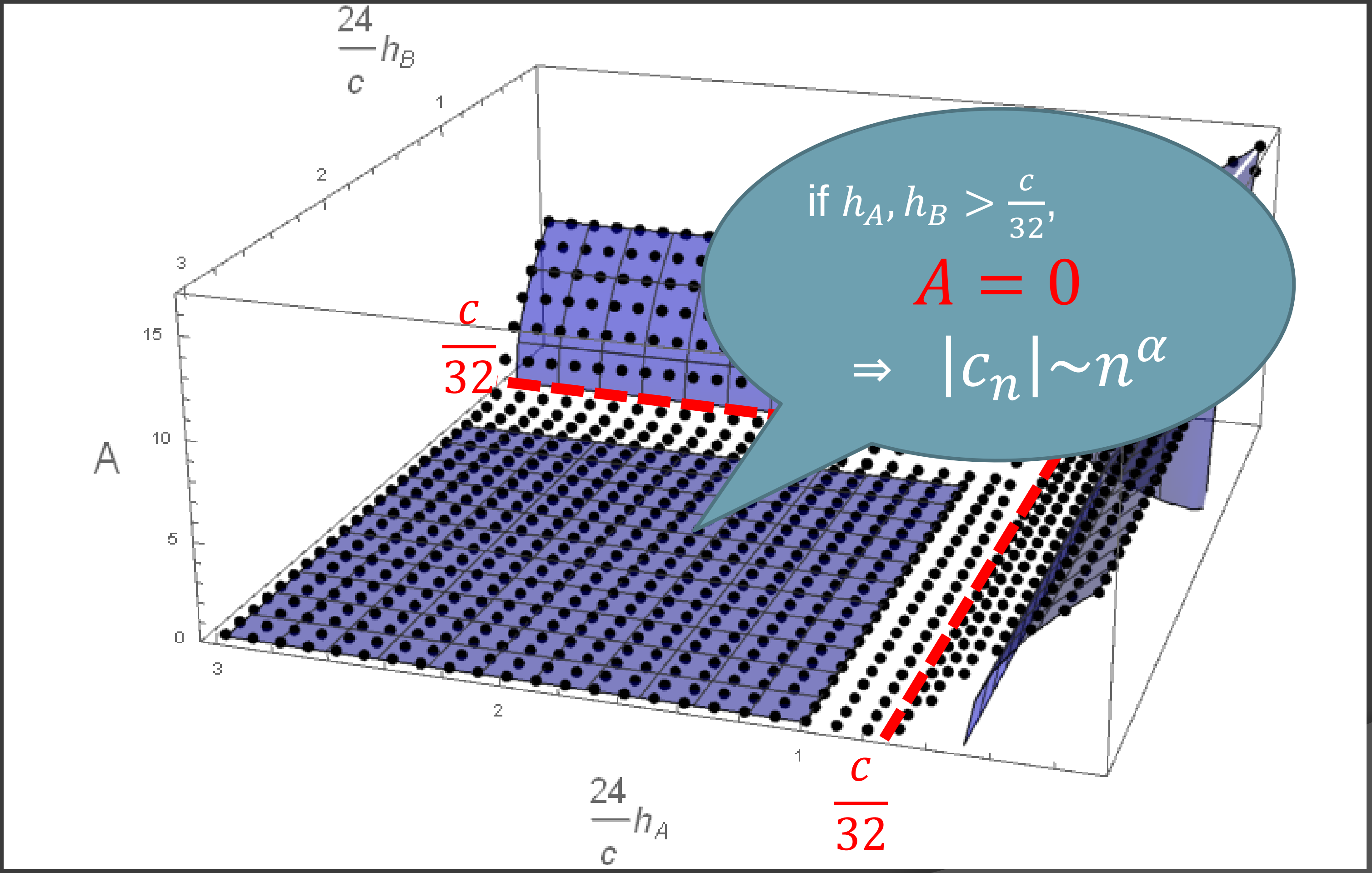
It's natural to fit it to the ansatz;

