SYK model, Coadjoint orbits and Liouville bulk dual

Gautam Mandal

YITP, Kyoto University,

"Holography and Quantum Dynamics", 11 November 2017

w/ P. Nayak and S. Wadia (arXiv:1702.04266)
w/ A. Gaikwad, L. Joshi and S. Wadia (1712......)
w/S. Das, P. Nayak, R. Poojary, N. Suryanarayana and S. Wadia (in progress)

SYK model

1D stat mech model with Majorana fermions and disorder

Sachdev-Ye, Kitaev

$$\sum_\tau \left(\sum_i \psi_i(\tau)\psi_i(\tau+a) + \sum_{i<j<k<l} J_{ijkl}\, \psi_i(\tau)\psi_j(\tau)\psi_k(\tau)\psi_l(\tau) \right)$$

$$\langle J_{ijkl} J_{ijkl} \rangle \sim J^2/N^3$$

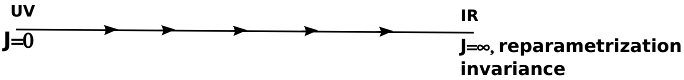
At large N, important observable is $G(\tau,\tau') = \sum_i \langle \psi_i(\tau)\psi_i(\tau') \rangle$

Schwinger-Dyson equations:

$$\frac{1}{G(i\omega)} = i\omega - \Sigma(i\omega), \quad \Sigma(\tau,\tau') = J^2 G(\tau,\tau')^3$$

SYK literature: *Polchinski-Rosenhaus, Maldacena-Stanford, Gross-Rosenhaus, Stanford, Berkooz-P.Narayan-Rozali-Simon, Verlinde, Polchinski-Shenker-...*

Variants (without disorder): *Gurau, Witten, Klebanov-Tarnopolsky*



At strong coupling, the SD equations exhibit reparametrization invariance under $\tau \to f(\tau)$ (Diff symmetry)

$$J^2 \int d\tau' G(\tau,\tau') G(\tau',\tau'')^3 = -\delta(\tau - \tau'') \tag{1}$$

where $G(\tau,\tau')$, behaves as a tensor of weight $\Delta = 1/4$:

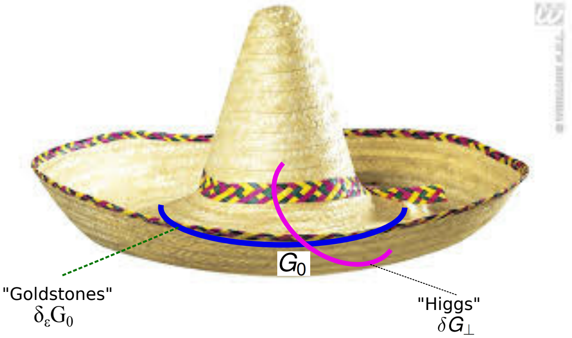
$$G \to G^f : G^f(f(\tau_1), f(\tau_2)) \left(f'(\tau_1) f'(\tau_2)\right)^{\Delta} = G(\tau_1,\tau_2)$$

$$\partial_\epsilon G(\tau,\tau') = [\epsilon(\tau)\partial_\tau + \epsilon(\tau)\partial_{\tau'} + \Delta(\partial_\tau \epsilon + \partial_{\tau'}\epsilon)] G$$

Solution of (1):

$$G(\tau,\tau') = G_0(\tau-\tau'),\ \ G_0(\tau) \sim (J|\tau|)^{-1/2}\mathrm{sgn}\tau$$

spontaneously breaks Diff (except an SL(2) subgroup), leading to ‘Goldstones’ of Diff/SL(2):



$$G = G_0 + \delta_{SL2}G + \delta G_{||} + \delta G_\perp$$

$$\delta G_{||} = \delta_\epsilon G_0$$

Explicit breaking of Diff

- The Goldstone modes have zero action! (contrast with $\int \partial_\mu \pi^2$ for pion physics).

Explicit breaking of Diff

- The Goldstone modes have zero action! (contrast with $\int \partial_\mu \pi^2$ for pion physics).
- The partition function diverges. To get finite physics, we turn on small $1/J$. This explicitly breaks Diff symmetry.

