Modular Hamiltonian of excited states, OPE blocks and Emergent bulk fields

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Based on the work with Gabor Sarosi (Vrije Universiteit)
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Intro(1)

- To understand the structure of a given space of density matrices, we sometimes need a proper measure of distance between two matrices.

・For example, consider a reduced density matrix ρ_{AB} on disjoint subsystems A and B and suppose that we want to know how much two subsystems A and B are entangled. One way to evaluate this is measuring the distances between ρ_{AB} and separable states.

$$\sigma_{AB} = \sum_a p_a \, \rho_A^a \otimes \rho_B^a$$

Intro(2)

- There are several distance measures known in the literature, like

Fidelity,

$$F(\rho||\sigma) \equiv \mathrm{tr}\sqrt{\sqrt{\sigma}\rho\sqrt{\sigma}}$$

or trace distance,

$$T(\rho||\sigma) = \frac{\mathrm{tr}\ |\rho - \sigma|^2}{\mathrm{tr}\ \rho_{(0)}^2}$$

- The one we would like to focus on this talk is relative entropy

$$S(\rho||\sigma) = \mathrm{tr}\left(\rho \log \rho\right) - \mathrm{tr}\left(\rho \log \sigma\right)$$

Relative entropy

Relative entropy satisfies several nice properties.

(1) Positive definite. $S(\rho||\sigma) \geq 0,$ $\quad S(\rho||\sigma) = 0 \Rightarrow \rho = \sigma$

(2)Monotonically decreasing under time evolutions, ie, for any CPTP map $\mathcal{N}_t$

$$S(\rho||\sigma) \geq S(\mathcal{N}_t\rho||\mathcal{N}_t\sigma)$$

(3) For RDMs of two regions A and B, $A \supset B$

$$S(\rho_A||\sigma_A) \geq S(\rho_B||\sigma_B)$$

Modular Hamiltonian (1)

- In some sense relative entropy is a generalization of free energy.

$$S(\rho||\sigma) = [\langle \rho K_\sigma \rangle - \langle \sigma K_\sigma \rangle] - [S(\rho) - S(\sigma)]$$
$$= \Delta \langle K_\sigma \rangle - \Delta S$$

$K_\sigma = -\log \sigma$ is called modular Hamiltonian of σ.

When $\sigma = e^{-\beta H}$ the relative entropy indeed reduced to free energy.

Modular Hamiltonian(2)

- If we are interested in the relative entropy of nearby states $S(\rho + \delta\rho || \rho)$, we can expand it with respect to $\delta\rho$. The first order term must

vanish because of the positivity => the first law like relation,

$$\delta S = \langle K_\rho \delta\rho \rangle$$

The quadratic term is sometimes called Fisher information,

$$F(\rho + \delta\rho || \rho) = \frac{d^2}{dt^2} S(\rho + t\delta\rho || \rho) \Big|_{t=0}$$

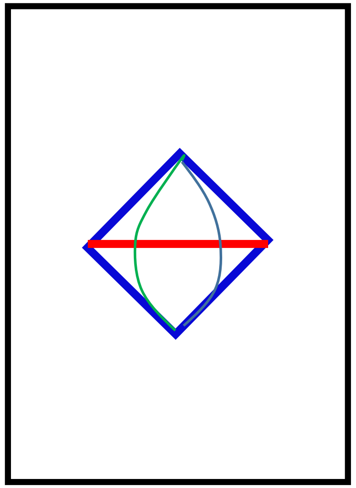
Modular Hamiltonian (3) :

- In this talk we are focusing on the relative entropy between two reduced density matrices $S(\rho_V||\rho_W)$ on a region A in CFT.

When the subsystem is a round sphere, the modular Hamiltonian of vacuum has a local expression,

$$K_{vac} = 2\pi \int drd\Omega_{d-2} \frac{R^2 - r^2}{2R} T_{00} + S_{EE}$$

And this generates the boost symmetry of the causal diamond of A.

