# Path Integral Complexity in (2d) quantum field theories

(some further developments)

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Holography, Quantum Entanglement and Higher Spin Gravity II, YITP 15.03.2018

# Based on:

1. AdS from Optimization of Path Integrals in Conformal Field Theories
   PRL 119 (2017) 7, 071602

2. Liouville Action as Path-Integral Complexity: From cTN to AdS/CFT
   JHEP11 (2017) 097

with Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi, Kento Watanabe (YITP)

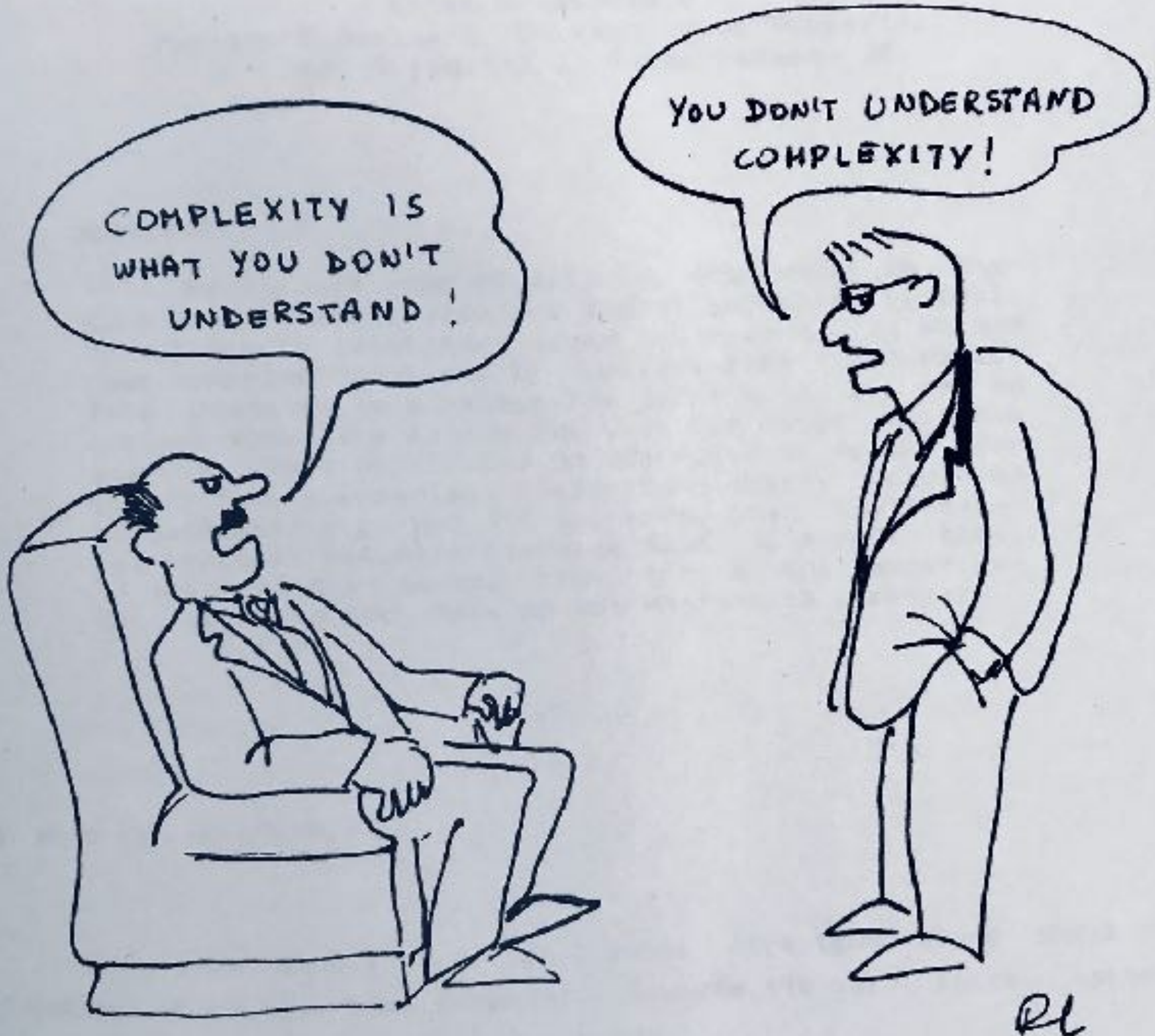
3. Path Integral Complexity and deformations

with Sumit Das (Kentucky U.), Arpan Bhattacharyya, Nilay Kundu, Masamichi Miyaji, Tadashi Takayanagi (YITP)

4. CFTs with W symmetries and Toda Action, work in progress

## Plan

- Introduction+Motivation (KENTO's TALK)
- Optimization and Liouville Action as Path Integral Complexity (KENTO)
- Deformations
- Symmetries and Toda Action
- Conclusions




A Scientific Discussion

## “Complexity” of states in QFT?

- “Hidden” information about the QFT wave functional
- Interacting QFT? Ansatz: MPS, MERA, AdS/CFT?
- Entanglement: Path Integrals and the replica trick!
- “Complexity” (Define a quantity through PI~Complex.)?