Explicit breaking of Diff

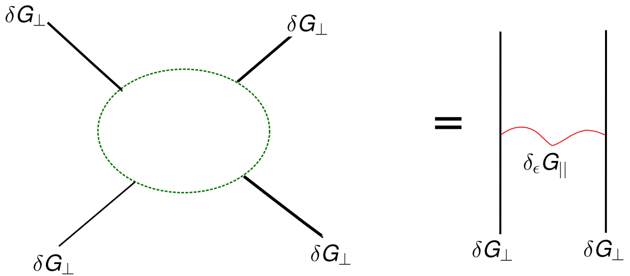
- The Goldstone modes have zero action! (contrast with $\int \partial_\mu \pi^2$ for pion physics).
- The partition function diverges. To get finite physics, we turn on small $1/J$. This explicitly breaks Diff symmetry.
- $G_0 \to G_0 + O(1/J) \Rightarrow$ Goldstones now pick up a finite action (equivalent to a "pion mass" term) given by the Schwarzian form

$$S \sim -\frac{N}{J} \int d\tau \{f,\tau\},\ \{f,\tau\} = \frac{f'''}{f'} - \frac{3}{2}\left(\frac{f''}{f'}\right)^2$$

At finite temperatures,

$$S \sim \frac{N}{J} \int d\tau \left[\left(\frac{f''}{f'}\right)^2 - \left(\frac{2\pi}{\beta}\right)^2 (f')^2\right]$$

Chaos in SYK



Thermalization, Chaos (Out-of-time-ordered)

maximal $\lambda_L = 2\pi/\beta$

Features of SYK

Universal low energy sector

$$S \sim -\frac{N}{J} \int d\tau \{f, \tau\}$$

Chaos (Liapunov exponent saturates the gravity bound)

$$\lambda = \frac{2\pi}{\beta}$$

Regge spectrum (different properties from O(N) or $N = 4$ SYM)

Zero temperature entropy

$$S_0 = N \left(\frac{1}{2} \log 2 + \text{model-dependent (function of } q) \right)$$

...

Towards a bulk dual

A 2D bulk dual that captures the universal low energy sector (and hopefully more)

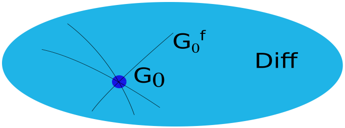
- In order to have non-trivial dynamics, need to explicitly break conformal invariance ($=$ Diff). this is done in the SYK model by turning on a small $1/J$. For a bulk representation of this:

- embed AdS_2 as the near-horizon geometry of a higher dimensional near-extremal black hole. Effective theory= dilaton gravity in 2D (Jackiw-Teitelboim). *Almheiri-Polchinski, Maldacena-Stanford-Yang*

Our approach

Our approach:
Low energy configurations of SYK are generated by $f(\tau) \in \text{Diff}$

$$G_0 \to G_0^f : G^f(f(\tau_1), f(\tau_2)) \left(f'(\tau_1) f'(\tau_2)\right)^{\Delta} = G_0(\tau_1, \tau_2)$$



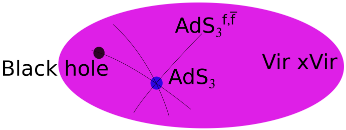
Diff-orbit of G_0

This is a space of zero modes (all flat directions). A potential is created by explicit breaking of Diff (a “pion mass” term):
$S = -\frac{N}{J} \int d\tau \{f, \tau\}$

We would like to find a bulk dual of this picture.

A bit of history: $CFT_2 \leftrightarrow AdS_3$

The Brown-Henneaux geometries



Vir x Vir-orbit of AdS_3

Here $f(z), \bar{f}(\bar{z})$ comprise 2D conformal transformations.
The metrics (in Euclidean signature) are explicitly given by

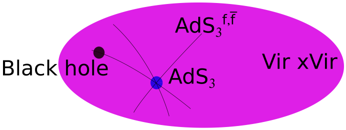
Banados 1999, see GM-Sinha-Sorokhoibam for eternal BTZ

$$ds^2 = \frac{d\zeta^2 + dz\, d\bar{z}}{\zeta^2} + L(z)dz^2 + \bar{L}(\bar{z})d\bar{z}^2 + \zeta^2 L(z)\bar{L}(\bar{z})dzd\bar{z}$$

where $L, \bar{L}$ are the hologrphic boundary stress tensors given by the Schwarzians $L(z) = \{f, z\},\ \bar{L}(\bar{z}) = \{\bar{f}, \bar{z}\}$

