Non-linear Gravity from Entanglement in Conformal Field Theories

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Based on

- T. Faulkner, F. Haehl, E. Hijano, OP, C. Rabideau & M. Van Raamsdonk, arXiv:1705.03026 [hep-th].
- T. Faulkner, F. Haehl, E. Hijano, OP, C. Rabideau & M. Van Raamsdonk, to appear.

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- In AdS/CFT, the entanglement structure of states with classical gravity duals is very special, as explained by the Ryu-Takayanagi formula [Ryu & Takayanagi '06].
- Conversely, constraints satisfied by the entanglement entropy in such states have been shown to give rise to bulk dynamics. [Van Ramsdonk '10, Takayanagi et al '12, Lashkari et al '13, Faulkner et al '13.]

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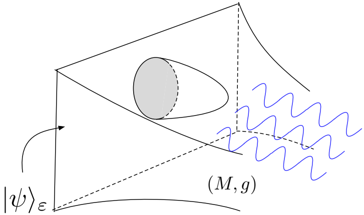
- It was shown that if (M, g) is an AAdS spacetime which correctly computes the entanglement entropies of $|\psi\rangle_\varepsilon$ for all ball-shaped regions in the CFT up to first order in ε via RT, then (M, g) satisfies the *linearized* Einstein equation.



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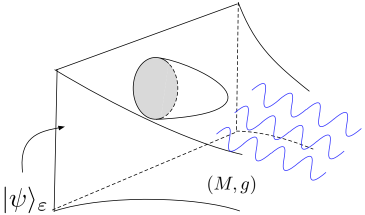


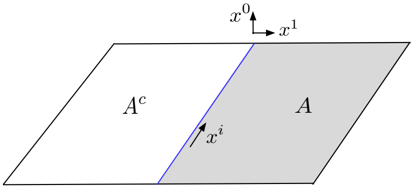
- This simply follows from the first law of entanglement in the CFT. (We will review the argument shortly.)

Objective

The main objective of today's talk will be to show that the analogous statement is also true at second order in state deformations: if (M, g) is an AAdS spacetime which correctly computes the entanglement entropies of $|\psi\rangle_\varepsilon$ for all ball-shaped regions in the CFT up to order ε^2 via RT, then (M, g) satisfies the second order Einstein equation.

Warm-up: Half-space Entanglement in Relativistic QFTs

- Let's take A to be a half-space. The Hilbert space splits as $\mathfrak{h}_\Sigma = \mathfrak{h}_A \otimes \mathfrak{h}_{A^c}$.



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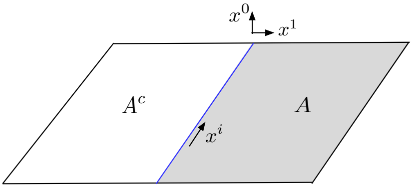

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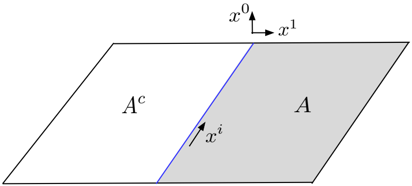

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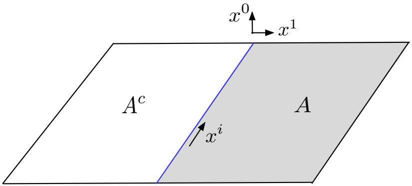

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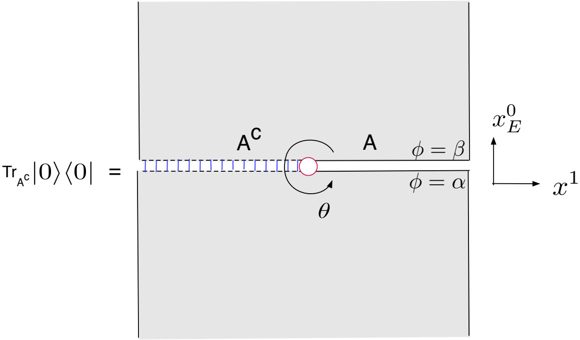
- The entanglement entropy is the Von-Neumann entropy of ρ_0:

$$S_{EE} = -\mathrm{Tr}_A\, \rho_0 \ln \rho_0.$$

Warm-up: Half-space Entanglement...

- Recall that the half-space reduced density matrix can be concretely obtained by the Euclidean path integral on $\mathbb{R}^d$ with a cut along A.