$$\underline{c_n = n^{\alpha} \exp\left(A\, n^{\beta} + B\right)}$$

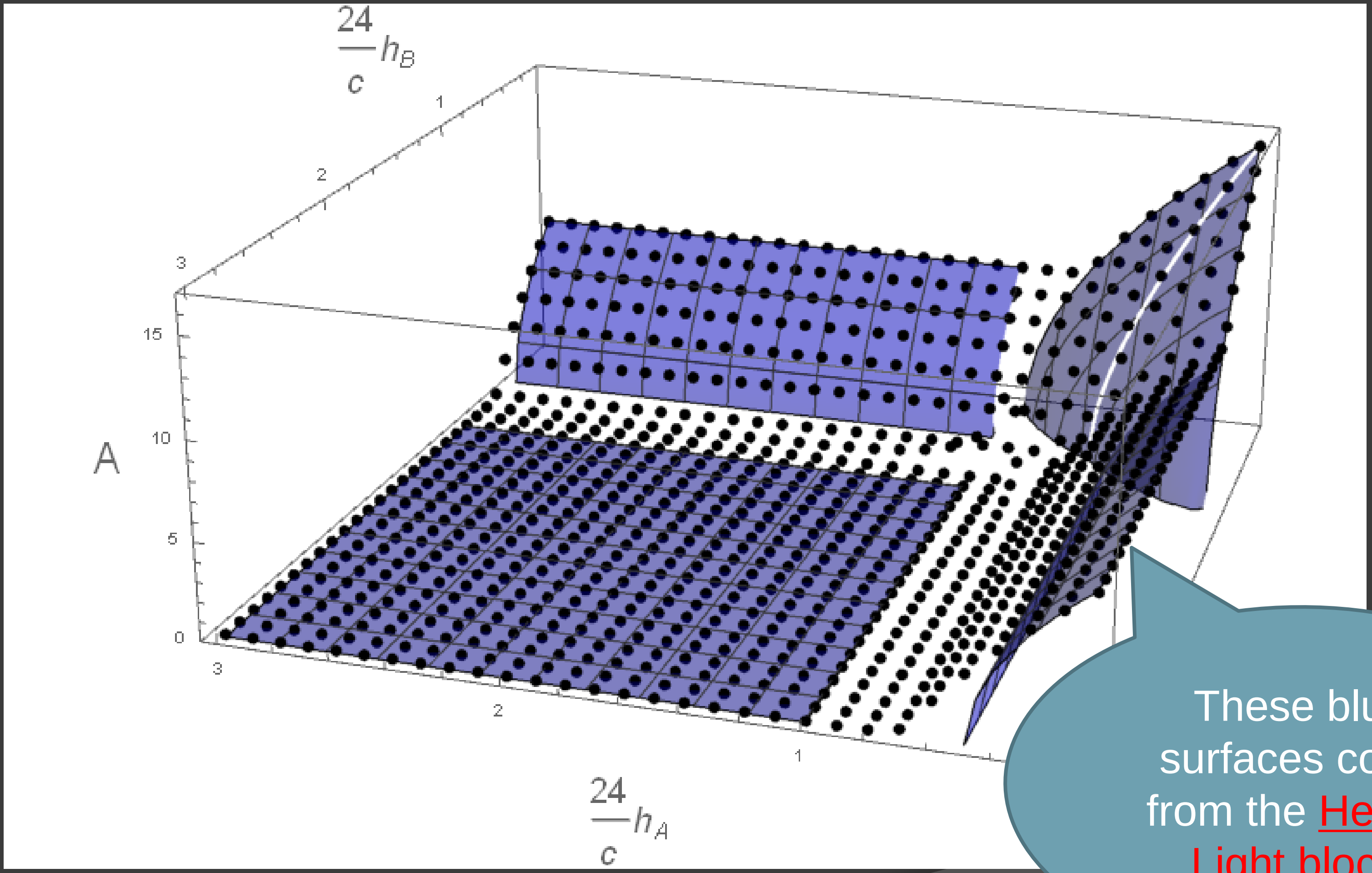
Method – numerical results



Method – numerical results

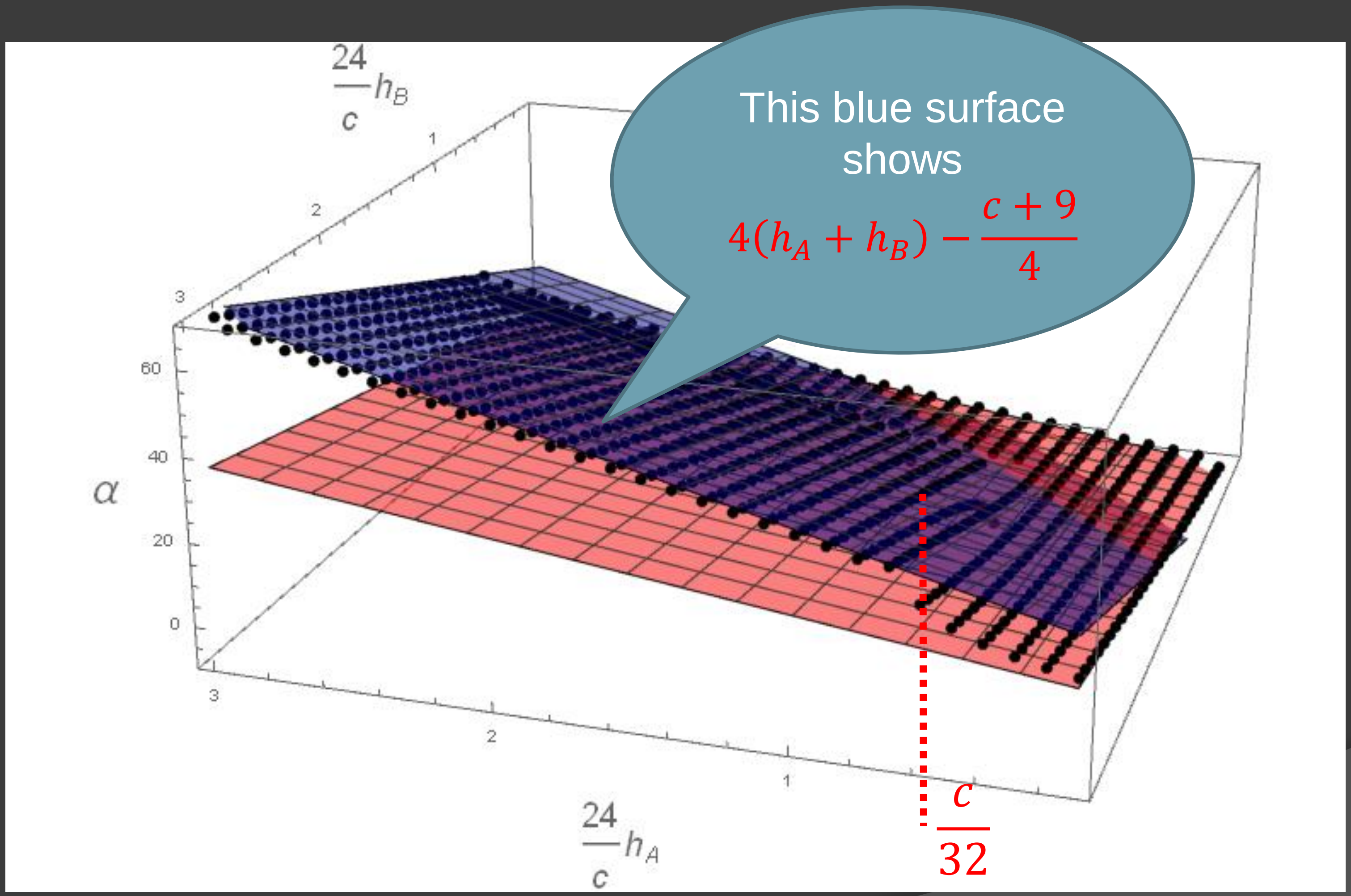


Method – numerical results

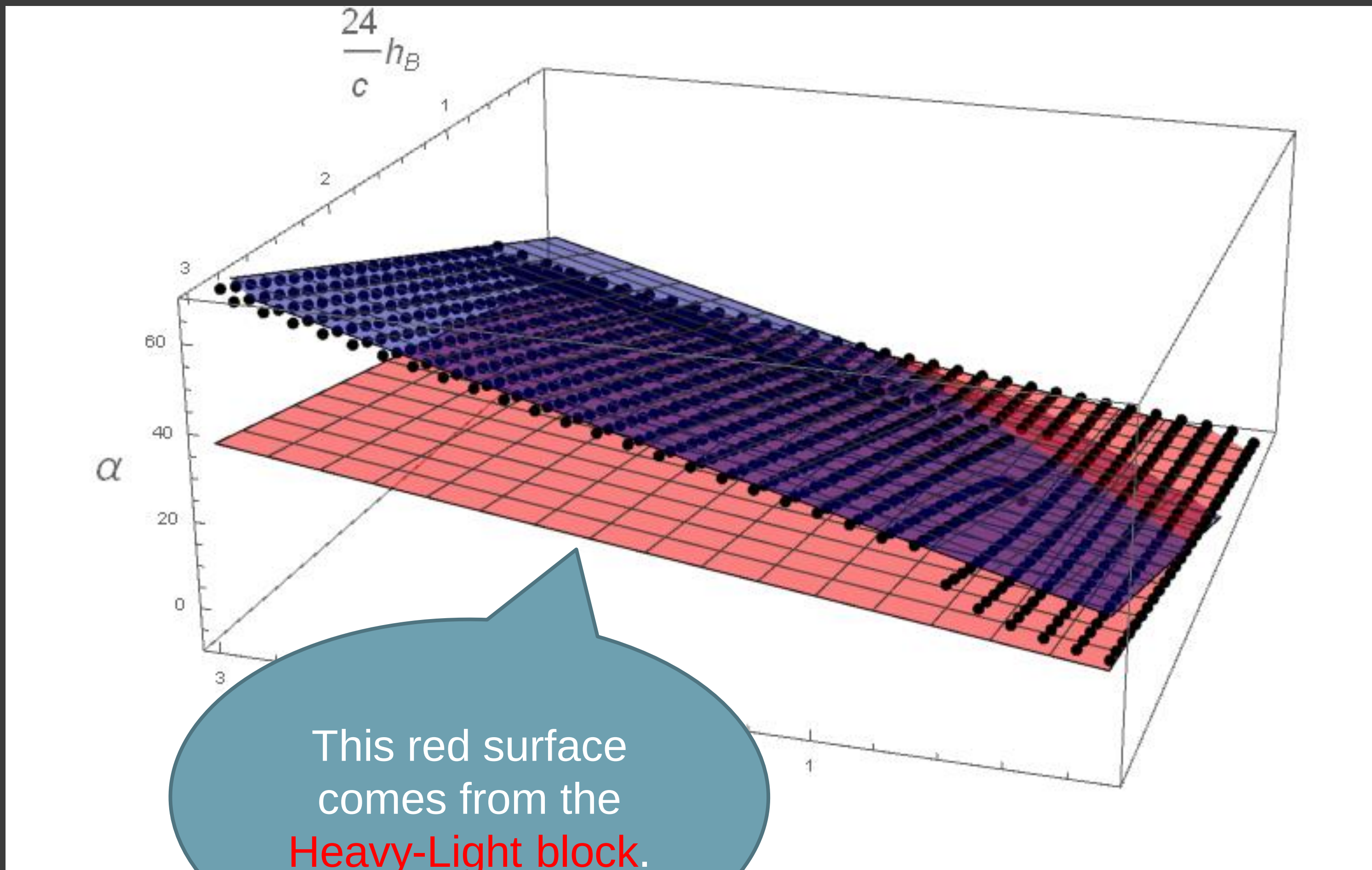


These blue surfaces come from the Heavy-Light block.

Method – numerical results



Method – numerical results



This red surface comes from the Heavy-Light block.