A bit of history: $CFT_2 \leftrightarrow AdS_3$

The Brown-Henneaux geometries



Vir x Vir-orbit of AdS_3

Here $f(z), \bar{f}(\bar{z})$ comprise 2D conformal transformations. The metrics (in Euclidean signature) are explicitly given by

Banados 1999, see GM-Sinha-Sorokhoibam for eternal BTZ

$$ds^2 = \frac{d\zeta^2 + dz\, d\bar{z}}{\zeta^2} + L(z)dz^2 + \bar{L}(\bar{z})d\bar{z}^2 + \zeta^2 L(z)\bar{L}(\bar{z})dzd\bar{z}$$

where $L, \bar{L}$ are the hologrphic boundary stress tensors given by the Schwarzians $L(z) = \{f, z\},\ \bar{L}(\bar{z}) = \{\bar{f}, \bar{z}\}$

Note that the orbit of AdS_3(Poincare) includes black holes (BTZ).

Brown-Henneaux metrics as Coadjoint orbits

According to Banados, the Vir $\times$ Vir transformations correspond to $SL_2 \times SL_2$ (large) gauge transformations in the Chern Simons formulation of 3D gravity with negative cosmological constant.

Brown-Henneaux metrics as Coadjoint orbits

According to Banados, the Vir $\times$ Vir transformations correspond to $SL_2 \times SL_2$ (large) gauge transformations in the Chern Simons formulation of 3D gravity with negative cosmological constant.

• The space of Brown-Henneaux geometries, therefore, can be regarded as $SL_2 \times SL_2$ orbits (in fact, they are coadjoint orbits of the corresponding 'loop groups').

Brown-Henneaux metrics as Coadjoint orbits

According to Banados, the Vir × Vir transformations correspond to $SL_2 \times SL_2$ (large) gauge transformations in the Chern Simons formulation of 3D gravity with negative cosmological constant.

• The space of Brown-Henneaux geometries, therefore, can be regarded as $SL_2 \times SL_2$ orbits (in fact, they are coadjoint orbits of the corresponding 'loop groups').

• It is an independent fact that coadjoint orbits admit a natural symplectic form (Kirillov). For coadjoint orbits of loop groups $\tilde{G}$, the action principle that follows gives the WZW action, based on the group G! *Rai and Rogers*

Brown-Henneaux metrics as Coadjoint orbits

According to Banados, the Vir $\times$ Vir transformations correspond to $SL_2 \times SL_2$ (large) gauge transformations in the Chern Simons formulation of 3D gravity with negative cosmological constant.

• The space of Brown-Henneaux geometries, therefore, can be regarded as $SL_2 \times SL_2$ orbits (in fact, they are coadjoint orbits of the corresponding 'loop groups').

• It is an independent fact that coadjoint orbits admit a natural symplectic form (Kirillov). For coadjoint orbits of loop groups $\tilde{G}$, the action principle that follows gives the WZW action, based on the group G! *Rai and Rogers*

• This, therefore, says that WZW action should describe the dynamics in the space of Brown-Henneau geometries.

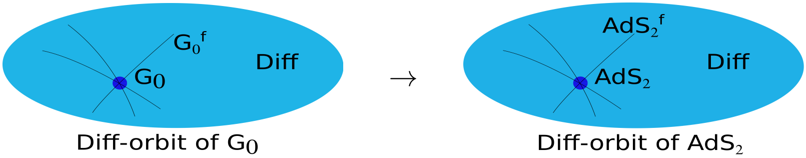
This is already anticipated in Witten's work: Chern-Simons $\leftrightarrow$ WZW (this predates AdS/CFT by a decade).