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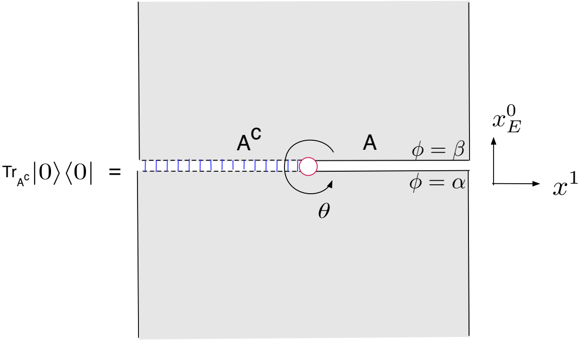

$$\langle\alpha|\rho_0|\beta\rangle = \int_\beta^\alpha [D\phi] e^{-S_{CFT}[\phi]}$$

- Note that there is a $U(1)$ isometry, i.e. rotations in the angular direction θ:

$$\xi = \partial_\theta.$$

Warm-up: Half-space Entanglement...

- The modular Hamiltonian can be obtained from slicing the path-integral along the angular direction θ. [Bisognano-Wichmann '76]

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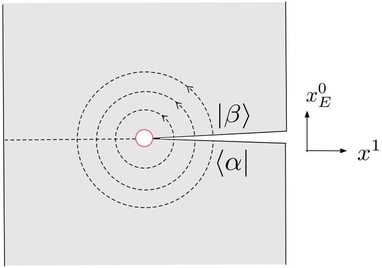

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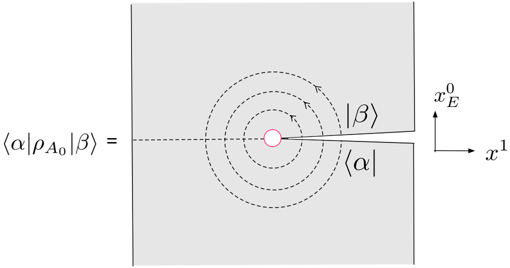

$$K_0 = \int d^{d-2}x^i \int_0^\infty dx^1\, x^1\, T_{00}(0, x^1, x^i) + c$$

- In this case, modular flow is local. In the Lorentzian description, it corresponds to boosts around the entanglement cut:

$$e^{isK_0} O(x^+, x^-, \vec{x}^i) e^{-isK_0} = O(x^+ e^s, x^- e^{-s}, \vec{x}^i).$$

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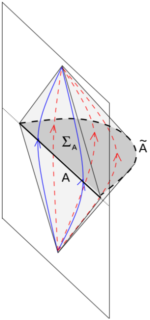
- So in the vacuum of a conformal field theory, modular flow for any ball-shaped region is also local and generated by a conformal isometry ξ [Casini-Huerta-Myers '11].
- In AdS/CFT, this means that there is an AdS isometry ξ_B which preserves the RT surface, and generates bulk modular flow.



Deforming away from the vacuum

- Now we wish to consider small deformations of the state away from the CFT vacuum:

$$|\psi\rangle_\varepsilon = |0\rangle + \varepsilon|\delta^1\psi\rangle + \frac{1}{2}\varepsilon^2|\delta^2\psi\rangle \cdots .$$

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- We can parametrize this by one-point functions if we like.
- Correspondingly, consider a general one-parameter family of geometries g_ε, with g_0 being AdS_{d+1}.
- It is convenient to pick a gauge where the extremal surface in g_ε is fixed to be the original AdS extremal-surface. This is often called the Hollands-Wald gauge [Lashkari & Van Raamsdonk '15].

A gravitational identity

- Our starting point now is the following gravitational identity [Hollands & Wald, '12]:

$$\boldsymbol{\omega}(\frac{dg_\varepsilon}{d\varepsilon}, \mathcal{L}_X g_\varepsilon) = d\boldsymbol{\chi}(\frac{dg_\varepsilon}{d\varepsilon}, X) - \mathcal{G}(g_\varepsilon, \frac{dg_\varepsilon}{d\varepsilon}, X),$$

  where the d-form

$$\boldsymbol{\omega}(\gamma^1, \gamma^2) = \frac{1}{16\pi}\boldsymbol{vol}_a^{(d)} P^{abcdef}\left(\gamma^2_{bc}\nabla_d\gamma^1_{ef} - \gamma^1_{bc}\nabla_d\gamma^2_{ef}\right)$$

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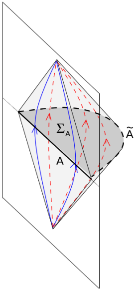
- This equation is valid for general vector fields X, but we will use it for $X = \xi_B$.
- Also, the equation is true for general ε, so we can study its consequences order by order in ε.