Method – comparing to the HHLL block

- Approximation at $z = 1 - \epsilon \ \ (\epsilon \ll 1)$

$$\sum_{n=0}^{\infty} c_n \, q^{2n} = \sum_{n=0}^{\infty} n^{\alpha} e^{A\sqrt{n}} e^{2\pi i n \tau(z)}$$

$$\simeq (\log \epsilon)^{-2\alpha - \frac{3}{2}} \, \epsilon^{-\frac{A^2}{8\pi^2}}$$

On the other hand, the HHLL block leads to

$$H(q) \simeq (\log \epsilon)^{\frac{c-1}{4} - 4(h_H + h_L)} \epsilon^{-h_L \left(1 - \sqrt{1 - \frac{24}{c} h_H}\right)}$$

Comparing these approximation forms leads to

Method – summary

$$|c_n| \sim n^{\alpha} \exp\left(A\, n^{\beta}\right)$$

h_A, h_B	A	α	β
$h_A, h_B > \frac{c}{32}$	0	$4(h_A + h_B) - \frac{c+9}{4}$	——
$h_A > \frac{c}{32}, h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1 - \frac{48}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < \frac{c}{32}, h_B > \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1 - \frac{48}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1 - \frac{24}{c}h_A\sqrt{1 - \frac{24}{c}h_B} - \frac{24}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_B < h_A < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1 - \frac{24}{c}h_B\sqrt{1 - \frac{24}{c}h_A} - \frac{24}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$

Method – summary

$$|c_n| \sim n^{\alpha} \exp(A$$

Comparing the approximation forms at $q = i$ in the same way.

h_A, h_B	$\boldsymbol{A}$		$\boldsymbol{\beta}$
$h_A, h_B > \frac{c}{32}$	0	$4(h_A + h_B) - \frac{c+9}{4}$	―
$h_A > \frac{c}{32}, h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1 - \frac{48}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < \frac{c}{32}, h_B > \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1 - \frac{48}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1 - \frac{24}{c}h_A\sqrt{1 - \frac{24}{c}h_B} - \frac{24}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_B < h_A < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1 - \frac{24}{c}h_B\sqrt{1 - \frac{24}{c}h_A} - \frac{24}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$

Method – summary

$|c_n| \sim n^{\alpha} \exp(A\, n$

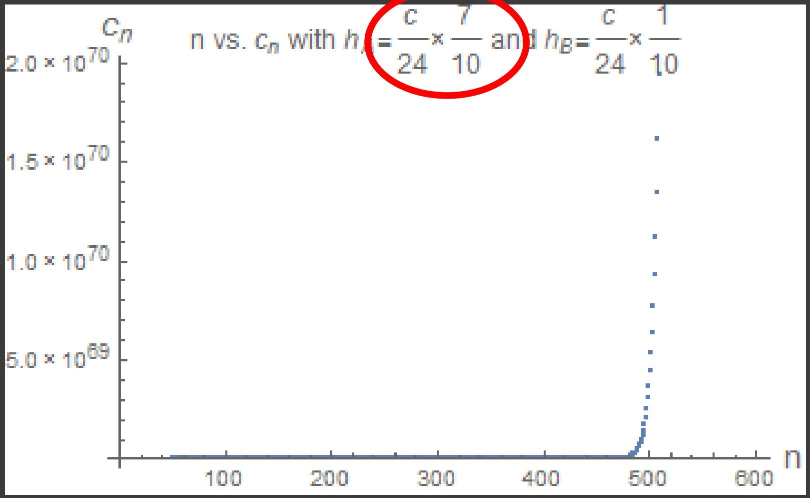
We can’t explain these fits from the known results.

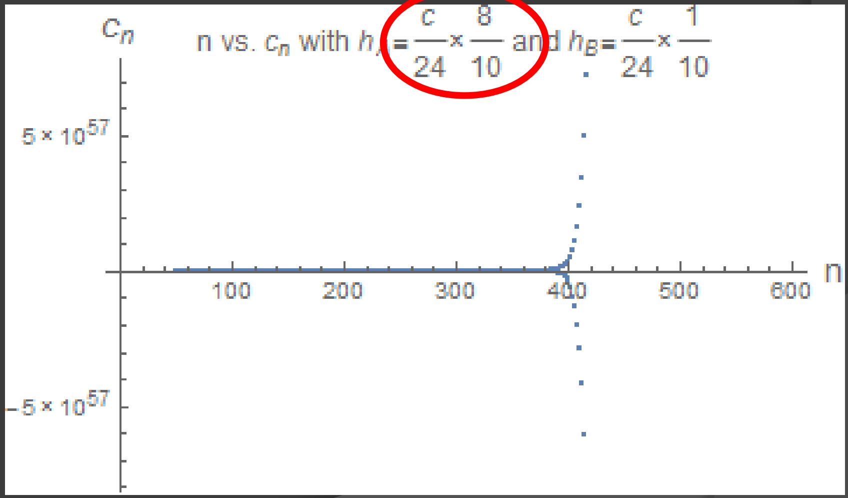
h_A, h_B	A	α	β
$h_A, h_B > \frac{c}{32}$	0	$4(h_A + h_B) - \frac{c+9}{4}$	——
$h_A > \frac{c}{32}, h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1-\frac{48}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < \frac{c}{32}, h_B > \frac{c}{32}$	$\pi\sqrt{\frac{c}{12}}\sqrt{1-\frac{48}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_A < h_B < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1-\frac{24}{c}h_A\sqrt{1-\frac{24}{c}h_B}-\frac{24}{c}h_B}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$
$h_B < h_A < \frac{c}{32}$	$\pi\sqrt{\frac{c}{3}}\sqrt{1-\frac{24}{c}h_B\sqrt{1-\frac{24}{c}h_A}-\frac{24}{c}h_A}$	$2(h_A + h_B) - \frac{c+5}{8}$	$\frac{1}{2}$

Method – numerical results

h_A, h_B	$\mathbf{sign}(c_n)$
$h_A, h_B > \frac{c}{32}$ or $h_A, h_B < \frac{c}{32}$	$+$
otherwise	$(-1)^n$

$$\frac{c}{24}\frac{7}{10} < \frac{c}{32} < \frac{c}{24}\frac{8}{10}$$





Method – numerical results

h_A, h_B	$\mathbf{sign}(\boldsymbol{c_n})$
$h_A, h_B > \frac{c}{32}$ or $h_A, h_B < \frac{c}{32}$	$+$
otherwise	$(-1)^n$

This sign pattern is consistent with

$$c_n = \frac{1}{n!}\left[\frac{c}{2}\left(1-\frac{32}{c}h_A\right)\left(1-\frac{32}{c}h_B\right)\right]^n + O(c^{n-1})$$

where $n \ll c$. (We don't give a proof but checked it by *Mathematica*.)

And our numerical results confirm that this sign pattern is saved for $n \gg c$.