Coajoint orbits of AdS_2

We have been able to construct the following coadjoint orbit of AdS_2 under Diff, given by the following exact metrics

GM-Nayak-Wadia 1702.04266

$$ds^2 \equiv \hat{g}^f_{\alpha\beta} dx^\alpha dx^\beta = \frac{1}{4\pi\mu\,\zeta^2}\left(d\zeta^2 + d\tau^2\left(1 - \zeta^2 \frac{\{f(\tau),\tau\}}{2}\right)^2\right)$$



Properties:

- All metrics represent ‘normalizable deformations’, are in the Fefferman-Graham gauge and are obtained by pulling back a Diff $f(\tau)$ from boundary to bulk by the exact map

$$\tilde{\tau} = f(\tau) - \frac{2\zeta^2 f''(\tau) f'(\tau)^2}{4f'(\tau)^2 + \zeta^2 f''(\tau)^2}, \quad \tilde{\zeta} = \frac{4\zeta f'(\tau)^3}{4f'(\tau)^2 + \zeta^2 f''(\tau)^2}$$

- There are ‘horizons’ in the new geometry, determined by

$$g_{\tau\tau} = \left(1 - \zeta^2 \frac{\{f(\tau), \tau\}}{2}\right)^2 = 0$$

The coadjoint orbit (Kirillov) action

What is the action in the space of these metrics?

The coadjoint orbit (Kirillov) action

What is the action in the space of these metrics?

- Since the Diff orbit is a coadjoint orbit, a natural action is given by the Kirillov action.

The coadjoint orbit (Kirillov) action

What is the action in the space of these metrics?

• Since the Diff orbit is a coadjoint orbit, a natural action is given by the Kirillov action.

• It is known for some time *Polyakov 1987, Rai-Rogers, Alexeev-Shatashvili* that Kirillov action in the space of 2D metrics is the induced gravity action of Polyakov

$$S[g] = \frac{1}{16\pi G_N}\int_M \sqrt{-g}\left(R\,\frac{1}{\Box}R - 16\pi\mu\right) + \frac{1}{4\pi G_N}\int_{\partial M}\sqrt{-\gamma}\mathcal{K}\,\frac{1}{\Box}R$$

where we have added a boundary term similar to the Gibbons-Hawking term.

The Liouville action

Choose the conformal gauge $g_{\alpha\beta} = e^{2\phi}\hat{g}_{\alpha\beta}$

$$S = -\frac{1}{4\pi\, G_N}\left[\int_M \sqrt{-\hat{g}}(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}) + \frac{1}{16\pi\, G_N}\int_M \sqrt{-\hat{g}}\hat{R}\,\frac{1}{\hat{\Box}}\hat{R}\right.$$
$$\left. + 2\int_{\partial M}\sqrt{-\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial M}\sqrt{-\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi\right]$$

The codjoint orbit metrics $\hat{g}_{\alpha\beta} = \hat{g}^f_{\alpha\beta}$, with $\phi = 0$, are classical solutions of the above action.

The Liouville action

Choose the conformal gauge $g_{\alpha\beta} = e^{2\phi}\hat{g}_{\alpha\beta}$

$$S = -\frac{1}{4\pi\ G_N}\left[\int_M \sqrt{-\hat{g}}(\hat{g}^{\alpha\beta}\partial_\alpha\phi\partial_\beta\phi + \hat{R}\phi + 4\pi\mu e^{2\phi}) + \frac{1}{16\pi\ G_N}\int_M \sqrt{-\hat{g}}\hat{R}\,\frac{1}{\hat{\Box}}\hat{R}\right.$$
$$\left. + 2\int_{\partial M}\sqrt{-\hat{\gamma}}\hat{\mathcal{K}}\phi + \int_{\partial M}\sqrt{-\hat{\gamma}}\hat{n}^\mu\phi\partial_\mu\phi\right]$$

The codjoint orbit metrics $\hat{g}_{\alpha\beta} = \hat{g}^f_{\alpha\beta}$, with $\phi = 0$, are classical solutions of the above action.

• To compute the potential in the space of these coadjoint orbits (Goldstones), we must compute the on-shell action. To do this, we need a regulator $\zeta = \delta$. The on-shell action, up to $O(\delta)$ terms, coincides with that of AdS_2. This reproduces the SYK result that the Goldstones have zero action.

