Entanglement of Purification in Free scalar field theory

Arpan Bhattacharyya

Yukawa Institute for Theoretical Physics, Kyoto

"*Entanglement of Purification in Free Scalar Field Theories*" arXiv:1802.09545
with *Tadashi Takayanagi* and *Koji Umemoto*

Introduction

$|\psi>_{AB}$

↓

For pure state Entanglement S_A entropy (EE) is good measure of correlation between subsystem

$$\rho = \sum_n p_n |\psi_n><\psi_n|$$

↓

For mixed state EE is not good measure of correlation anymore. We need other measure.

One such is : *Entanglement of Purification (EoP)* (Terhal,Horodecki Leung DiVincenzo '02)

Given a mixed state: ρ_{AB} extend the Hilbert space $H_A \otimes H_B \rightarrow H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$

$\rho_{A\tilde{A}B\tilde{B}} = |\psi_{A\tilde{A}B\tilde{B}}><\psi_{A\tilde{A}B\tilde{B}}|$ is pure and $Tr_{\tilde{A}\tilde{B}}\rho_{A\tilde{A}B\tilde{B}} = \rho_{AB}$

Then EoP is given by : $$E_P(\rho_{AB}) \equiv \min_{\text{all possible } |\psi_{A\tilde{A}B\tilde{B}}>} S_{A\tilde{A}}.$$

Various properties:

(Koji's talk)

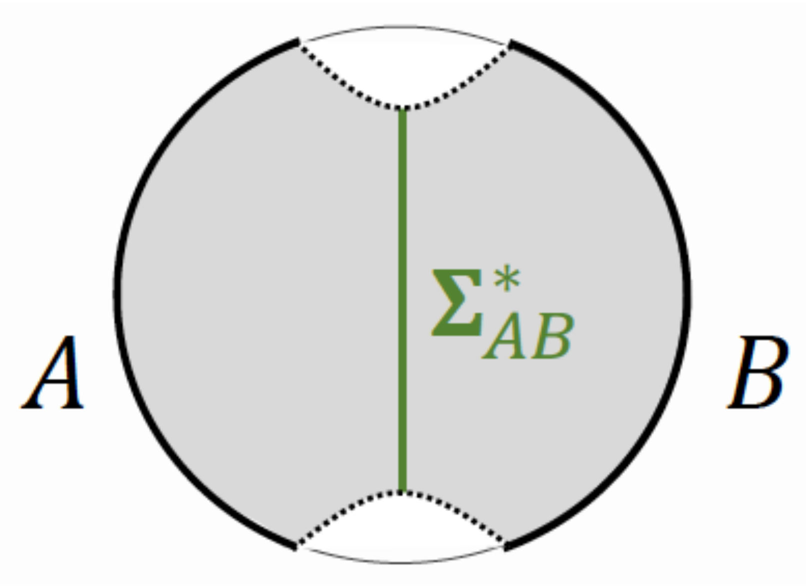
$$E_P(\rho_{AB}) > 0$$

$$E_P(\rho_{AB}) = 0 \quad \text{when} \quad \rho_{AB} = \rho_A \otimes \rho_B$$

$$\frac{I(A:B)}{2} \leq E_P(A:B) \leq min\left\{S_A, S_B\right\}$$

For tripartite states: $$\frac{I(A:B_1)+I(A:B_2)}{2} \leq E_P(A:B_1B_2)$$

Holography



(Takayanagi Umemoto 1708.09393, To appear in Nature Physics, Nguyen et al 1709.07424)

$$E_W(A:B) = \frac{Area(\Sigma^*_{AB})}{4G_N} = E_P(A:B)$$

Free Scalar Field Theory

- We want to compute the EoP for the ground states of 1+1 dimensional scalar field theory

- We will discretize the theory on lattice

The Hamiltonian: $$H_0 = \frac{1}{2}\int dx\Big[\pi^2 + (\partial_x\phi)^2 + m^2\phi^2\Big]$$

The Discretization gives: $$x = an, \phi_n = \phi(na), \phi_n = a\pi(na)$$

"a" is the lattice spacing and "n" denotes the site.

$$n = 1, 2, \cdots N$$

Cannonical quantization: $$[\phi_n, \pi_{n'}] = i\delta_{n,n'}$$

We also impose periodic boundary conditions: $\phi_{n+N} = \phi_n, \phi_{n+N} = \pi_n$

The rescaled Hamiltonian:

$$H = \sum_{n=1}^{N} \frac{1}{2}\pi_n^2 + \sum_{n,n'=1}^{N} \frac{1}{2}\phi_n V_{nn'} \phi_{n'}$$

After doing an Fourier Transform:

$$V_{nn'} = N^{-1} \sum_{k=1}^{N} [a^2 m^2 + 2(1 - cos(2\pi\, k/N))] e^{2\pi i k(n-n')/N}$$

(Bombelli et al '86)

Ground state wavefunction for this model:

$$\psi_0[\phi] = \mathcal{N}_0 e^{-\frac{1}{2}\sum_{n,n'=1}^{N} \phi_n W_{nn'} \phi_{n'}}$$

where, $W_{nn'} = \frac{1}{N} \sum_{k=1}^{N} \sqrt{a^2 m^2 + 2(1 - cos(2\pi\, k/N))}\, e^{2\pi\, i\, k(n-n')/N}$

We set $a = 1$ by rescaling mass

- In all our subsequent analysis we set total number of lattice site

$$N = 60$$

- And consider five different mass $m = 0.0001, 0.001, 0.01, 0.1, 1$

To compute the entanglement entropy:

$$H_{tot} = H_A \otimes H_B$$

$$\Psi_{AB} = \mathcal{N}_{AB} \cdot \exp\left[-\frac{1}{2}(\phi_A \ \phi_B)\begin{pmatrix} A & B \\ B^T & C \end{pmatrix}\begin{pmatrix} \phi_A \\ \phi_B \end{pmatrix}\right].$$

- We define:

$$W = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}, \quad W^{-1} = \begin{pmatrix} D & E \\ E^T & F \end{pmatrix}$$

(Bombelli et al '86)