Result – Renyi entropy

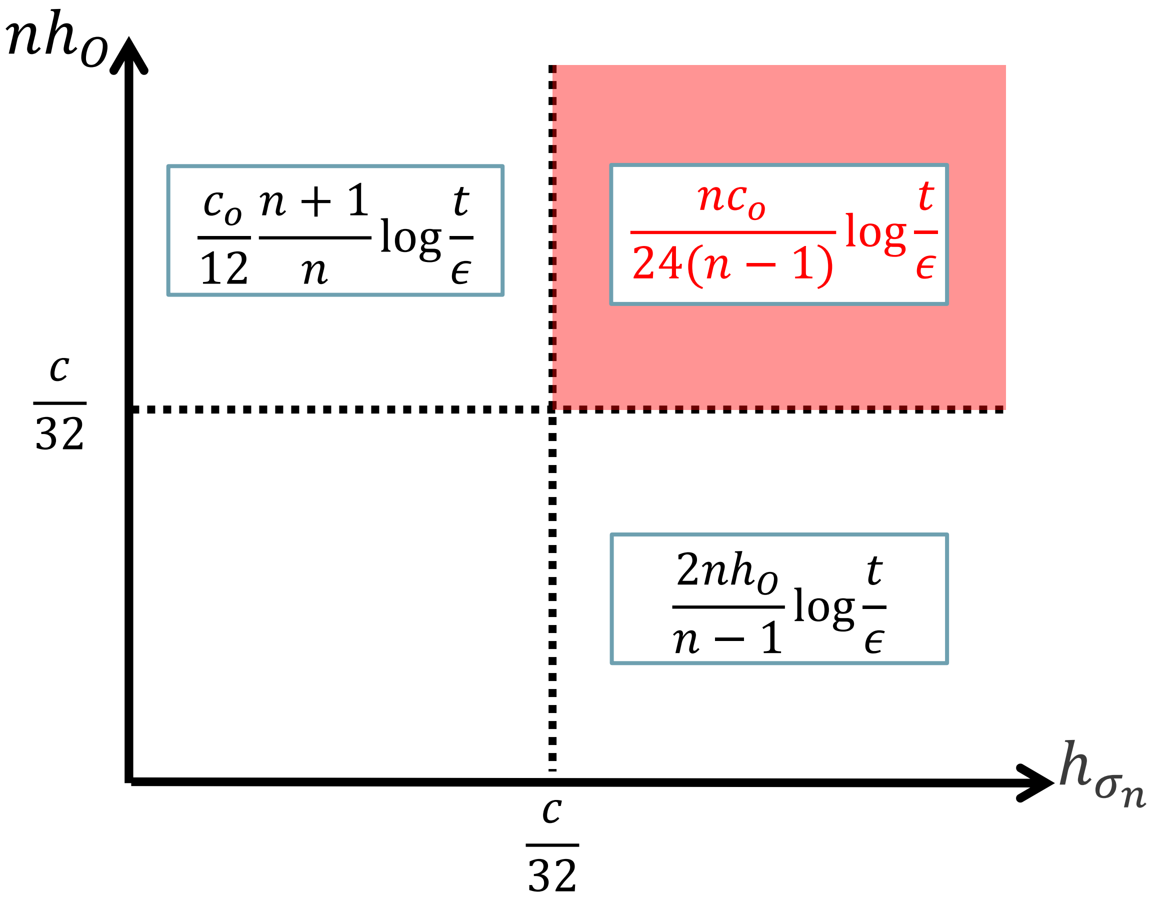
$$\Delta S_A^{(n)} = \frac{1}{1-n} \log\left(|z(t)|^{4\, h_{\sigma_n}} \left| F_{h_{O^n}\, h_{O^n}}^{h_{\sigma_n} h_{\sigma_n}}(0|z(t)) \right|^2 \right)$$

Cation:

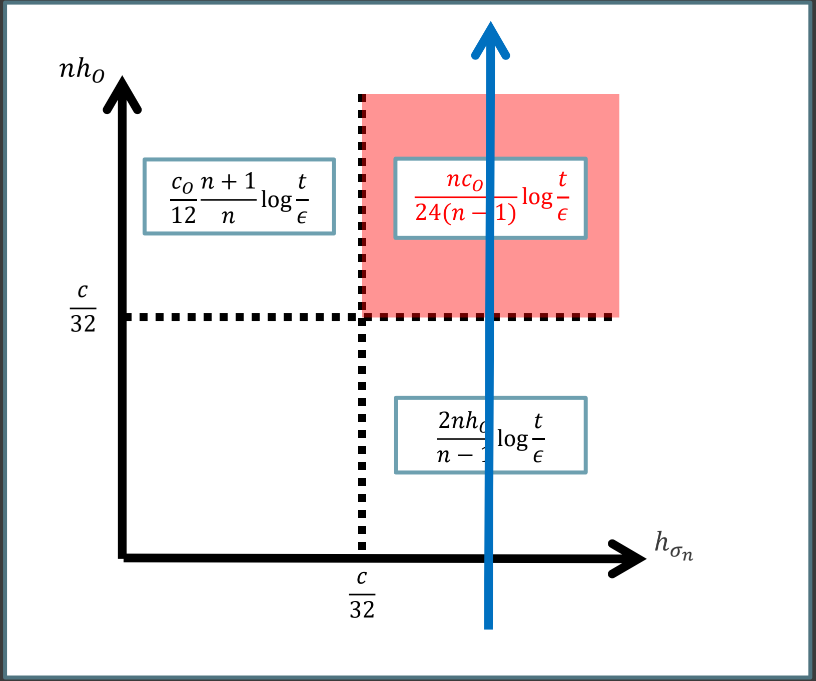
If we consider the excited REE on the CFT with central charge c, then the CFT where the above block defined on has the central charge nc. In the following, we use c as the later and we describe the former as c_o (original central charge), in that