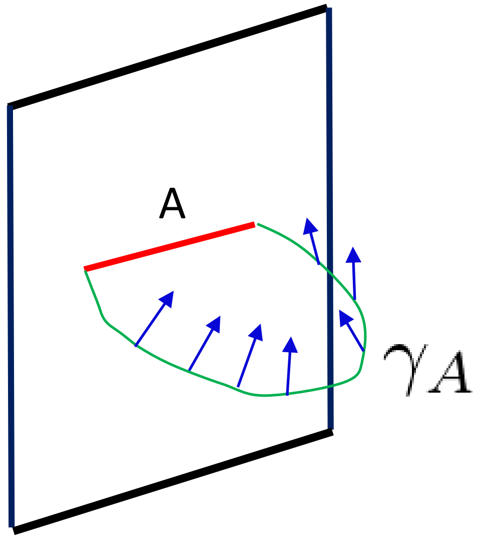


A holographic interpretation of vacuum MH

- There is a nice holographic interpretation of the vacuum modular flow.

・The bulk RT surface γ_A can be regarded as the bifurcation surface of the topological black hole. The timelike killing vector of the black hole generates the vacuum modular flow at the boundary . [Casini Huerta Myers}

First law of entanglement <-> First law of the black hole
->linearized Einstein equations.



Modular Hamiltonians of excited states

- On the contrary to the nice story of vacuum modular Hamiltonian, the mH of an excite state in general non local, and hard to derive the exact answer.

Nevertheless there are nice holographic results for them. One is the JLMS conjecture,

$$K_V = \frac{A}{4G} + K_V^{\mathrm{bulk}} + \cdots$$

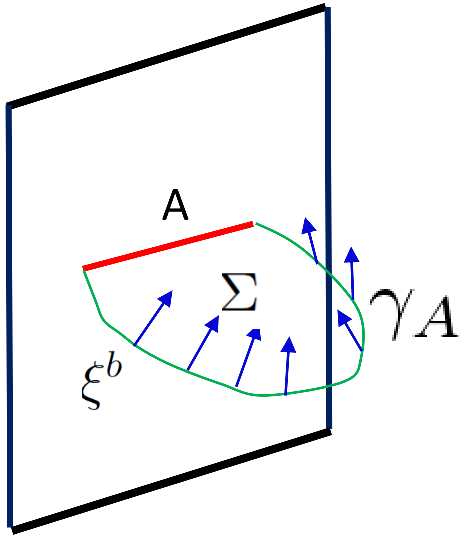
Where A is the area operator, whose expectation value computes the area of the bulk RT surface, is the mH of the bulk QFT.

Modular Hamiltonians of excited states

A related statement is Fisher information = Bulk canonical energy.

Suppose that we are interested in the entanglement entropy of a slightly excited state $|V\rangle$, whose RDM can be split into $\rho_V = \rho_0 + \delta\rho$

If we expand the entropy with respect to $\delta\rho$ then the quadratic term is given by the bulk canonical energy. [van Raamsdonk, Lashkari]