Liouville solution and Explicit symmetry breaking

• Around the AdS_2 background, the Liouville field ϕ satisfies the well-known equation of motion:

$$2\widehat{\Box}\phi = \widehat{R} + 8\pi\mu e^{2\phi}$$

together with the Virasoro constraints

$$\partial_z^2\phi - (\partial_z\phi)^2 + 2\frac{\partial_z\phi}{z+\bar{z}} = 0, \quad \{z \leftrightarrow \bar{z}\}$$

• It turns out that the solution set gets fixed by just three real constants (which parameterize a hyperboloid) $\phi = \frac{a+b\tau+c\tau^2}{\zeta} + ...$
The terms ... are small near the boundary and are not important for our purpose here.

• Out of the parameters a, b, c, only a corresponds to the physics of the SYK model. We will identify $a \sim 1/J$. For consistency, note that (i) a has dimension of length, and J has dimension of mass; (ii) a represents a non-normalizable mode (as it deforms the asymptotically AdS_2 geometry) and $1/J$ represents an irrelevant deformation from the IR CFT.

With the above solution for the Liouville field, we get

$$S_{\text{on-shell}} = \frac{a}{2G_N} \int d\tau \{f, \tau\}$$

We thus recover the Schwarzian Goldstone action ("pion mass" term) of SYK from the Liouville bulk dual, modulo the identification

$$a \sim 1/J, \; G_N \sim N$$

up to an overall numerical factor.

Thermodynamics

- To compute the thermodynamics from the Liouville model, let us go to Euclidean AdS_2 and apply to it the coadjoint orbit transformation $f(\tau)$ which compactifies real line into a circle

$$f(\tau) = \tan \frac{\pi \tau}{\beta}$$

This gives $d\hat{s}_f^2 = \frac{1}{4\pi\zeta^2}\left[d\zeta^2 + d\tau^2\left(1 - \pi^2 \frac{\zeta^2}{\beta^2}\right)\right]$.
The geometries are capped: $g_{\tau\tau}$ vanishes at $\zeta = \beta/\pi$.

Thermodynamics

- To compute the thermodynamics from the Liouville model, let us go to Euclidean AdS_2 and apply to it the coadjoint orbit transformation $f(\tau)$ which compactifies real line into a circle

$$f(\tau) = \tan \frac{\pi\tau}{\beta}$$

This gives $d\hat{s}_f^2 = \frac{1}{4\pi\zeta^2}\left[d\zeta^2 + d\tau^2\left(1 - \pi^2\frac{\zeta^2}{\beta^2}\right)\right]$.
The geometries are capped: $g_{\tau\tau}$ vanishes at $\zeta = \beta/\pi$.

- The on-shell action for this geometry is (modulo some important subtleties of counterterms which we have not completely understood)

$$S_{\mathrm{on-shell}} = \log Z = \beta F = \frac{1}{G_N}\left[\log(4\pi) - 3/2 + \frac{a}{2G_N\beta} + O(a^2)\right]$$

which shows the same features as SYK.

Charged model

In stead of Majorana, consider Dirac fermions

$$\sum_{\tau}\left(\sum_{i}\psi_i^\dagger(\tau)\psi_i(\tau+a)+\sum_{i<j<k<l}J_{ijkl}\,\psi_i^\dagger(\tau)\psi_j^\dagger(\tau)\psi_k(\tau)\psi_l(\tau)\right)$$

Dyson-Schwinger equations

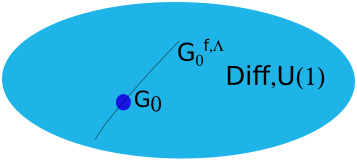
$$\frac{1}{G(i\omega)}=i\omega+\mu-\Sigma(i\omega),\quad \Sigma(\tau)=J^2G(\tau)^2G(-\tau)$$

Davison-Fu-Georges-Gu-Jensen-Sachdev 2016

Diff-U(1) symmetry

At strong coupling, there is now an emergent (Diff,U(1)) symmetry. $G \to G^{f,\Lambda}$:

$$G^{f,\Lambda}(f(\tau_1), f(\tau_2)) \left(f'(\tau_1) f'(\tau_2)\right)^{\Delta} = G(\tau_1, \tau_2) \exp[i\Lambda(\tau_2) - i\Lambda(\tau_1)]$$