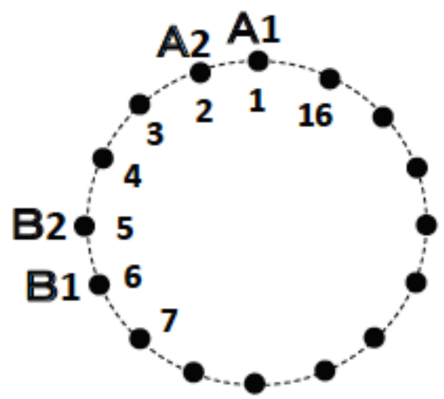
$$S_A = S_B = \sum_{i=1}^{|A|} f(\lambda_i)$$

where,

$$f(x) = \log \frac{\sqrt{x}}{2} + \sqrt{1+x} \cdot \log\left(\frac{1}{\sqrt{x}} + \frac{\sqrt{1+x}}{\sqrt{x}}\right)$$

- and, λ_i are eigenvalues of $\Lambda = -E.B^T = D.A - 1$

Computation of EoP: Gaussian Ansatz



Next we prepare our system. We identify two subsystem A and B and then trace out the rest.

$$\Psi_0[\phi_{AB},\phi_C] = \mathcal{N}_0 \cdot \exp\left[-\frac{1}{2}(\phi_{AB},\phi_C)\begin{pmatrix} P & Q \\ Q^T & R \end{pmatrix}\begin{pmatrix} \phi_{AB} \\ \phi_C \end{pmatrix}\right].$$

$$\rho_{AB}[\phi_{AB},\phi'_{AB}] \propto \exp\left[-\frac{1}{2}(\phi_{AB},\phi'_{AB})\begin{pmatrix} P-\frac{1}{2}QR^{-1}Q^T & -\frac{1}{2}QR^{-1}Q^T \\ -\frac{1}{2}QR^{-1}Q^T & P-\frac{1}{2}QR^{-1}Q^T \end{pmatrix}\begin{pmatrix} \phi_{AB} \\ \phi'_{AB} \end{pmatrix}\right]$$

Now to compute $E_P(\rho_{AB})$ we first purify the system by adding ancilla system

(AB, Takayanagi, Umemoto, *arXiv: 1802.09545*)

such that:

$H_{tot} = H_A \otimes H_B \otimes H_{\tilde{A}} \otimes H_{\tilde{B}}$ ***is pure***

Assumption: $|\psi_{A\tilde{A}B\tilde{B}}>$ ***optimal purified state is Gaussian***

$$\Psi_{A\tilde{A}B\tilde{B}}[\phi_{AB},\phi_{\tilde{A}\tilde{B}}] = \exp\left[-\frac{1}{2}(\phi_{AB},\phi_{\tilde{A}\tilde{B}})\begin{pmatrix} J & K \\ K^T & L \end{pmatrix}\begin{pmatrix} \phi_{AB} \\ \phi_{\tilde{A}\tilde{B}} \end{pmatrix}\right]$$

We should recover original density matrix after tracing out ancilla

$$\boxed{J = P, \quad KL^{-1}K^T = QR^{-1}Q^T}$$

Now we have to compute $S_{A\tilde{A}}$ and $E_P(\rho_{AB}) = \min S_{A\tilde{A}}$

To compute $S_{A\tilde{A}}$

$$\Psi_{A\tilde{A}B\tilde{B}} = \mathcal{N} \cdot \exp\left[-\frac{1}{2}(\phi_{A\tilde{A}}\ \phi_{B\tilde{B}}) W_{A\tilde{A}B\tilde{B}} \begin{pmatrix} \phi_{A\tilde{A}} \\ \phi_{B\tilde{B}} \end{pmatrix}\right].$$

$$W = \begin{pmatrix} J_{AA} & K_{A\tilde{A}} & J_{AB} & K_{A\tilde{B}} \\ K_{\tilde{A}A} & L_{\tilde{A}\tilde{A}} & K_{\tilde{A}B} & L_{\tilde{A}\tilde{B}} \\ J_{BA} & K_{B\tilde{A}} & J_{BB} & K_{B\tilde{B}} \\ K_{\tilde{B}A} & L_{\tilde{B}\tilde{A}} & K_{\tilde{B}B} & L_{\tilde{B}\tilde{B}}. \end{pmatrix} \equiv \begin{pmatrix} A & B \\ B^T & C \end{pmatrix}$$

We observe the following transformations: (AB, Takayanagi, Umemoto, *arXiv:1802.09545*)

$$\hat{P} = \begin{pmatrix} I_{|A|} & 0 \\ 0 & P \end{pmatrix}, \quad \hat{Q} = \begin{pmatrix} I_{|B|} & 0 \\ 0 & Q \end{pmatrix}$$ two non degenerate matrices

$$J \to J, \quad K \to K \begin{pmatrix} P^T & 0 \\ 0 & Q^T \end{pmatrix}, \quad L \to \begin{pmatrix} P & 0 \\ 0 & Q \end{pmatrix} L \begin{pmatrix} P^T & 0 \\ 0 & Q^T \end{pmatrix}.$$

Keep $J = P, \quad KL^{-1}K^T = QR^{-1}Q^T$ invariant

Then:

$$A \to \hat{P}A\hat{P}^T, \quad B \to \hat{P}B\hat{Q}^T, \quad C \to \hat{Q}C\hat{Q}^T, \quad D \to (\hat{P}^T)^{-1}D\hat{P}^{-1}, \quad E \to (\hat{P}^T)^{-1}E\hat{Q}^{-1}, \quad F \to (\hat{Q}^T)^{-1}F\hat{Q}$$

Consequently: $\Lambda = -E \cdot B^T \quad \Lambda \to (P^T)^{-1}\Lambda P^T$ $S_{A\tilde{A}}$ remains same:

We can use these transformations to simplify the matrices J,K and L

Minimal Gaussian Ansatz

(AB, Takayanagi, Umemoto, *arXiv:1802.09545*)

Still the problem of minimization is difficult as the dimensions $|\tilde{A}|, |\tilde{B}|$

in principle can be arbitrary. We still need to optimize over $|\tilde{A}|^2 + |\tilde{B}|^2$ numer of parameters

Assumption: $|A| = |\tilde{A}|, |B| = |\tilde{B}|$ "Minimal Gaussian Ansatz"

For this case: $$K_{(AB),(\tilde{A}\tilde{B})} = \begin{pmatrix} I_{|A|} & K_{A\tilde{B}} \\ K_{B\tilde{A}} & I_{|B|} \end{pmatrix}$$