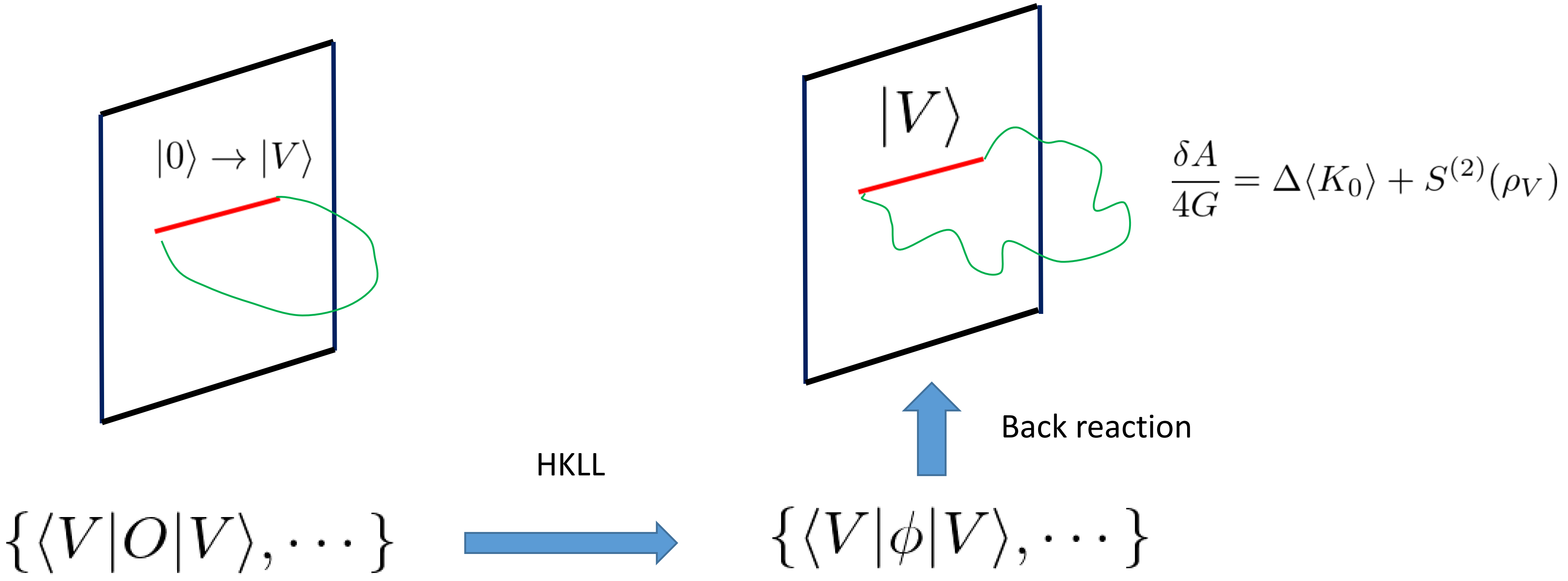


[Nozaki Numasawa Prudenziatti Takayanagi]

[Lin marcori Ooguri Stoica]

$$S_V^{(2)} = -2\pi \int_\Sigma d\Sigma^a \xi^b T_{ab}(\langle V|\phi|V\rangle)$$

Fisher information= Canonical energy



The excitation turns on non trivial CFT 1pt functions

Bulk profile

This formula captures the first non trivial back reaction effect in the bulk spacetime.

Summary so far

- Two nice holographic results

(1) JLMS : $$K_V = \frac{A}{4G} + K_V^{\mathrm{bulk}} + \cdots$$

(2) Fisher information =Canonical energy $$S_V^{(2)} = -2\pi \int_\Sigma d\Sigma^a \xi^b T_{ab}(\langle V|\phi|V\rangle)$$

Both of them involve excited state modular Hamiltonians.
Can we derive these nice results from CFT side ?

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Our work!

Outline of this talk(1):

- In this work we present a novelway of calculating the entanglement entropy and the modular Hamiltonian (mH)of excited states in CFT.
- We first develop a general prescription to perturbatively calculate the mH

$K_\rho = -\log\rho$ when $\rho = \rho_0 + \delta\rho$ and mH $K_0 = -\log\rho_0$ of the reference state is known,

$$K_\rho = K_0 + \sum_{n=1}^{\infty} (-1)^n \delta K^{(n)}$$

The result is general, and applicable to any theory. This formula also gives a formal series expansion of von Neumann entropy

$$S(\rho) = \langle \rho K_\rho \rangle \quad = \sum_{n=1}^{\infty} \delta S^{(n)}(\delta\rho)$$

Outline of the talk(2) ;CFT part

- We then formally apply this formula to CFT, by taking ρ, ρ_0 to be the RDMs of an excited state and vacuum on a ball shaped region A respectively.