First order argument

- At first order (i.e. setting $\varepsilon = 0$), integrating this equation along the region Σ_A (the original entanglement wedge), we get

$$\frac{1}{4G_N}\delta^{(1)}A_{\text{ext}} - \delta^{(1)}\langle K_0\rangle = \int_{\Sigma_A} n^a \xi_B^b EOM_{ab}^{(1)}.$$



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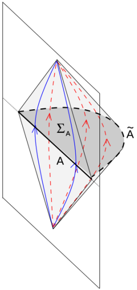

- Using the input that this geometry computes entropies using the RT formula, we can rewrite this as

$$\delta^{(1)}S_{EE} - \delta^{(1)}\langle K_0\rangle = \int_{\Sigma_A} n^a \xi_B^b EOM^{(1)}_{ab}.$$

First order argument...

However, by the first law of entanglement in the CFT, we have

$$\delta^{(1)} S_{EE} - \delta^{(1)} \langle K_0 \rangle = 0,$$

and so we conclude that

$$\int_{\Sigma_A} n^a \xi_B^b EOM_{ab}^{(1)} = 0$$

for all ball shaped regions in the CFT. It can be shown that this necessarily implies the linearized Einstein equation

$$EOM_{ab}^{(1)} = 0.$$

Second order

- At second order, the gravitational identity reads:

$$\int_{\Sigma_A} \boldsymbol{\omega}(\delta^{(1)}g, \mathcal{L}_{\xi_B}\delta^{(1)}g) = \frac{1}{4G_N}\delta^{(2)}A_{\text{ext}} - \delta^{(2)}\langle K_0\rangle - \int_{\Sigma_A} n^a \xi_B^b EOM_{ab}^{(2)},$$

where note that the LHS is non-trivial, but depends only on $\delta^{(1)}g$ (which satisfies linearized EE).

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  where note that the LHS is non-trivial, but depends only on $\delta^{(1)}g$ (which satisfies linearized EE).

- On the other hand, from the CFT side all we can say is

$$\delta^{(2)}S_{EE} - \delta^{(2)}\langle K_0\rangle = \delta^{(2)}S(\rho||\rho_0)$$

  where the RHS is the relative entropy between the excited state and the vacuum at $O(\varepsilon^2)$, and in fact only depends on $\delta^{(1)}\rho$.

Second order

So if we could prove somehow, from purely CFT arguments and the *first order* Einstein equation for $\delta^{(1)}g$, that

$$\delta^{(2)}S(\rho||\rho_0) = \int_{\Sigma_A} \boldsymbol{\omega}(\delta^{(1)}g, \mathcal{L}_{\xi_B}\delta^{(1)}g),$$

then we would deduce that the bulk geometry satisfies the Einstein equation to second order. This will be our goal in the rest of the talk.

Relative Entropy: CFT calculation

- Recall that the relative entropy is given by

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- Writing $\rho = \rho_0 + \varepsilon \delta^{(1)}\rho$, and expanding this expression out, we obtain the first non-trivial term at $O(\varepsilon^2)$ [Faulkner '14]

$$\delta^{(2)}S(\rho||\rho_0) = \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\epsilon}{2}\right)} \mathrm{Tr}\left(\rho_0 e^{isK_0} \rho_0^{-1} \delta^{(1)}\rho e^{-isK_0} \rho_0^{-1} \delta^{(1)}\rho\right)$$

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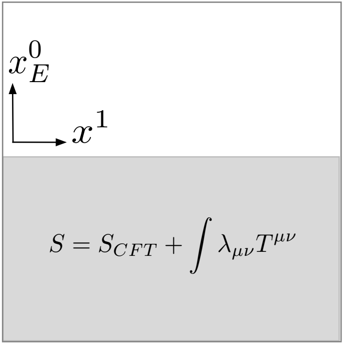
- As mentioned before, the relative entropy at this order only depends on $\delta^{(1)}\rho$.
- In order to proceed, we will focus on excited states in the CFT which can be constructed using the Euclidean path integral.