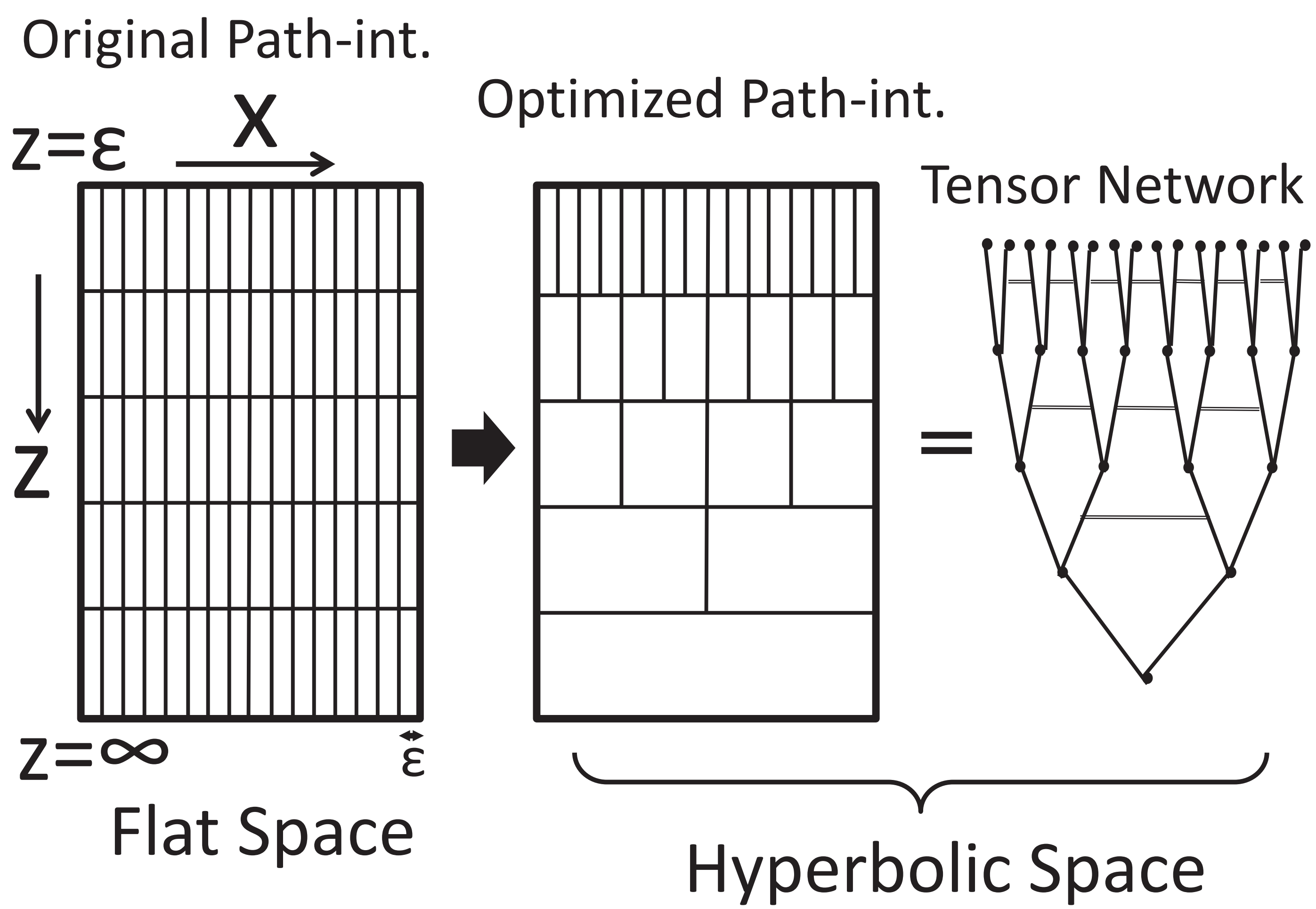
The basic tool to “define/compute” wave functions in QFT is the Euclidean PI

$$\Psi[\varphi_0(x)] = \int_{\varphi(0,x)=\varphi_0(x)} D\varphi\, e^{-S_E}$$

How can we optimize it and/or quantify its complexity?
Which notion of complexity is meaningful for Path Integrals…?