In a CFT, $\delta\rho$ of interest is given by a sum of OPE blocks of primary operators. From this, it follows that $\delta S^{(n)}(\delta\rho)$ is given by an integral 2n+2 point function along vacuum modular flow.

Outline of the talk (3)

We can rewrite each in the expansion holographically , as they are fixed just by conformal symmetry.

At quadratic order n=2, the holographic expression is of the JLMS form for the modular Hamiltonian, and the canonical energy for the entanglement entropy.

Expanding the log

- We can the modular Hamiltonian by using the formula,

$$-\log\rho = \int_0^\infty d\beta \left(\frac{1}{\beta+\rho} - \frac{1}{\beta+1} \right)$$

The result is

$$K_\rho = K_0 + \sum_{n=1}^{\infty} (-1)^n \int_{-\infty}^{\infty} ds_1 ... ds_n \mathcal{K}_n(s_1, ..., s_n) \prod_{k=1}^{n} (e^{-\left(\frac{is_k}{2\pi}+\frac{1}{2}\right)K_0} \delta\rho e^{\left(\frac{is_k}{2\pi}-\frac{1}{2}\right)K_0})$$

This tells us n-th order term of the mH is given by evolving the Perturbation $\delta\rho$ by the unperturbed mH K0, and integrating it along the flow.

Expanding the log

- The explicit expression of the kernel is

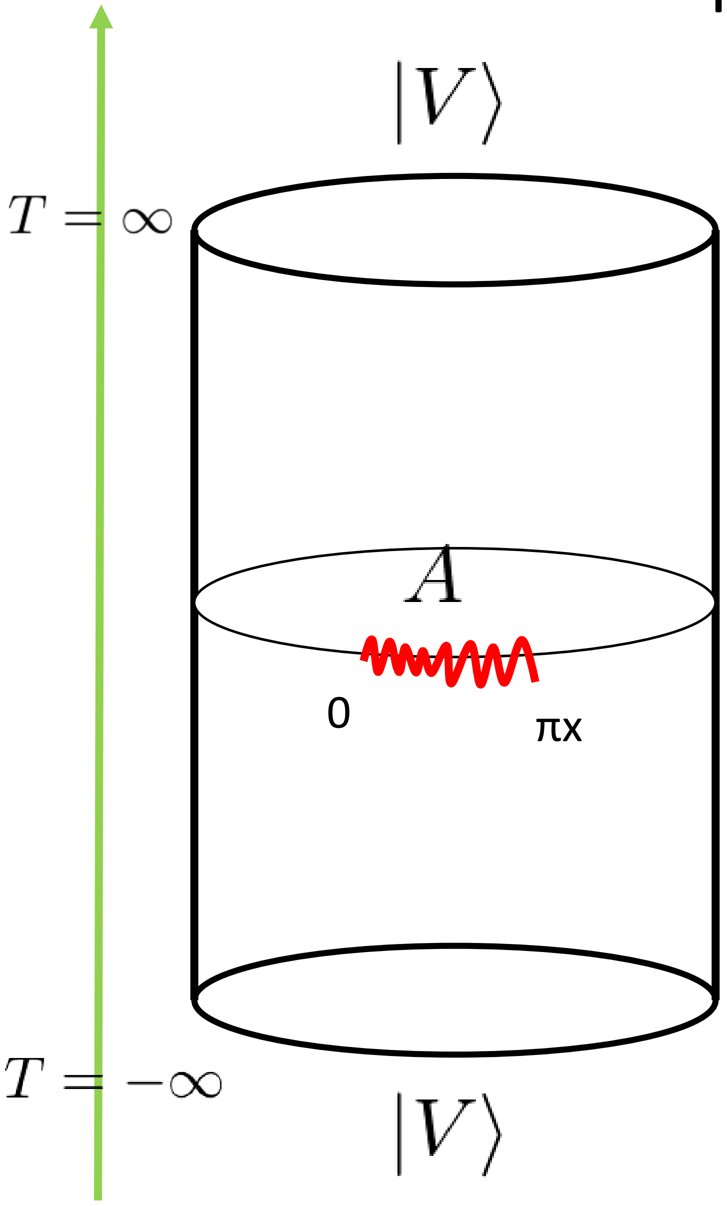
$$\mathcal{K}_n(s_1, \ldots, s_n) = \frac{(2\pi)^2}{(4\pi)^{n+1}} \frac{i^{n-1}}{\cosh \frac{s_1}{2} \cosh \frac{s_n}{2} \prod_{k=2}^{n} \sinh \frac{s_k - s_{k-1}}{2}}$$