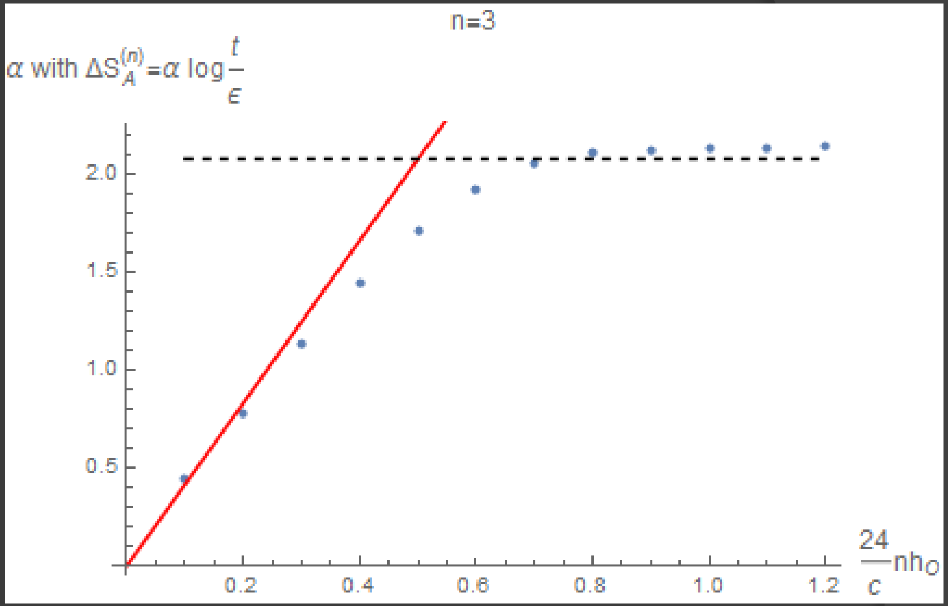
$$c = nc_o$$

Result – Renyi entropy



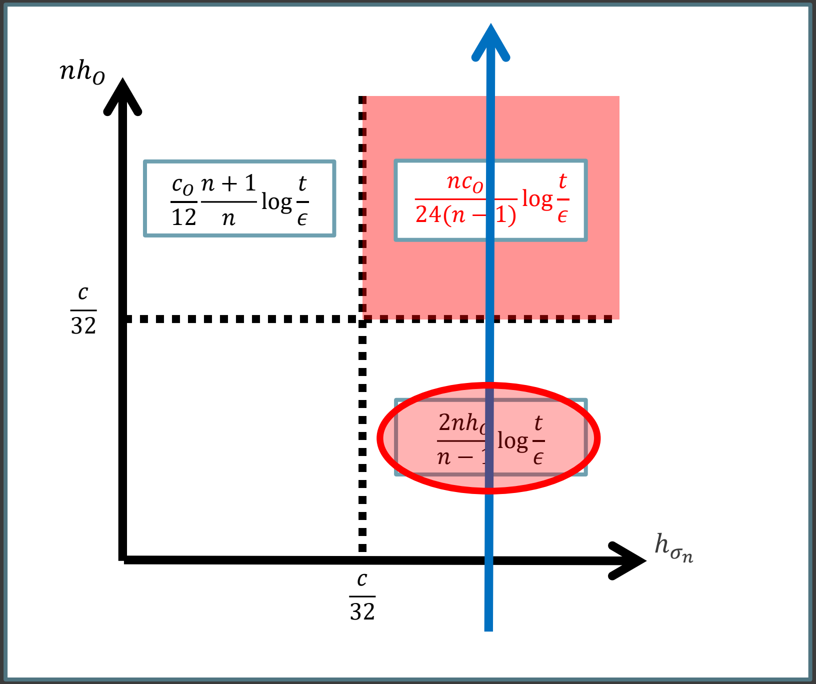
Result – Renyi entropy

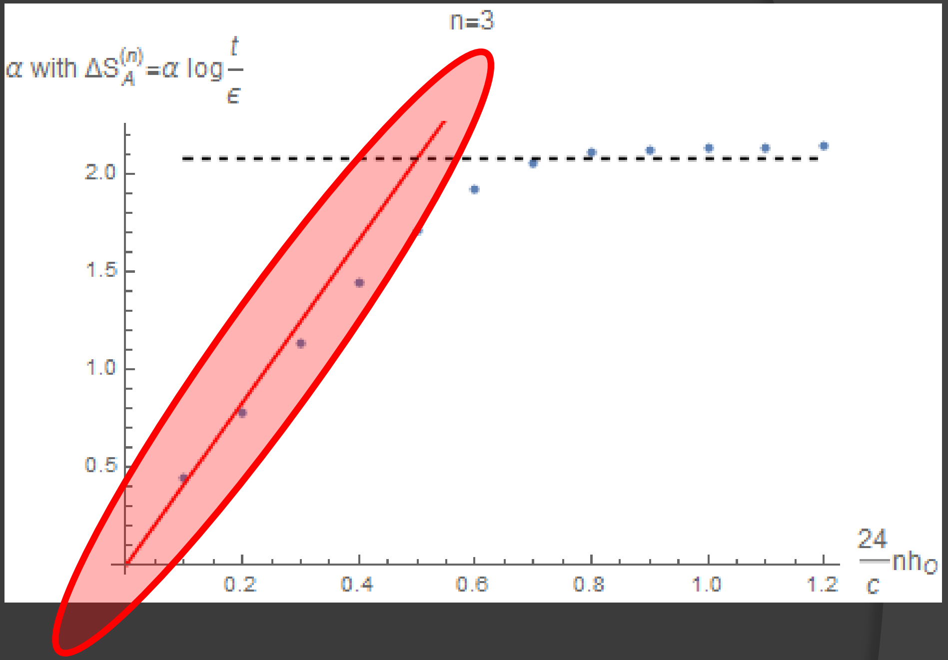




The right figure shows the h_O dependence along the blue line on the left.
This plot is derived directly from the numerical conformal block provided by using the recursion formula.

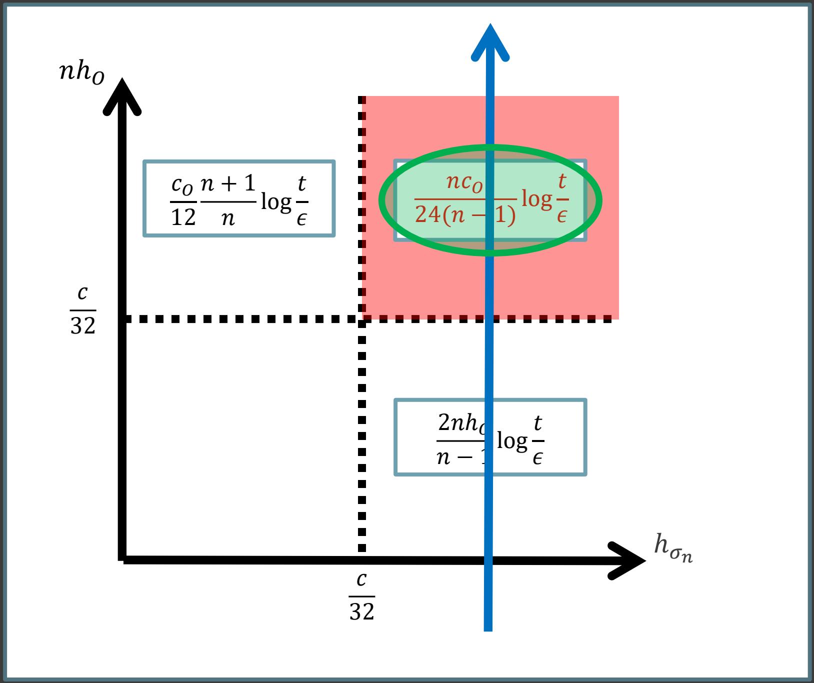
Result – Renyi entropy

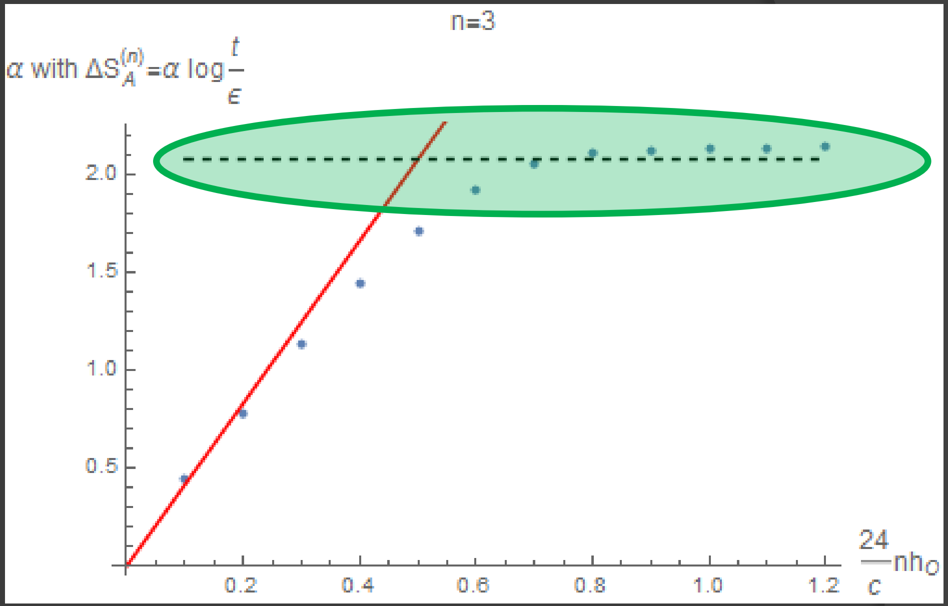




The right figure shows the h_O dependence along the blue line on the left.
This plot is derived directly from the numerical conformal block provided by using the recursion formula.