The matrix $L_{(\tilde{A}\tilde{B}),(\tilde{A}\tilde{B})}$ completely determined by *K*

$J_{(AB),(AB)}$ also fixed by the constraints discussed previously

So we have to minimize $S_{A\tilde{A}}$ *over* $2|A||B|$ *number of parameters*

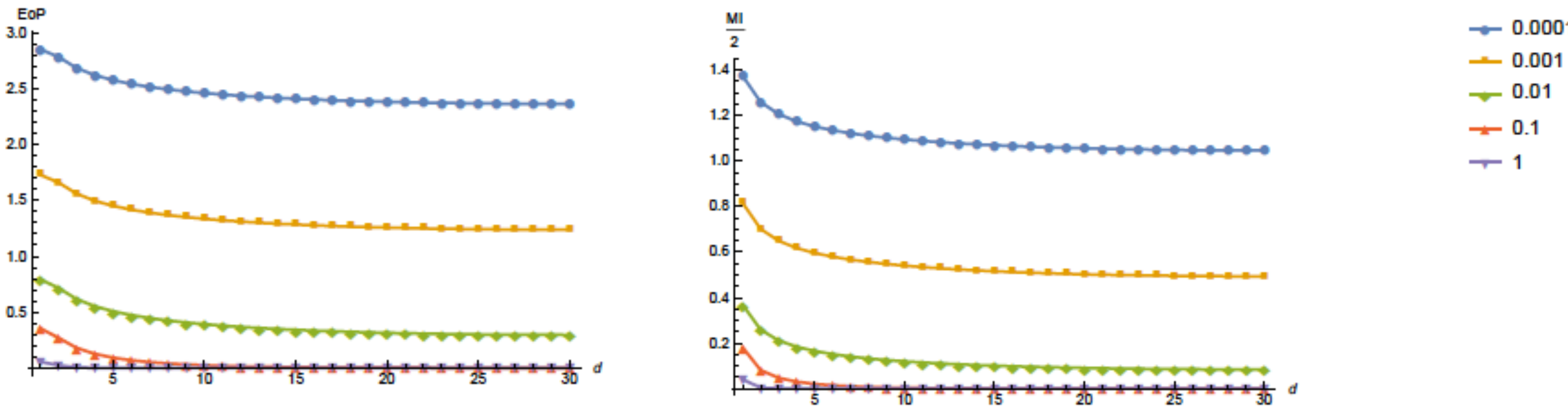
We do the numerical analysis now for $(|A|, |B|) = (1,1), (1,2) \text{and} (2,2)$

and total lattice site N=60

EoP: Results

(AB, Takayanagi, Umemoto, *arXiv:1802.09545*)

$E_P(\rho_{AB})$ when $(|A|, |B|) = (1, 1)$



Monotonically decreasing, as the mass is lowered it changes from power law to exponential decay

$$E_P(\rho_{AB}) > \frac{I(A:B)}{2}$$

For this case: $$K_{(AB),(\tilde{A}\tilde{B})} = \begin{pmatrix} 1 & x_1 \\ x_2 & 1 \end{pmatrix}$$

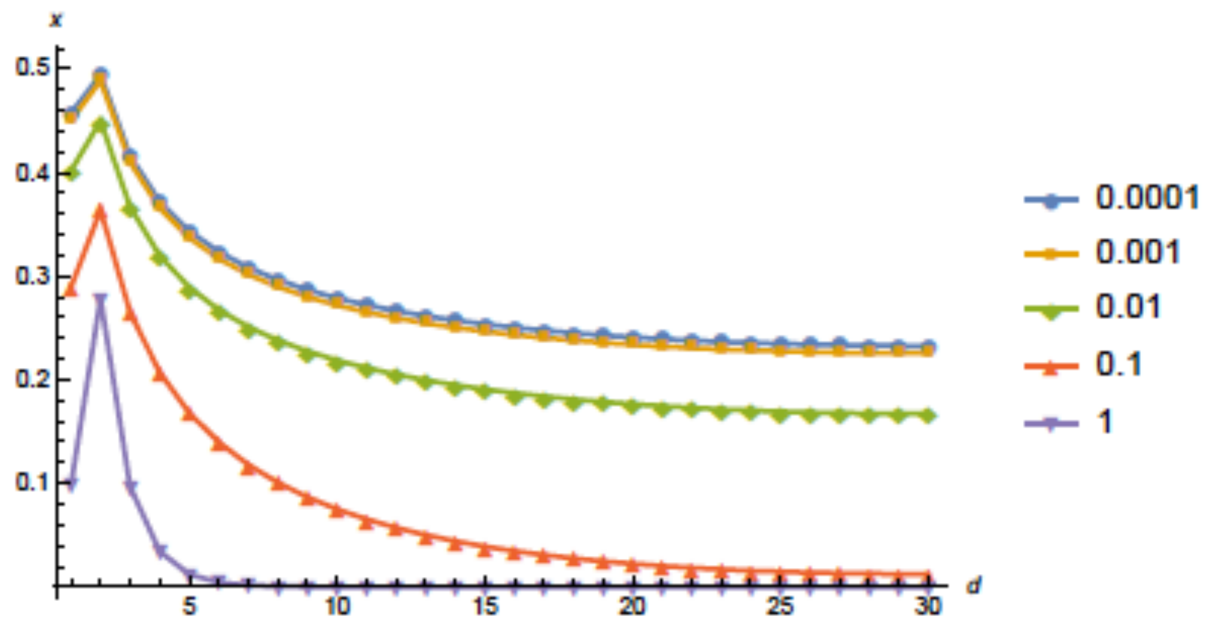
we optimize over this two parameter and find $x_2 = x_1$