Return of the Path Integral

- The states of interest are prepared by performing the Euclidean path-integral on the lower half space with the action deformed by an $O(\varepsilon)$ source for the Stress tensor:

$$\psi_\varepsilon[\phi^{(0)}] = \int_{\phi(0,\mathbf{x})=\phi^{(0)}(\mathbf{x})} [D\phi] e^{-S_{CFT} - \int_{-\infty}^{0} dx_E^0 \int d^{d-1}\mathbf{x}\, \lambda_{\mu\nu}(x) T^{\mu\nu}(x)}.$$



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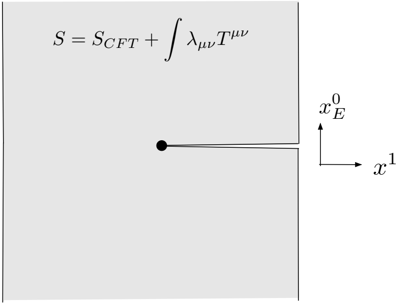

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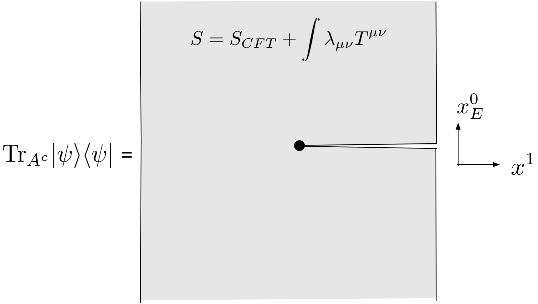

- Expanding the path-integral in $\lambda_{\mu\nu}$, we obtain

$$\rho = \rho_0 + \int d^d x \lambda_{\mu\nu}(x) \rho_0 T^{\mu\nu}(x) + O(\lambda^2).$$

Back to Relative Entropy

- Coming back to the relative entropy, we obtain

$$\delta^{(2)} S(\rho||\rho_0) = \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\epsilon}{2}\right)} \lambda_{\mu\nu}(x_a)\lambda_{\rho\sigma}(x_b)\left\langle T^{\mu\nu}(x_a^{s})T^{\rho\sigma}(x_b)\right\rangle$$

where

$$T_{\mu\nu}(x_a^s) = e^{isK_0} T_{\mu\nu}(x_a) e^{-isK_0}$$

is the modular-flowed stress tensor. This is a local operator because modular flow for ball-shaped regions is local in the CFT vacuum.

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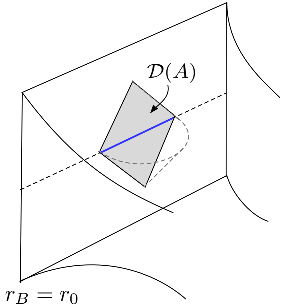
- We wish to rewrite this formula in "gravitational terms".
- This is not too hard, because only the CFT stress-tensor 2-point function appears in this calculation, which is universal in CFTs.

Rewriting in Gravitational terms

- Imagine a fiducial AdS spacetime with $a_* \equiv c_d \frac{\ell^{d-1}}{G_N} = C_T$. We claim that the stress tensor two-point function for *any* CFT with this C_T can be written as

$$\left\langle T^{\mu\nu}(x_a) T^{\rho\sigma}(x_b) \right\rangle = \int_{\mathcal{D}(A), r_B = r_0} \boldsymbol{\omega}(K_E^{\mu\nu}, K_R^{\rho\sigma})$$



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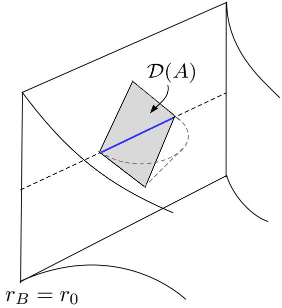
- Imagine a fiducial AdS spacetime with $a_* \equiv c_d \frac{\ell^{d-1}}{G_N} = C_T$. We claim that the stress tensor two-point function for *any* CFT with this C_T can be written as

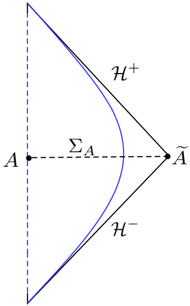
$$\left\langle T^{\mu\nu}(x_a) T^{\rho\sigma}(x_b) \right\rangle = \int_{\mathcal{D}(A), r_B = r_0} \boldsymbol{\omega}(K_E^{\mu\nu}, K_R^{\rho\sigma})$$



- Here K_E and K_R are the Euclidean & Retarded bulk-to-boundary propagators sourced at x_a and x_b respectively.

Rewriting in Gravitational terms...

- Next, we note that the symplectic flux density is a conserved quantity, so we can push it into the bulk.



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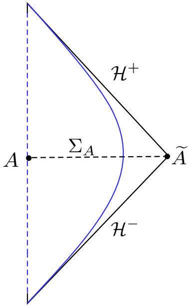

- So the relative entropy essentially has two contributions:

$$\delta^{(2)} S(\rho||\rho_0) = \int_{-\infty}^{\infty} \frac{ds}{4\sinh^2\left(\frac{s+i\epsilon}{2}\right)} \int_{\mathcal{H}^+\cup\mathcal{H}^-} \boldsymbol{\omega}(h_E, h_R^s)$$

where $h = \int \lambda K$ is the bulk-to-boundary propagator integrated against the boundary source λ.