- Some special cases:

$$\mathcal{K}_1(s_1) = \frac{1}{(2\cosh \frac{s_1}{2})^2},$$

$$\mathcal{K}_2(s_1, s_2) = \frac{1}{16\pi} \frac{i}{\cosh \frac{s_1}{2} \cosh \frac{s_2}{2} \sinh \frac{s_2 - s_1}{2}}$$

The CFT set up(1)



- A CFT on a Cylinder $\mathbb{R}\times S^{d-1}$
- A subsystem $[0,\theta_0]\times S^{d-2}$
- Excited states $|V\rangle$ at $T=\pm\infty$
- Reduced density matrices $\rho_V=\mathrm{tr}_{A_c}|V\rangle\langle V|$

The CFT setup(2)

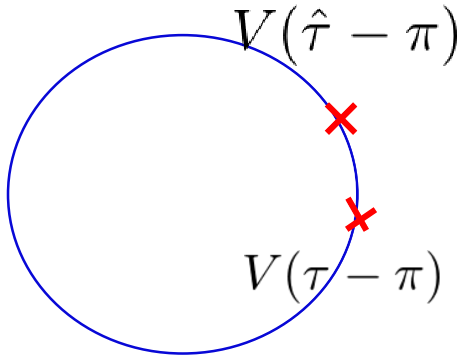
- One can map the cylinder with cut to $S^1 \times H^{d-1}$ with the metric,

$$ds^2_{\Sigma_n} = d\tau^2 + du^2 + \sinh^2 u d\Omega^2_{d-2} \qquad \tau \sim \tau + 2\pi$$

- There the RDM has a nice form in terms of the local operator

Corresponding to the excited state,

$$\rho_V = \frac{e^{-\pi K} V(\tau - \pi) V(\hat{\tau} - \pi) e^{-\pi K}}{\langle V(\tau) V(\hat{\tau}) \rangle_{S^1 \times H^{d-1}}} \qquad \tau = \pi - \theta_0 \quad \hat{\tau} = \pi + \theta_0$$



K is the generator of the translation along and coincide with vacuum mH.

CFT setup(3)

- In the small subsystem size limit $\theta_0 \to 0$ one can expand the product by OPE blocks of primaries,

$$V(\tau-\pi)V(\hat{\tau}-\pi) = \langle V(\tau)V(\hat{\tau})\rangle_{S^1\times H^{d-1}} \sum_k C^k_{VV} B_k(\tau-\pi, \hat{\tau}-\pi)$$

where the index k runs for all primaries. We are then able to write the density matrix as $\rho_V = \rho_{\text{vac}} + \delta\rho$,

$$\delta\rho = e^{-\pi K} \sum_{k\neq vac} C^k_{VV} B_k(\tau-\pi, \hat{\tau}-\pi) e^{-\pi K}$$

CFT set up (4)

- Using the general formula, we derive the (formal)perturbative expression of the excited state mH,

$$K_V = K + \sum_{k \neq vac} C_{VV}^{\mathcal{O}_k} \int_{-\infty}^{\infty} \frac{ds}{\cosh^2 \frac{s}{2}} B_{\mathcal{O}_k}(\tau - \pi + is, \hat{\tau} - \pi + is) + \cdots$$

and for entanglement entropy, $\delta S_V = \sum_{m=2}^{\infty} \delta S_V^{(m)}$

$$\delta S_V^{(2)} = -\sum_k (C_{VV}^k)^2 \int \frac{ds}{8\cosh^2 \frac{s}{2}} \mathcal{F}_k(\tau, \hat{\tau}, \tau - \tau_s, \hat{\tau} - \tau_s)$$