This can be understood because of a Z_2 which exchanges A and B

$$(A, \tilde{A}) \to (B, \tilde{B})$$

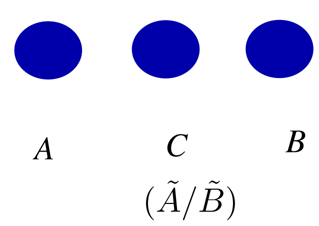
Results: Continued

We also plot this parameter against the distance between A and B



Interestingly there is peak at d=2

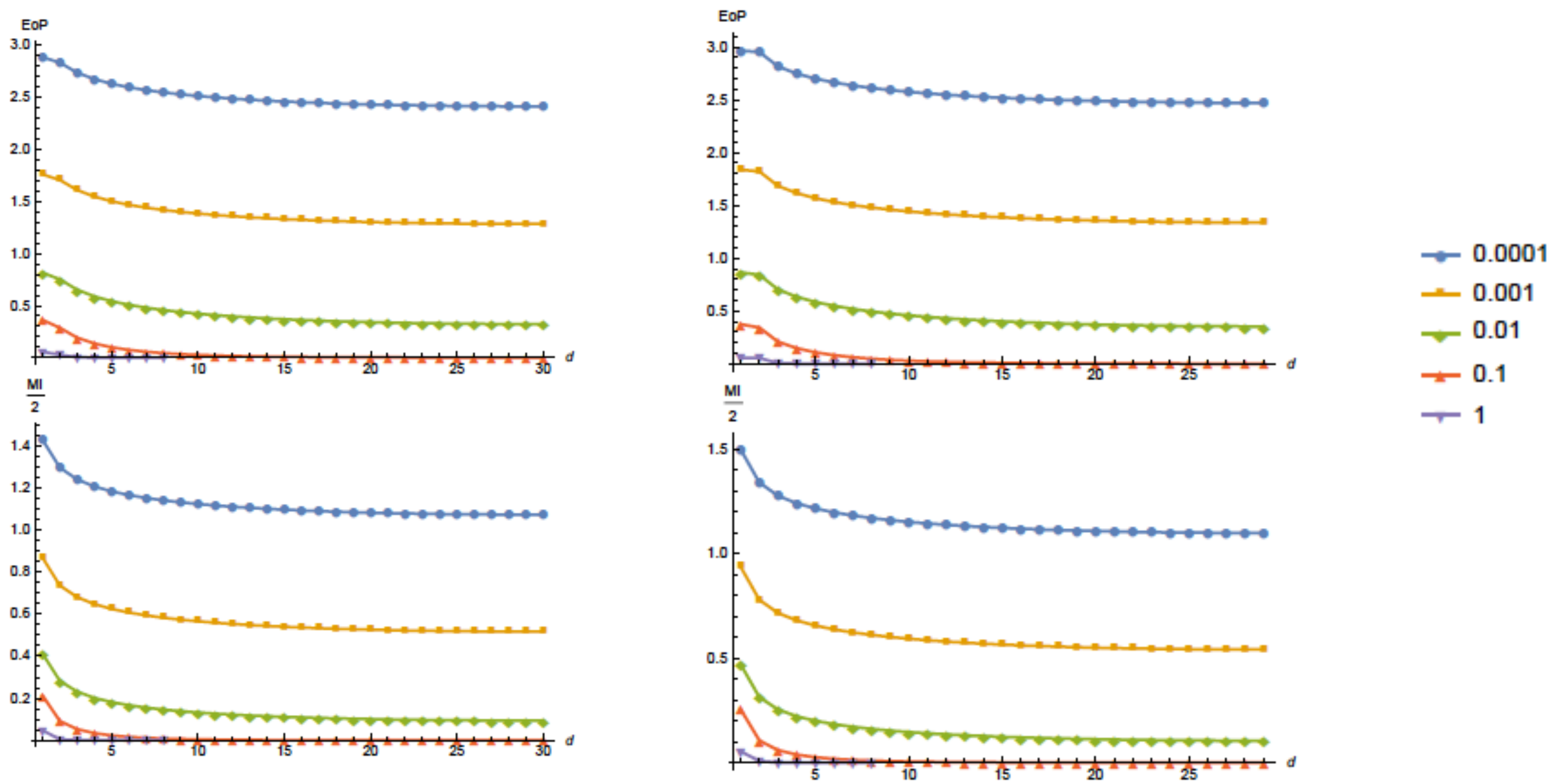
at d=2 there is vacant site between "A" and "B"



The correlation between **A** and $\tilde{A}$ and **B** and $\tilde{B}$ gets enhanced

Results: Continued

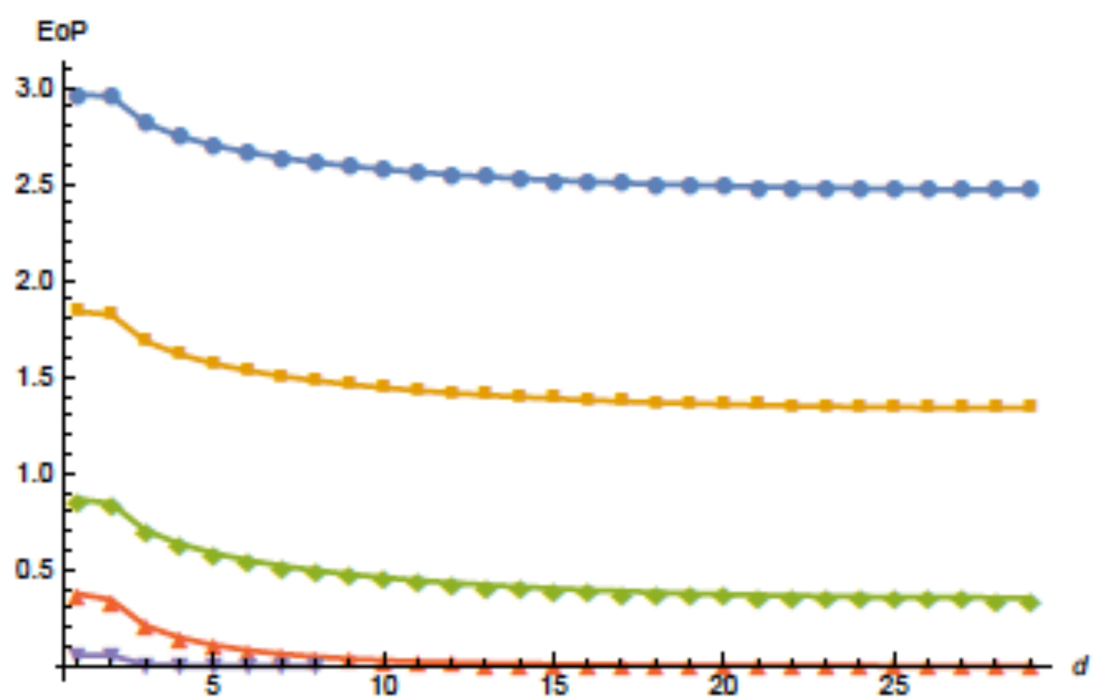
We increase the size of “A” and “B”