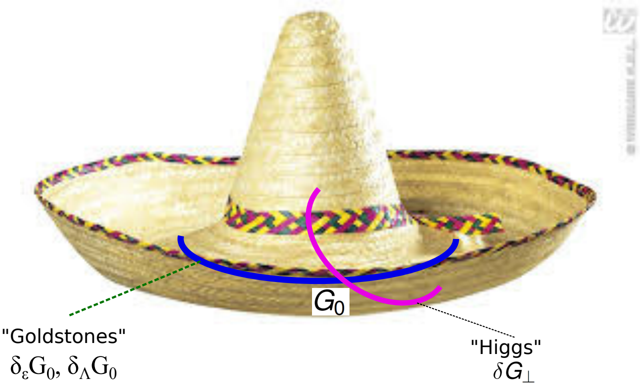
Diff,U(1)-orbit of G_0

$$\partial_\epsilon G(\tau, \tau') = [\epsilon(\tau)\partial_\tau + \epsilon(\tau)\partial_{\tau'} + \Delta(\partial_\tau \epsilon + \partial_{\tau'} \epsilon)]G, \quad \delta f = \epsilon$$

$$\partial_\Lambda G(\tau, \tau') = i[\Lambda(\tau') - \Lambda(\tau)]G(\tau, \tau'$$

pseudo-Goldstones

Vacuum solution breaks the Diff-U(1) symmetry.

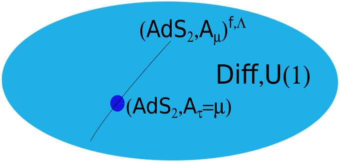


Action for the pseudo-Goldstones (generalized Schawrzian)

$$\frac{S}{N} = -\frac{\gamma}{4\pi^2}\int d\tau \{\tau+\epsilon(\tau),\tau\} + \frac{K}{2}\int d\tau\,[\partial_\tau\Lambda + \mu\partial_\tau\epsilon]^2$$

Symmetries

The bulk configurations should be described by Diff-U(1) orbits of AdS(2) with $A_\tau = \mu$:



Diff,U(1)-orbit of AdS_2

where, as before, the Diff and U(1) transformations should be 'large gauge transformations' which are nontrivial at the boundary.

Gaikwad-Joshi-GM-Wadia

Bulk action

A natural choice of the dynamics which gives rise to such orbits is a U(1) gauge theory coupled to Polyakov (Liouville) gravity

$$S_{gravity,gauge} = S_{Polyakov}[g_{\mu\nu}] + S_2$$

where

$$S_2 = \frac{k}{G_N}\left[\int_M [2\,\chi\,F_{\tau\zeta}]\,d\tau d\zeta + \int_{\partial M} A_\tau \chi\,d\tau\; - \;\frac{1}{2}\int_{\partial M}\sqrt{\gamma}\,(\gamma^{\tau\tau}A_\tau A_\tau - \chi^2)\,d\tau\right]$$

which can be obtained from a KK reduction of 3D Chern-Simons theory. The boundary term in S_2 can be obtained either from the 3D boundary term or independently by demanding a consistent variational principle.

Bulk action

A natural choice of the dynamics which gives rise to such orbits is a U(1) gauge theory coupled to Polyakov (Liouville) gravity

$$S_{gravity,gauge} = S_{Polyakov}[g_{\mu\nu}] + S_2$$

where

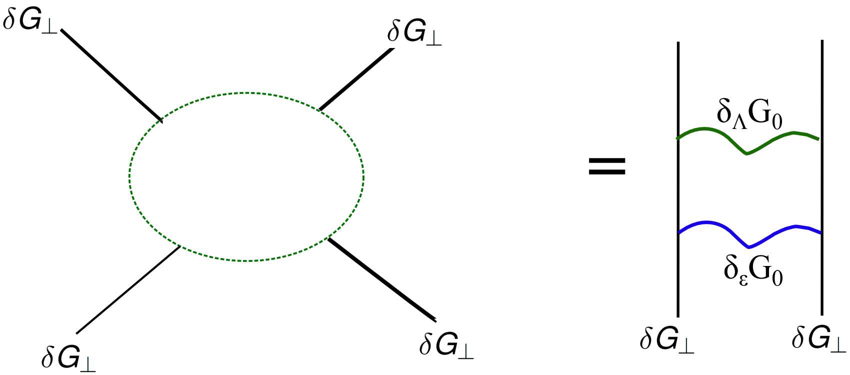
$$S_2 = \frac{k}{G_N}\left[\int_M [2\,\chi\, F_{\tau\zeta}]\, d\tau d\zeta + \int_{\partial M} A_\tau \chi\, d\tau\; - \;\frac{1}{2}\int_{\partial M}\sqrt{\gamma}\,(\gamma^{\tau\tau} A_\tau A_\tau - \chi^2)\, d\tau\right]$$

which can be obtained from a KK reduction of 3D Chern-Simons theory. The boundary term in S_2 can be obtained either from the 3D boundary term or independently by demanding a consistent variational principle.