Where F is the 4pt conformal block of the primary $\mathcal{O}_k$

Comparison with the replica trick

- We can derive the same formula for $\delta S_V^{(2)}$ from the conventional replica trick,

$$S_A(\rho_V) = \lim_{n\to 1} \frac{1}{1-n} \mathrm{Tr}\rho_V^n \qquad \mathrm{Tr}\rho_V^n = \frac{\langle \prod_{k=0}^{n-1} V(\tau_k)V(\hat{\tau}_k)\rangle_{\Sigma_n}}{\prod_{k=0}^{n-1}\langle V(\tau_k)V(\hat{\tau}_k)\rangle_{\Sigma_1}}$$

by picking up the leading contribution to the correlation function by OPE.

We can also check that in the small subsystem size limit $\theta_0 \to 0$, this formula reproduces the known results in the limit .

Rewriting the EE holographically

- We can rewrite each term in $\delta S_V^{(2)}$ holographically . The formula was

$$\delta S_V^{(2)} = -(C_{VV}^{\mathcal{O}})^2 \int_{-\infty}^{\infty} \frac{ds}{8\cosh^2\frac{s}{2}} \mathcal{F}_{\mathcal{O}}(\tau, \hat{\tau}, \tau - \tau_s, \hat{\tau} - \tau_s), \quad \tau_s = \pi - is.$$

$$= -C(\delta\tau, \partial_a) C(\delta\tau, \partial_b) \int_{-\infty}^{\infty} \frac{ds}{8\cosh^2\frac{s}{2}} \langle \mathcal{O}(\tau_a + \tau_s, Y_a) \mathcal{O}(\tau_b, Y_b) \rangle_{\Sigma_1} \Big|_{(\tau_a, Y_a) = (\tau_b, Y_b) = (0,0)}$$

In the second line we write the 4pt block F in terms of the 2 pt function <OO>

And the the $C(\delta\tau, \partial_a)$ diffrerential operator summing up the descendants,

$$V(\tau)V(\hat{\tau}) = \langle V(\tau)V(\hat{\tau})\rangle_{\Sigma_1} C(\delta\tau, \partial_a)\mathcal{O}(\tau_a, Y_a) + \text{..........}$$

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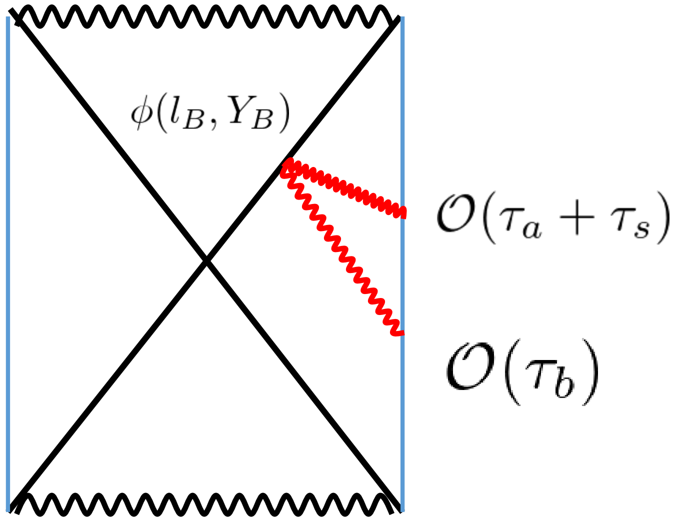
And the $C(\delta\tau, \partial_a)$ differential operator summing up the descendants,

$$V(\tau)V(\hat{\tau}) = \langle V(\tau)V(\hat{\tau})\rangle_{\Sigma_1} C(\delta\tau, \partial_a)\mathcal{O}(\tau_a, Y_a) + \dots\dots\dots$$