Result – Renyi entropy

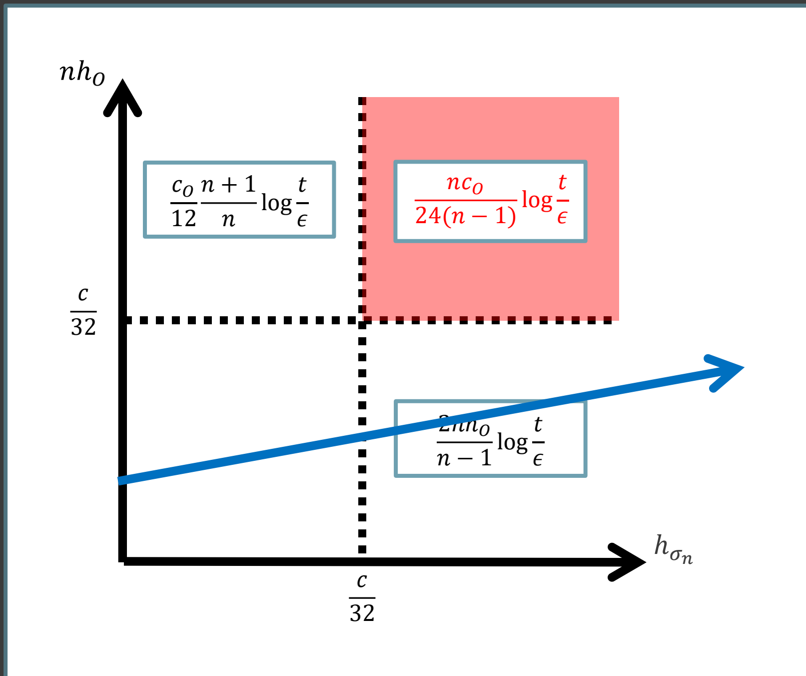


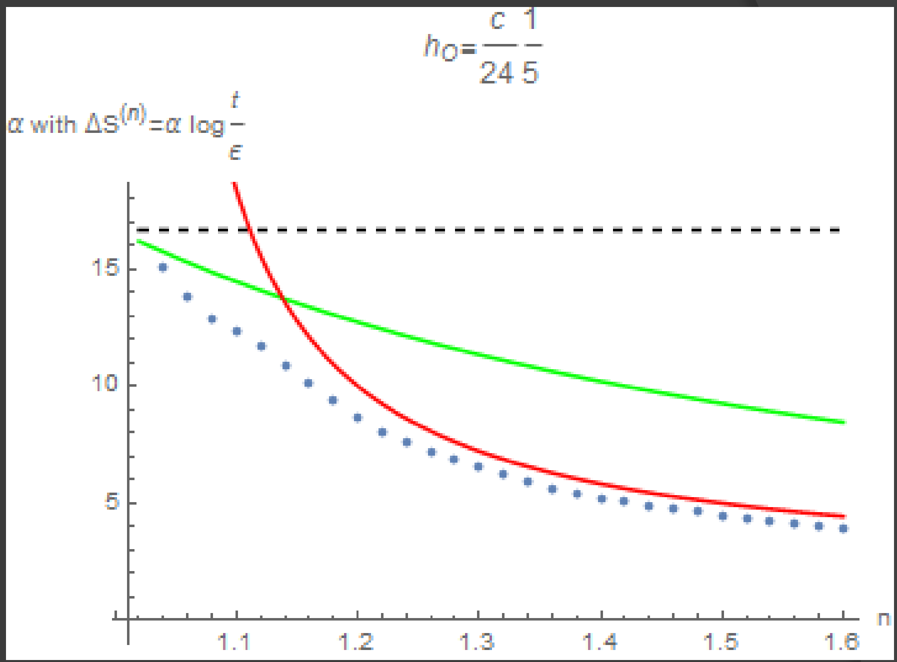


For a heavy operator, the Renyi entropy in holographic CFTs is given by

$$\frac{nc_O}{24(n-1)}\log\frac{t}{\epsilon}$$

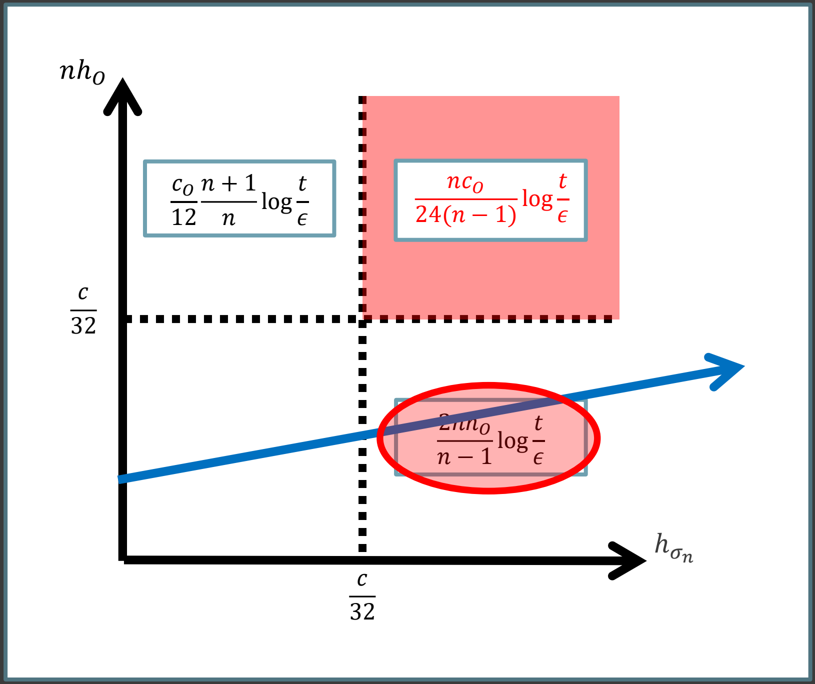
Result – Renyi entropy

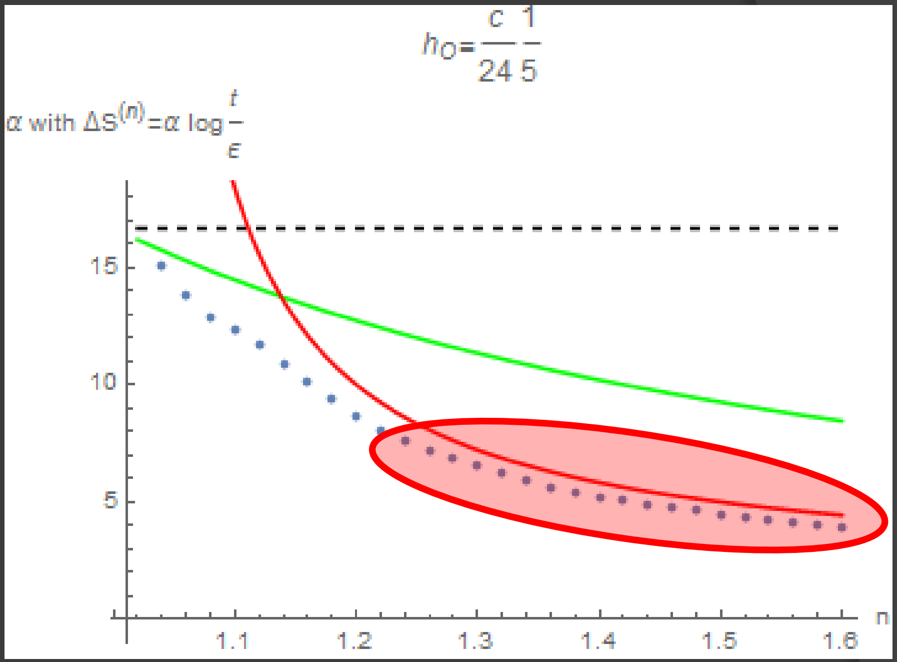




The right figure shows the n dependence along the blue line on the left.
This plot is derived directly from the numerical conformal block provided by using the recursion formula.