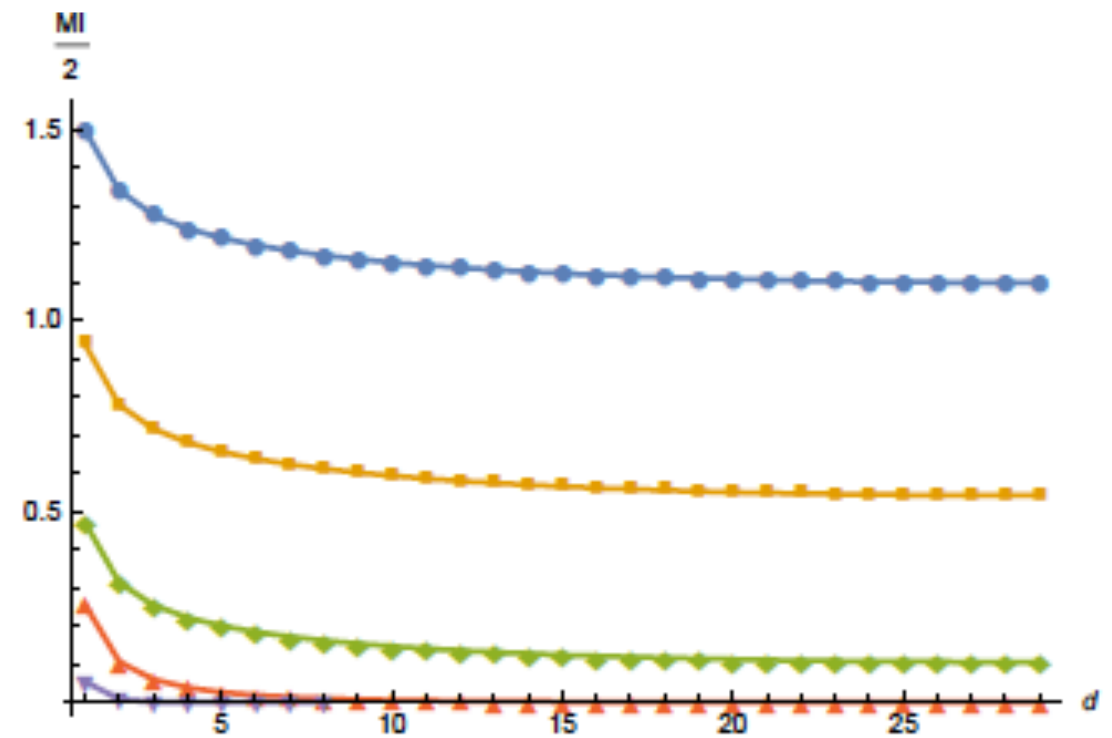


$(|A|, |B|) = (1, 2)$ $(|A|, |B|) = (2, 2)$

From this it is easy to confirm EoP is extensive

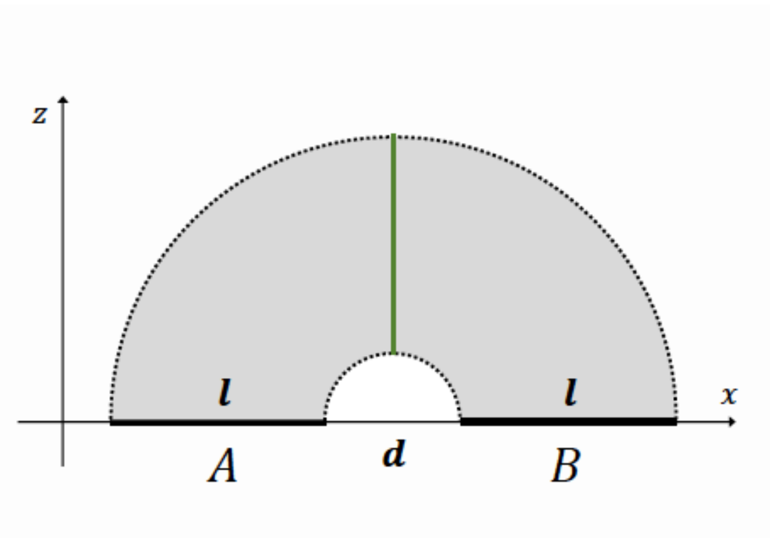
$$(|A|,|B|) = (2,2)$$

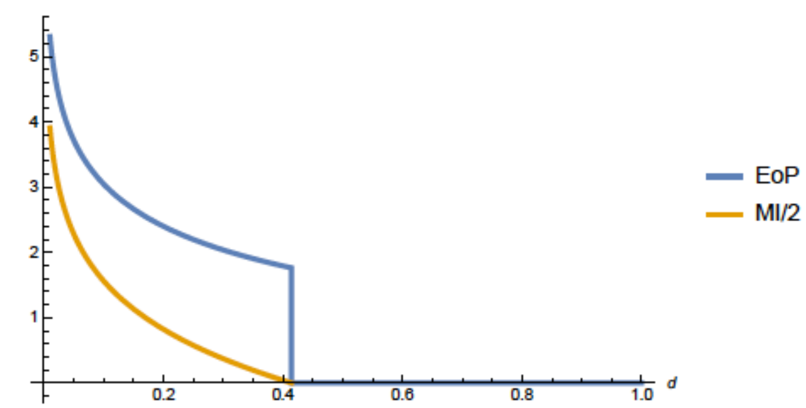




We observe that unlike Mutual information EoP remains almost same for d=1 and d=2

This looks qualitatively very similar to what happens in holography





$$E_W(A:B) = \begin{cases} \frac{c}{6}\log\left[1+\frac{2l}{d}\right], & d < (\sqrt{2}-1)l, \\ 0 & d > (\sqrt{2}-1)l. \end{cases}$$

For our case $d_c = 1$ very close (consistent) to the holographic result.