Rewriting EE (1)

Faulkner proved following identity, which is converting the integral of CFT 2 pt function <OO>along the vacuum modular flow to the integral of bulk to boundary propagator on the horizon of bulk topological black hole,

$$\int_{-\infty}^{\infty} \frac{ds}{\cosh^2 \frac{s}{2}} \langle \mathcal{O}(\tau_a+\tau_s)\mathcal{O}(\tau_b)\rangle = \int dl_B dY_B \frac{\partial}{\partial l_B}\langle \phi(l_B, Y_B)\mathcal{O}(\tau_a)\rangle \frac{\partial}{\partial l_B}\langle \phi(l_B, Y_B)\mathcal{O}(\tau_b)\rangle$$



Rewriting EE(2)

- If we act the differential operator to the bulk to bdy propagator,

$$C(\delta\tau, \partial_a)\langle\phi(l_B, Y_B)\mathcal{O}(\tau_a, Y_a)\rangle_{\Sigma_1} \quad = \frac{\langle\phi(l_B, Y_B)V(\tau)V(\hat{\tau})\rangle_{\Sigma_1}}{\langle V(\tau)V(\hat{\tau})\rangle_{\Sigma_1}} \quad \equiv \langle\phi(l_B, Y_B)\rangle_V$$

We get the expectation value of the bulk scalar field. Using this we derive a bulk expression of the quadratic term,

$$\delta S_V^{(2)} = -2\pi \int dl_B l_B \int dY_B (\partial_{l_B}\langle\phi\rangle_V)^2$$

rewriting modular Hamiltonian

We can also rewrite the CFT expression of the modular Hamiltonian,

$$K_V = 2\pi K - \quad C^{\mathcal{O}}_{VV} \int_{-\infty}^{\infty} ds \frac{B_{\mathcal{O}}(\tau - \pi + is, \hat{\tau} - \pi + is)}{(\cosh \frac{s}{2})^2}$$

Then, we get

$$K_V = 2\pi \left(K - \int_{\Sigma} d\Sigma^a \xi^b T_{ab}(\phi) \right) + 2\pi \int_{\Sigma} d\Sigma^a \xi^b T_{ab}(\phi - \langle \phi \rangle_V) + \delta S^{(2)}_V + \cdots$$

The first term canbe identified with area operator, and the second term is the mH of the bulk excited state dual to the CFT state $|V\rangle$

Cubic order term of EE

- Similarly we can evaluate the cubic order term of the EE

$$\delta S_V^{(3)} = -(C_{VV}^{\mathcal{O}})^3 \int_{-\infty}^{\infty} ds_1 ds_2 \mathcal{K}_2(s_1, s_2) \times \frac{i(s_2 - s_1)}{2\pi} \langle B_{\mathcal{O}}(\tau - \tau_{s_1}, \hat{\tau} - \tau_{s_1}) B_{\mathcal{O}}(\tau - \tau_{s_2}, \hat{\tau} - \tau_{s_2}) B_{\mathcal{O}}(\tau, \hat{\tau}) \rangle_{\Sigma_1}$$

In the small subsystem size limit we can evaluate the integral,

$$\delta S_V^{(3)} = (2\theta_0)^{3\Delta} (C_{VV}^{O})^3 C_{OOO} \frac{\Gamma(\frac{1+\Delta}{2})^3}{12\pi\Gamma(\frac{3+3\Delta}{2})}.$$

This again agree with the holographic calculation[Casini, Galante Myers], in the presence of the bulk cubic interaction.

$$\mathcal{L}_{bulk} = (\partial\phi)^2 - \kappa\phi^3$$

Conclusions

- Holographic expression of excited state modular Hamiltonian and Entanglement entropy from CFT from vacuum modular flow.

- Can we drive a holographic expression of cubic term?

- Can we perform a similar analysis for mutual information?