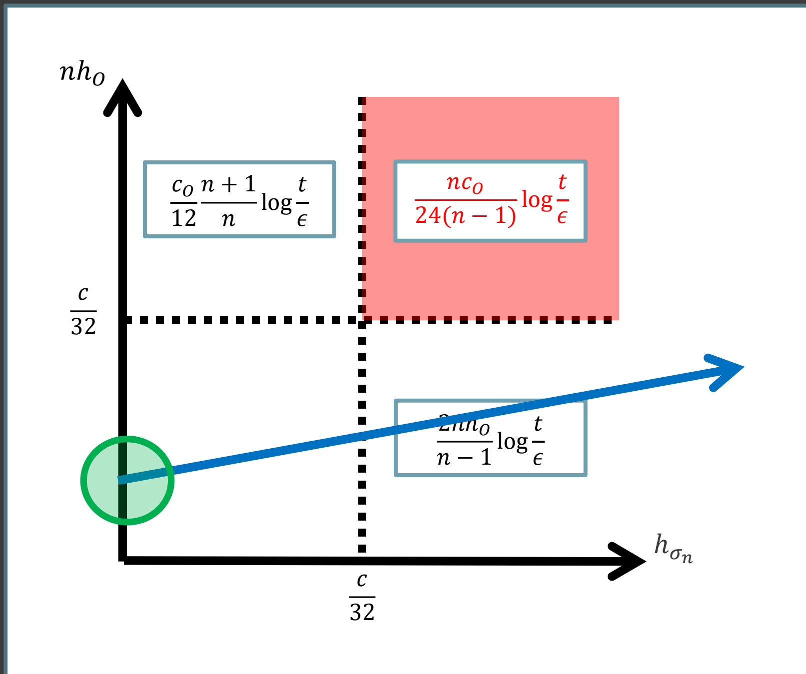
Result – Renyi entropy

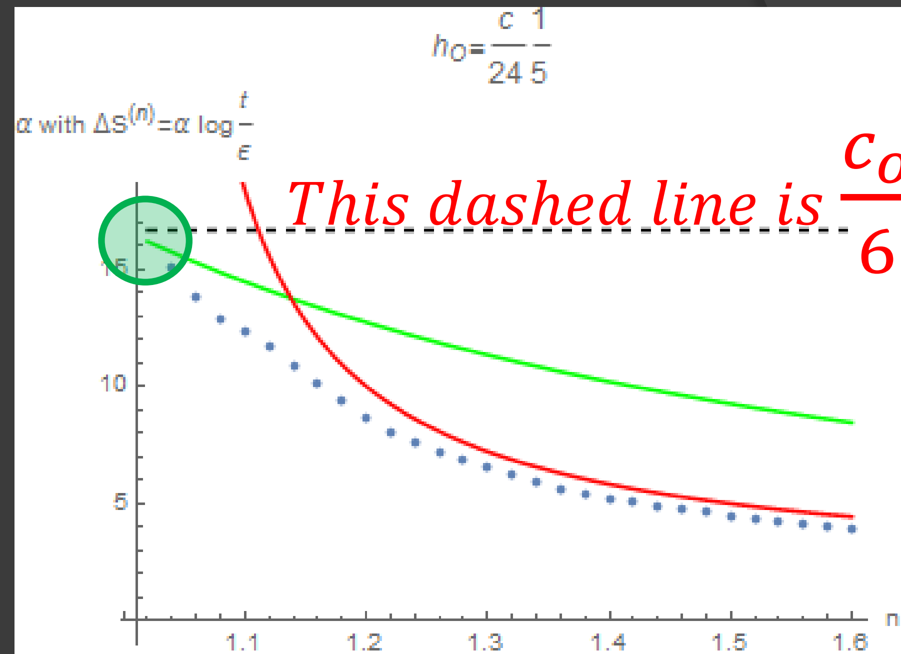




The right figure shows the n dependence along the blue line on the left.
This plot is derived directly from the numerical conformal block provided by using the recursion formula.

Result – Renyi entropy

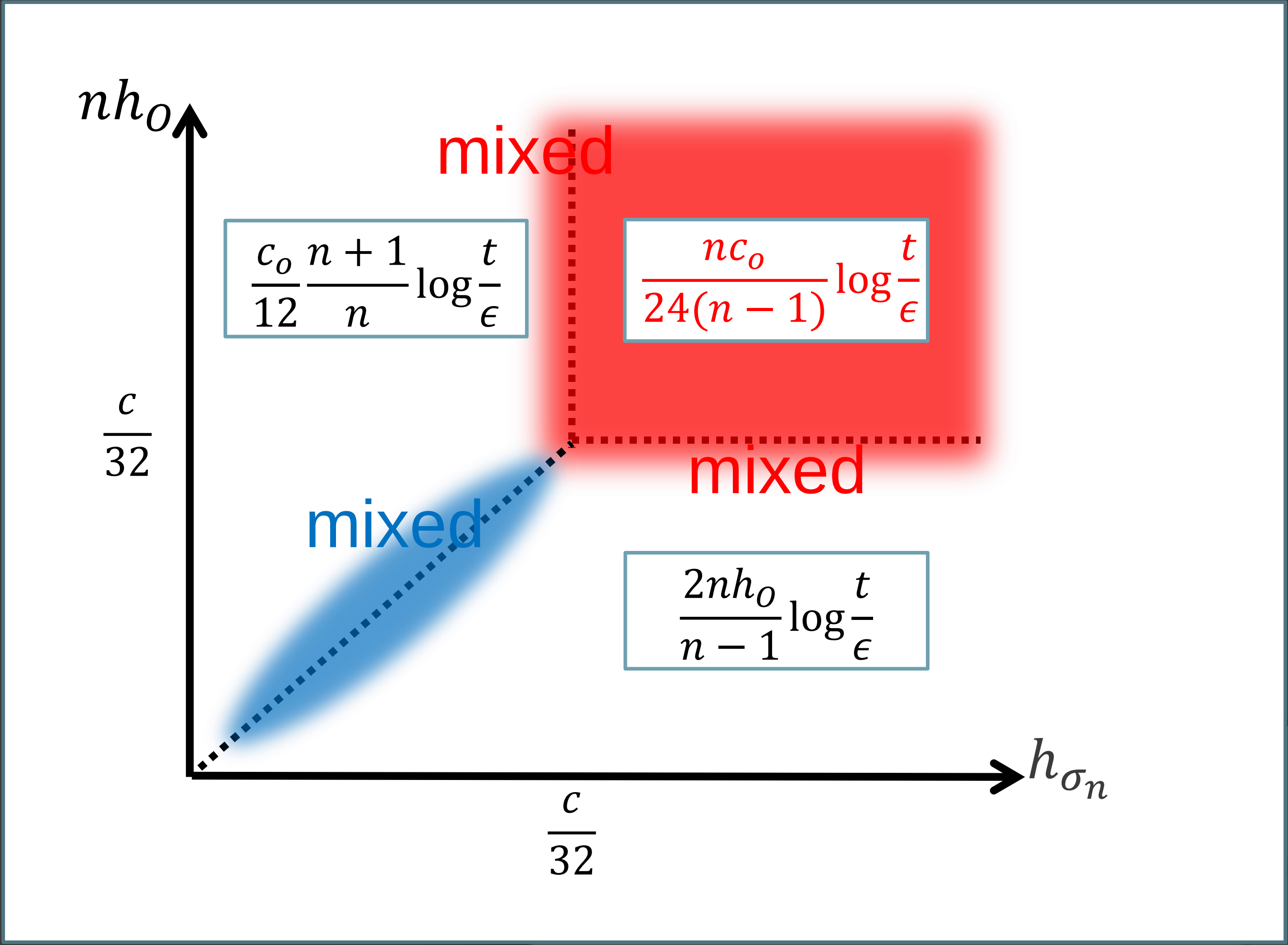




As well as a heavy op., for a light op. the entanglement entropy in holographic CFTs is given by

$$\frac{c_O}{6}\log\frac{t}{\epsilon}$$

Result – Renyi entropy



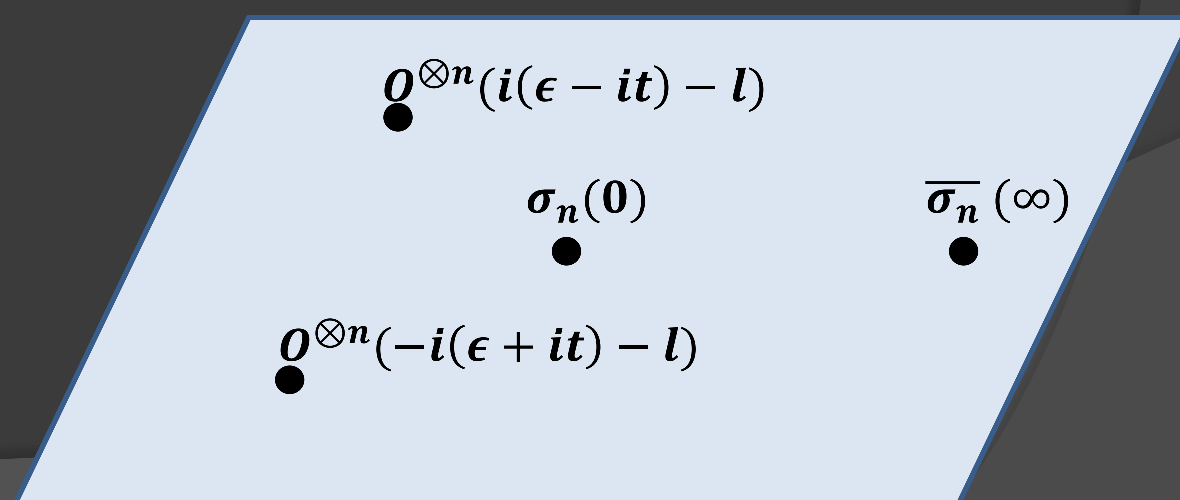
Summary

- We conjectured the formula for general conformal blocks in holographic CFTs
- By using it, we evaluated the excited Renyi entropy for any operators.
- We hope to prove our conjectures analytically.
- It's interesting to interpret the value $\frac{c}{32}$ physically.