Various Inequalities

Desirable Features of Measures of Quantum entanglement for mixed states:

(Koashi winter '03, Cornelio, de-Oliveira '09)

Monogamy (tripartite state): $E_{\#}(A:B_1B_2) \geq E_{\#}(A:B_1) + E_{\#}(A:B_2)$

Strong superadditivity: $E_{\#}(A_1A_2:B_1B_2) \geq E_{\#}(A_1:B_1) + E_{\#}(A_2:B_2)$

If Monogamy is satisfied then Strong superadditivity automatically satisfies

Holographic mutual information always satisfies monogamy
thus strong supperaddivity

(Hayden, Headrick, Maloney '11)

Holographic EoP always satisfies *strong supperaddivity*
but for generic quantum states it violates

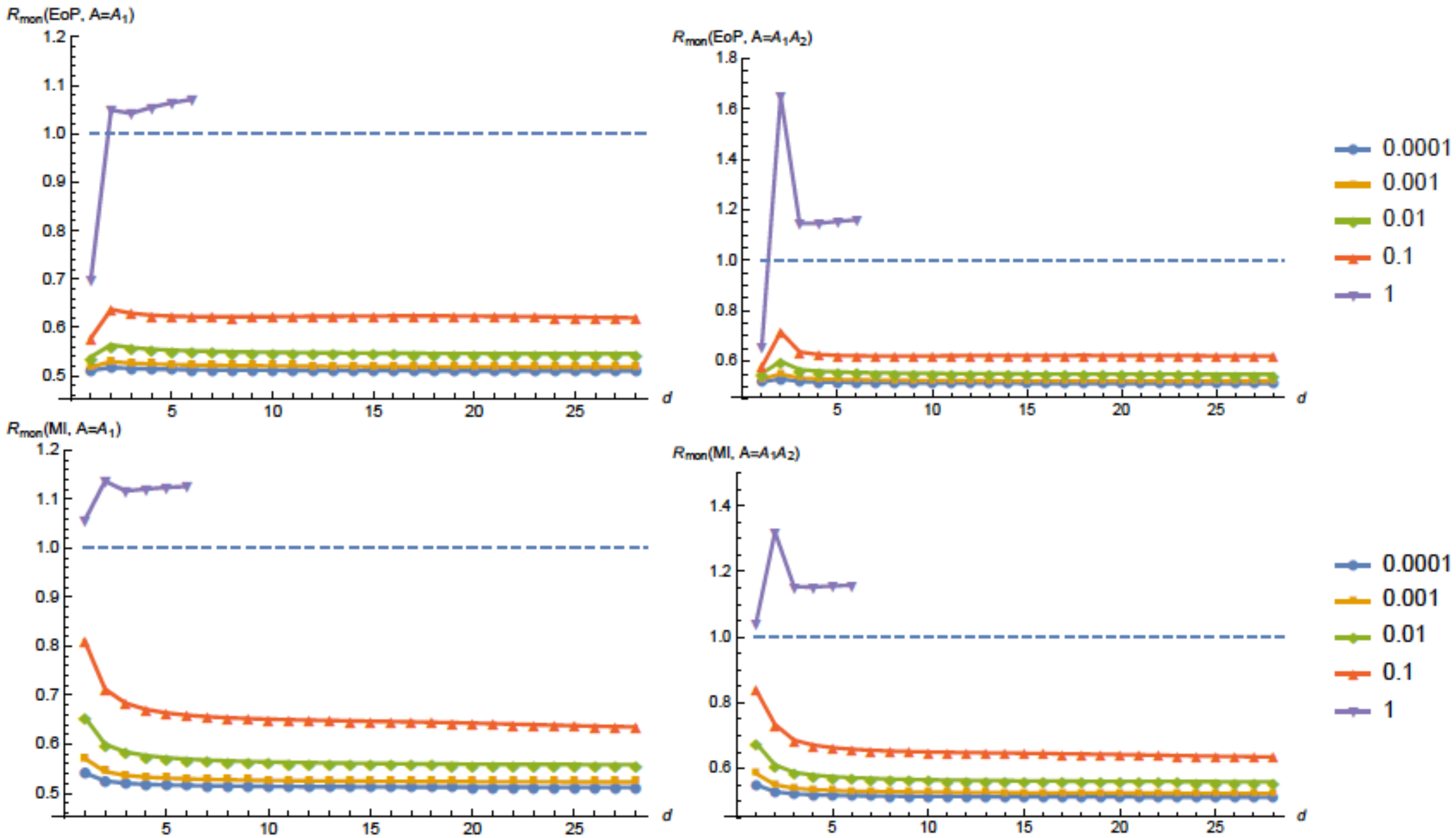
For pure tripartite: EoP satsfies "*Polygamy*" relation

$$E_P(A:B_1B_2) \leq E_P(A:B_1) + E_p(A:B_2)$$

EoP: Monogamy/Polygamy

$$R_{mon} = \frac{E_P(A:B_1B_2)}{E_P(A:B_1) + E_P(A:B_2)}$$

(AB, Takayanagi, Umemoto, *arXiv:1802.09545*)



Bounded from below $R_{mon} \geq \frac{1}{2}$ *(our result is consistent with this)*

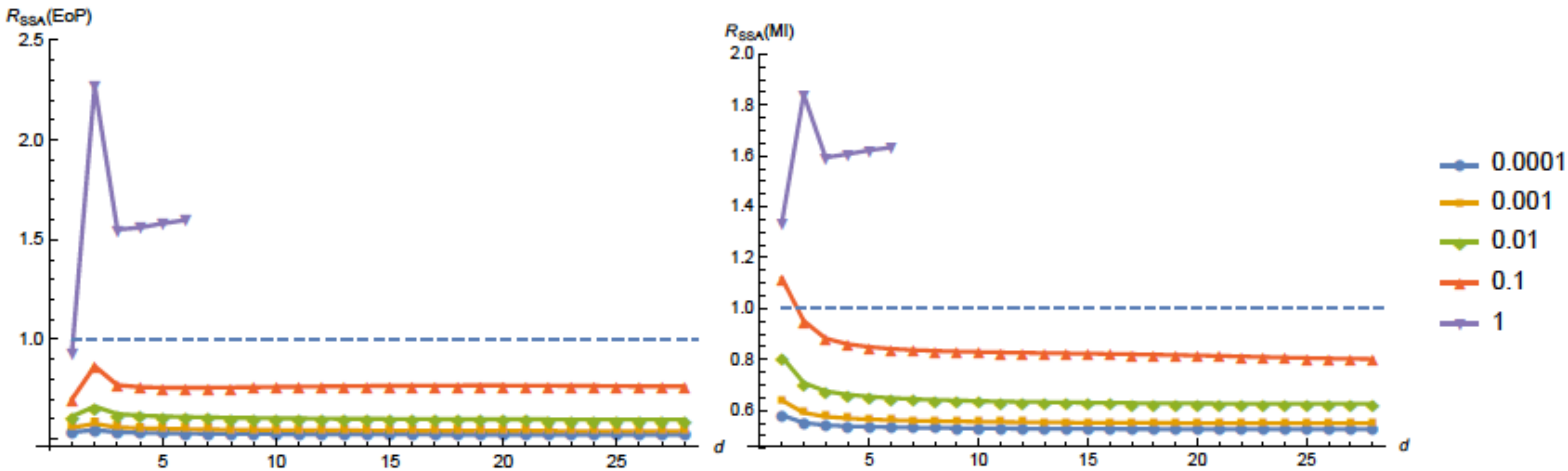
It violates strictly monogamy rather it satisfies polygamy $R_{mon} < 1$

except for m=1 it satisfies "monogamy"