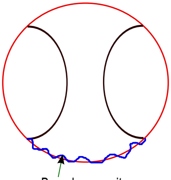
The on-shell value of S_2 precisely gives the charge contribution to the pseudo-Goldstone action

$$N\frac{K}{2}\int d\tau\, [\partial_\tau \Lambda + \mu \partial_\tau \epsilon]^2$$

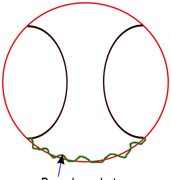
Chaos in charged SYK



Chaos in charged bulk



Action $\sim \int d\tau \;\; \epsilon \left(\partial_\tau^4 - (2\pi/\beta)^2 \partial_\tau^2\right) \epsilon, \Rightarrow e^{2\pi t/\beta}, \lambda_L = 2\pi/\beta$



Action $\sim \int d\tau \;\; \Lambda(\partial_\tau^2)\Lambda, \Rightarrow \lambda_L = 0$

Pure photon sector does not have any quantum chaos.

Conclusion

- Identified coadjoint orbits of AdS_2 under Diff as the space of Goldstones of SYK.

Conclusion

- Identified coadjoint orbits of AdS_2 under Diff as the space of Goldstones of SYK.
- Identified 2D Liouville action as a coadjoint orbit action which governs these Goldstones.

Conclusion

- Identified coadjoint orbits of AdS_2 under Diff as the space of Goldstones of SYK.
- Identified 2D Liouville action as a coadjoint orbit action which governs these Goldstones.
- The on-shell action of AdS_2+ (Diff transformations) is the same as that of AdS_2 (they differ by a regulator term which vanishes in the continuum limit).

Conclusion

- Identified coadjoint orbits of AdS_2 under Diff as the space of Goldstones of SYK.
- Identified 2D Liouville action as a coadjoint orbit action which governs these Goldstones.
- The on-shell action of AdS_2+ (Diff transformations) is the same as that of AdS_2 (they differ by a regulator term which vanishes in the continuum limit).
- AdS_2+ Diff + Non-normalizable Liouville mode generate a non-zero on-shell gravity action. This coincides with the Schwarzian of SYK.

Conclusion

- Identified coadjoint orbits of AdS_2 under Diff as the space of Goldstones of SYK.
- Identified 2D Liouville action as a coadjoint orbit action which governs these Goldstones.
- The on-shell action of AdS_2+ (Diff transformations) is the same as that of AdS_2 (they differ by a regulator term which vanishes in the continuum limit).
- AdS_2+ Diff + Non-normalizable Liouville mode generate a non-zero on-shell gravity action. This coincides with the Schwarzian of SYK.
- This technique reproduces the features of the low temperature thermodynamics of SYK from the Liouville bulk dual.

- We have been able to generalize the bulk dual to the case of charged SYK models.

- We have been able to generalize the bulk dual to the case of charged SYK models.

Follow-up:

- How does one understand the hard modes in the dual geometry?

- We have been able to generalize the bulk dual to the case of charged SYK models.

Follow-up:

- How does one understand the hard modes in the dual geometry?
- Simple model: Need to couple matter to Liouville to obtain non-trivial excitations. *See also Gross-Rosenhaus*

• We have been able to generalize the bulk dual to the case of charged SYK models.

Follow-up:

• How does one understand the hard modes in the dual geometry?

• Simple model: Need to couple matter to Liouville to obtain non-trivial excitations. *See also Gross-Rosenhaus*

• More comprehensive: hard modes have an approximately linear spectrum. Could come from a dimensional reduction *See Das-Jevicki-Suzuki arXiv:1704.07208, Das-Nayak-Poojary-Suryanarayana-GM-Wadia*