Appendix

The Renyi entropy after a local quench can be expressed by

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \frac{\langle O^n O^n \sigma_n \bar{\sigma}_n \rangle}{\langle O^n O^n \rangle \langle \sigma_n \bar{\sigma}_n \rangle}$$



Appendix

The Renyi entropy after a local quench can be expressed by

$$\Delta S_A^{(n)} = \frac{1}{1-n}\log\frac{\langle O^n O^n \sigma_n \bar{\sigma}_n\rangle}{\langle O^n O^n\rangle\langle\sigma_n\bar{\sigma}_n\rangle}$$

By conformal map,

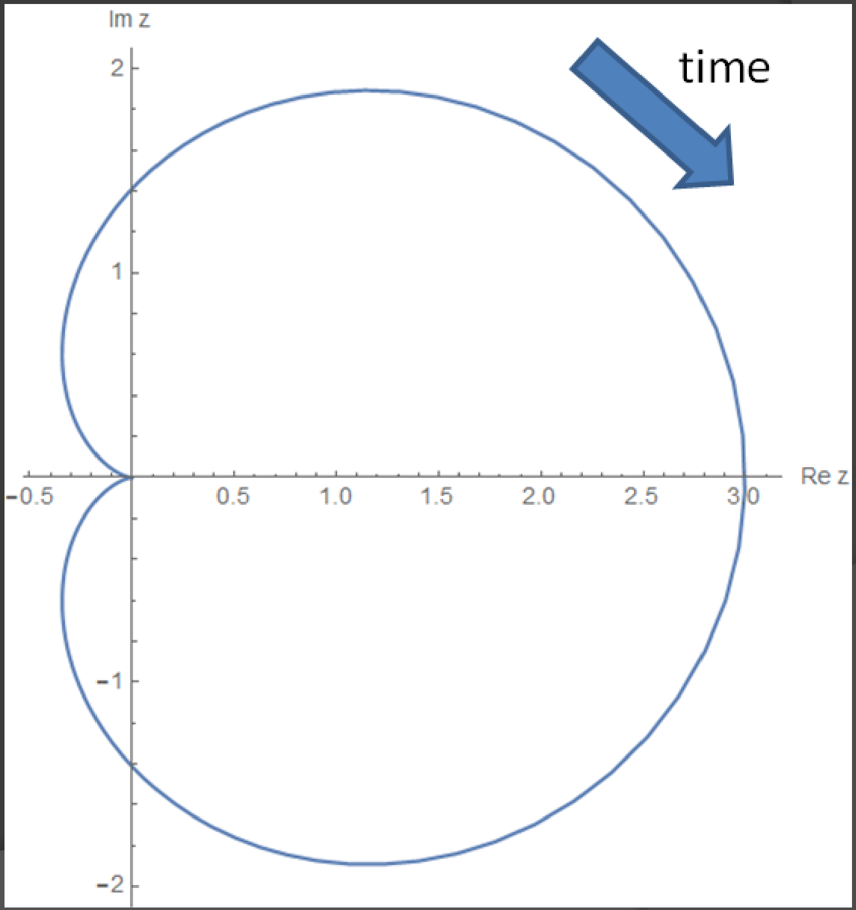
$$\frac{\langle O^n O^n \sigma_n \bar{\sigma}_n\rangle}{\langle O^n O^n\rangle\langle\sigma_n\bar{\sigma}_n\rangle} = \left|z^{2h_{\sigma_n}}\right|^2 \langle O^n(\infty) O^n(1) \sigma_n(z) \bar{\sigma}_n(0)\rangle$$

Appendix

The cross ratio is given by

$$z = \frac{2i\epsilon}{l - t + i\epsilon}, \bar{z} = \frac{-2i\epsilon}{l + t - i\epsilon}$$

The time dependence of the holomorphic part z moves along the blue Line on the right figure.

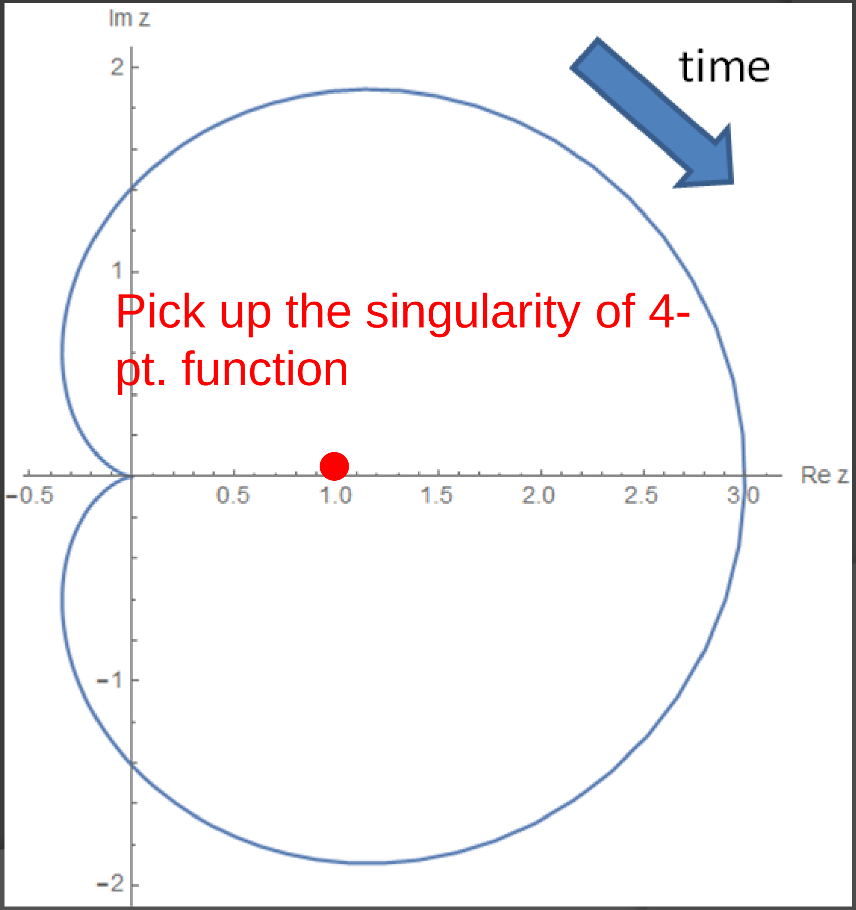


Appendix

The cross ratio is given by

$$z = \frac{2i\epsilon}{l - t + i\epsilon}, \bar{z} = \frac{-2i\epsilon}{l + t - i\epsilon}$$

The time dependence of the holomorphic part z moves along the blue Line on the right figure.



Appendix

If we describe the function $f(z)$ picking up the monodromy at $z = 1$ as f_{mono}(z), the excited Renyi entropy at late time can be re-expressed as

$$\Delta S_A^{(n)} = \frac{1}{1-n} \log \left(\left| z^{2h_{\sigma_n}} \right|^2 \langle O^n(\infty) O^n(1) \sigma_n(z) \bar{\sigma}_n(0) \rangle \right)$$

$$\rightarrow \frac{1}{1-n} \log \left(|z(t)|^{4\,h_{\sigma_n}} \left| F^{h_{\sigma_n} h_{\sigma_n}}_{h_{O^n}\, h_{O^n}}{}_{mono} (0|z(t)) \right|^2 \right)$$