EoP: Strong Superadditivity

$$R_{SSA} = \frac{E_P(A_1A_2 : B_1B_2)}{E_P(A_1 : B_1) + E_P(A_2 : B_2)}$$

(AB, Takayanagi, Umemoto, *arXiv:1802.09545*)



Bounded from below $R_{SSA} \geq \frac{1}{2}$ *(our result is again consistent with this)*

It strictly violates SSA rather it satisfies polygamy $R_{SSA} < 1$

except for m=1 it satisfies this shows self consistency of our numerics

Numerical Evidence for Minimal Ansatz

Consider first $|A| = |B| = 1$

Consider a bigger anstaz $|\tilde{A}| = |\tilde{B}| = 2$

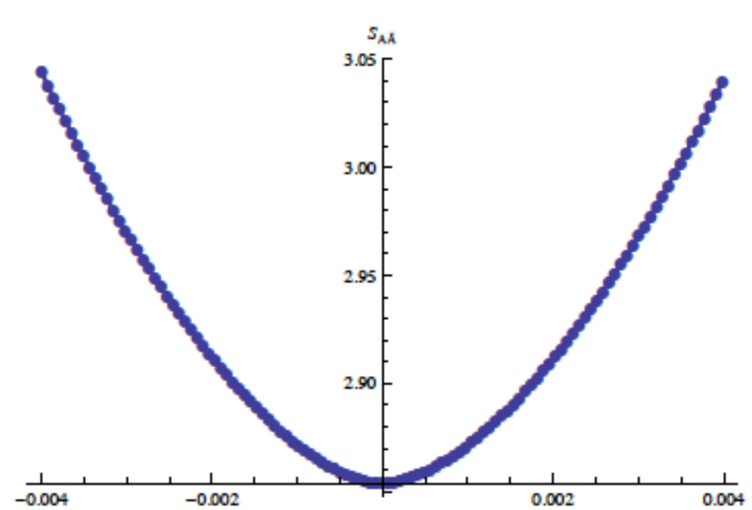
$$W_{A\tilde{A}B\tilde{B}} = \begin{pmatrix} J_{AA} & K_{A\tilde{A}_1} & K_{A\tilde{A}_2} & J_{AB} & K_{A\tilde{B}_1} & K_{A\tilde{B}_2} \\ K_{\tilde{A}_1 A} & L_{\tilde{A}_1\tilde{A}_1} & L_{\tilde{A}_1\tilde{A}_2} & K_{\tilde{A}_1 B} & L_{\tilde{A}_1\tilde{B}_1} & L_{\tilde{A}_1\tilde{B}_2} \\ K_{\tilde{A}_2 A} & L_{\tilde{A}_2\tilde{A}_1} & L_{\tilde{A}_2\tilde{A}_2} & K_{\tilde{A}_2 B} & L_{\tilde{A}_2\tilde{B}_1} & L_{\tilde{A}_2\tilde{B}_2} \\ J_{BA} & K_{B\tilde{A}_1} & K_{B\tilde{A}_2} & J_{BB} & K_{B\tilde{B}_1} & K_{B\tilde{B}_2} \\ K_{\tilde{B}_1 A} & L_{\tilde{B}_1\tilde{A}_1} & L_{\tilde{B}_1\tilde{A}_2} & K_{\tilde{B}_1 B} & L_{\tilde{B}_1\tilde{B}_1} & L_{\tilde{B}_1\tilde{B}_2} \\ K_{\tilde{B}_2 A} & L_{\tilde{B}_2\tilde{A}_1} & L_{\tilde{B}_2\tilde{A}_2} & K_{\tilde{B}_2 B} & L_{\tilde{B}_2\tilde{B}_1} & L_{\tilde{B}_2\tilde{B}_2} \end{pmatrix}$$

Minimal ansatz is contained in this $|\tilde{A}| = |\tilde{B}| = 1$

$$W^{min}_{A\tilde{A}B\tilde{B}} = \begin{pmatrix} J_{AA} & K_{A\tilde{A}_1} & 0 & J_{AB} & K_{A\tilde{B}_1} & 0 \\ K_{\tilde{A}_1 A} & L_{\tilde{A}_1\tilde{A}_1} & 0 & K_{\tilde{A}_1 B} & L_{\tilde{A}_1\tilde{B}_1} & 0 \\ 0 & 0 & L_{\tilde{A}_2\tilde{A}_2} & 0 & 0 & L_{\tilde{A}_2\tilde{B}_2} \\ J_{BA} & K_{B\tilde{A}_1} & 0 & J_{BB} & K_{B\tilde{B}_1} & 0 \\ K_{\tilde{B}_1 A} & L_{\tilde{B}_1\tilde{A}_1} & 0 & K_{\tilde{B}_1 B} & L_{\tilde{B}_1\tilde{B}_1} & 0 \\ 0 & 0 & L_{\tilde{B}_2\tilde{A}_2} & 0 & 0 & L_{\tilde{B}_2\tilde{B}_2} \end{pmatrix}.$$

$\tilde{A}_2, \tilde{B}_2$ decouples from $A, B, \tilde{A}_1, \tilde{B}_1$

We expand $W_{A\tilde{A}B\tilde{B}}$ around the minimal ansatz value always found the answer is greater than the minimal ansatz value



Conclusion and Outlook

- We have initiated a study of EoP in quantum field theory and checked its various property

- Extend it for fermion theory : Probably for that we can go beyond Gaussian approximation

- We have been restricted by small system size. Need to use more efficient techniques to overcome this. Use tensor network?