The s integral

- Now we wish to perform the s-integral:

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- Quick and dirty way: The integral vanishes on $\mathcal{H}^-$ because K_R is the retarded propagator. On $\mathcal{H}^+$, we pick up the double pole from $\sinh^{-2}(s/2)$, which gives

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- Now pushing the symplectic flux to Σ_A, we get the required result.

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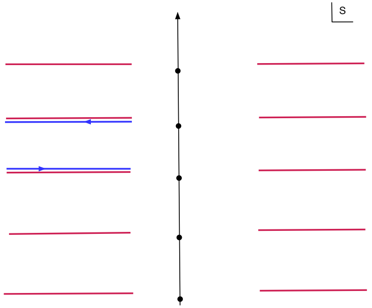
- The analytic structure of the integrand in the complex s-plane is as follows:



(where we have used $h_R \sim (h_+ - h_-)$ and the KMS condition.)

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- Actually, an important point to note here is that this analytic structure is true in a particular gauge, called the generalized de-Donder gauge:
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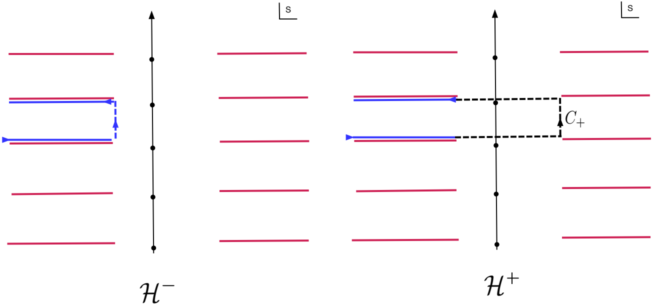
$$\nabla_a^{(0)} h^{ab} = h^a{}_a = 0.$$

- This gauge is most convenient for our calculations because of the simple analytic structure. To match on to the result in the Hollands-Wald gauge, we will have to perform a suitable gauge transformation at the end.

The s integral...

- Coming back to the s-integral, it is easiest to evaluate by completing the contours in the following way:



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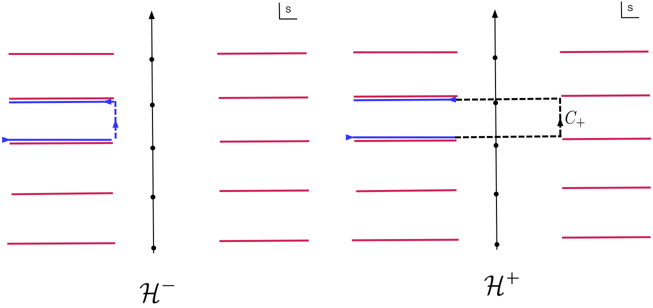

- In the end, the final result looks like

$$\delta^{(2)} S = \int_{\Sigma_A} \boldsymbol{\omega}(h, \mathcal{L}_{\xi_B} h) + \int_{\mathcal{H}^+} \boldsymbol{\omega}(h, \mathcal{L}_V g^{(0)}) + \int_{\mathcal{H}^-} \boldsymbol{\omega}(h, \mathcal{L}_V g^{(0)}).$$

where V is a vector field which can be determined in terms of h.

Relative Entropy

- Actually, V turns out to be precisely the gauge transformation between the de-Donder gauge and the Hollands-Wald gauge:

$$\gamma_{ab} = h_{ab} + 2\nabla^{(0)}_{(a} V_{b)}.$$

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- So finally, we can transform our result to the Hollands-Wald gauge.
- Happily, our result rewritten in this gauge becomes

$$\delta^{(2)} S(\rho||\rho_0) = \int_{\Sigma_A} \boldsymbol{\omega}(\gamma, \mathcal{L}_{\xi_B}\gamma)$$

which is precisely what we set out to show.

Conclusion

- If (M, g) is an AAdS spacetime which computes the entanglement entropies of all ball-shaped regions in a CFT correctly via the Ryu-Takayanagi formula up to second order in state deformations, and if $a_* = C_T$, then (M, g) must satisfy the Einstein equation to second-order in the bulk metric deformation.

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- The analogous statement for higher-derivative theories can also be proven [Faulkner, Haehl, Hijano, OP, Rabideau, Van Raamsdonk – to appear].