- How to define it for CFT? (Tamaoka's talk)

- Use path integral approach ?

Many more

Thank you

Mutual Information

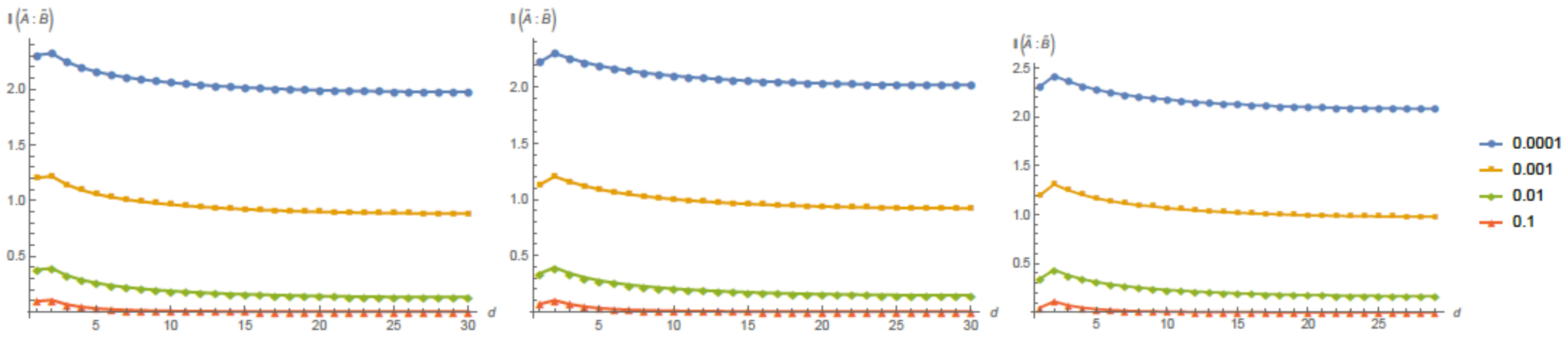
We compute various types of Mutual information

$$S_{A\tilde{A}} = \frac{1}{2}I(A\tilde{A}:B\tilde{B}) = \frac{1}{2}I(A:B\tilde{B}) + \frac{1}{2}I(\tilde{A}:B\tilde{B}) \geq \frac{1}{2}I(A:B) + \frac{1}{2}I(\tilde{A}:\tilde{B})$$

$$I(\tilde{A}:\tilde{B})_{hol} = 0$$

It seems for holography the minimization procedure is realized maximally

$$I(\tilde{A}:\tilde{B}):$$



shows peak at d=2, this is consistent with the earlier observation that

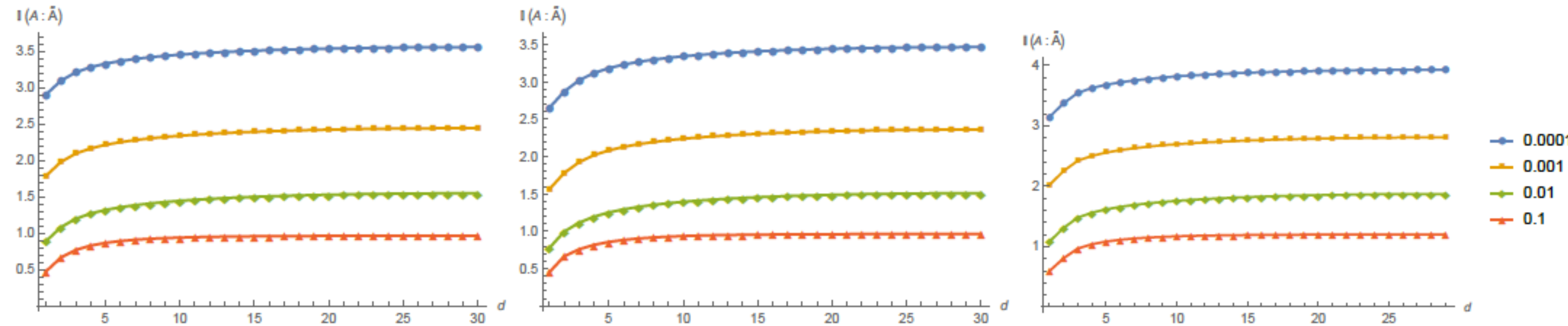
at d=2 there is vacant site between "A" and "B"

The correlation between **A** and $\tilde{A}$ and **B** and $\tilde{B}$ gets enhanced

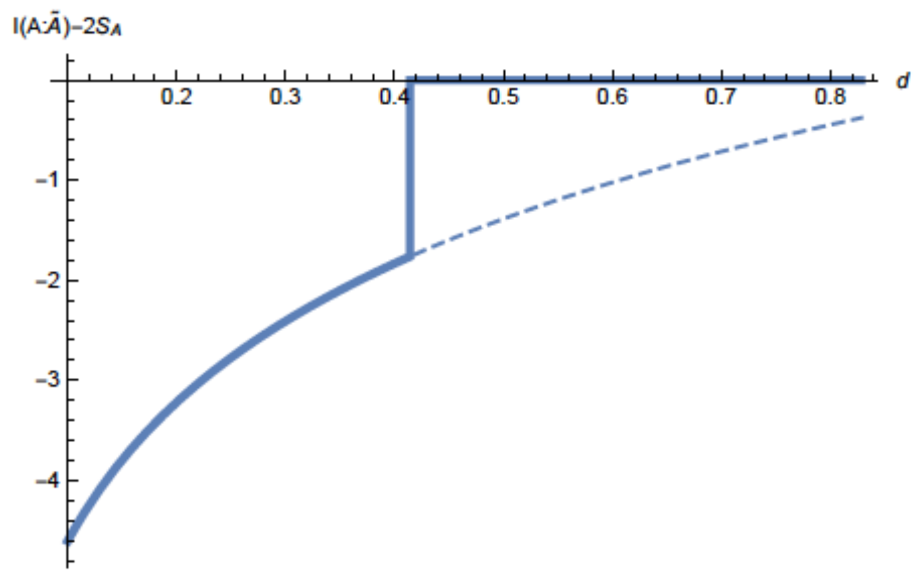
Mutual Information: Continued

$I(A:\tilde{B}):$ shows the same behaviour

$I(A:\tilde{A}):$



Holography



because of the small system size we donot observe this phase transitions