# IDEA (cTN):

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]




## Optimization:

$$\frac{\Psi_g}{\Psi_{flat}} = e^{I_\Psi[g]}$$

Minimize "Path Integral Complexity"

## 2D CFTs and Liouville

[PC,N.Kundu,M.Miyaji,T.Takayanagi,K.Watanabe '17]

Background metric for path integral $z = -\tau$

$$ds^2 = e^{2\phi(z,x)}(dz^2 + dx^2)$$

Once we introduce the background metric

$$[D\varphi]_{g_{ab}=e^{2\phi}\delta_{ab}} = e^{S_L[\phi]-S_L[0]} \cdot [D\varphi]_{g_{ab}=\delta_{ab}}$$

With the Liouville action

[Polyakov'81]

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx \int_{\epsilon}^{\infty} dz \left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu e^{2\phi}\right]$$

The wave functional is

$$\Psi_{g_{ab}=e^{2\phi}\delta_{ab}}(\tilde{\varphi}(x)) = e^{S_L[\phi]-S_L[0]} \cdot \Psi_{g_{ab}=\delta_{ab}}(\tilde{\varphi}(x))$$

# cTN Optimization <=> Minimizing PI complexity

Optimized metric satisfies Liouville equation with the appropriate b.c.

$$4\partial_w \partial_{\bar{w}} \phi = e^{2\phi} \qquad e^{2\phi(z=\epsilon,x)} = 1/\epsilon^2$$ Original cut-off

Solutions: surfaces of constant curvature (Liouville equation)

$$R\left[e^{2\phi}\hat{g}\right] = e^{-2\phi}\left(R\left[\hat{g}\right] - 2\hat{\nabla}\phi\right) = -\frac{\mu}{2}$$

A general solution:

$$e^{2\phi} = \frac{4A'(w)B'(\bar{w})}{(1 - A(w)B(\bar{w}))^2} \qquad \begin{aligned} w &= z + ix \\ \bar{w} &= z - ix \end{aligned}$$

Optimized metrics agree with time-slice of dual AdS/CFT solutions.

# Liouville Action as Path Integral Complexity

$$I_L[e^{2\phi}g,g] = \frac{c}{24\pi}\int_{\mathcal{M}} d^2x\sqrt{g}\Big[g^{ab}\partial_a\phi\partial_b\phi + (e^{2\phi}-1) + R_g\phi\Big] + \frac{c}{12\pi}\int_{\partial\mathcal{M}} ds\sqrt{h}K_g\phi.$$

Features (?) :

1) Chain Rule

$$I_L[g_1,g_2] + I_L[g_2,g_3] = I_L[g_1,g_3]$$

2) Continuous TN

$$S_L[\phi] = \frac{c}{24\pi}\int_{-\infty}^{\infty} dx\int_{\epsilon}^{\infty} dz\,\left[(\partial_x\phi)^2 + (\partial_z\phi)^2 + \mu e^{2\phi}\right]$$

Curvature

(~Number of Isometries [Czech'17])

Volume

(~Number of tensors)

# Generalizations I: Deformations

# Deformations

[PC, A.Bhattacharyya, S.R.Das, N.Kundu, M.Miyaji,T.Takayanagi,'18]

Setup (d=2) and a class of deformations

$$S = S_{CFT}(\varphi) + \epsilon^{d-\Delta}\int dx^{d-1}dz\sqrt{g}\lambda(x,z)O(x,z)$$

1. Choose the background metric:

$$ds^2 = e^{2\phi(x,z)}(dz^2 + dx^a dx^a) \qquad\qquad e^{2\phi(z=\epsilon,x)} = 1/\epsilon^2$$

2. We choose the coupling

[Osborn'91]

$$\lambda(x,z) = \lambda(\lambda_0, \phi(x,z)) \qquad\qquad \lambda(z=\epsilon) = \lambda_0$$

such that the deformed action is still invariant.

3. We claim that

$$\Psi^{\lambda_\phi}_{g=e^{2\phi}}[\varphi(x)] = e^{N[\phi,\lambda_\phi]-N[\phi_0,\lambda_0]}\cdot\Psi^{\lambda_0}_{g=e^{2\phi_0}}[\varphi(x)]$$

Path Integral Complexity for the deformed model

# Deformations: Free scalar

$$S = \frac{1}{2}\int dxdz\sqrt{g}\partial_a\varphi\partial_a\varphi + \frac{1}{2}\int dxdz\sqrt{g}\lambda(x,z)\varphi(x,z)^2$$

with coupling

$$\lambda_\phi(x,z) = \lambda_0 e^{-2\phi(x,z)}$$

As usually we write

$$\varphi(x,z) = \bar{\varphi}(x,z) + \eta(x,z)$$

And derive the proportionality factor

$$\Psi^{\lambda_\phi}_{g=e^{2\phi}}[\varphi(x)] = e^{N[\phi,\lambda_\phi]-N[0,\lambda_0]} \cdot \Psi^{\lambda_0}_{g=e^{2\phi_0}}[\varphi(x)]$$

with determinant that depends on the background field

$$e^{N[\phi,\lambda_\phi=\lambda_0 e^{-2\phi}]} = \int \prod_{x,z}[D\eta(x,z)] e^{-\frac{1}{2}\int dxdz\left[(\partial\eta)^2+\lambda_0\eta^2\right]}\Bigg|_{\eta(x,z=0)=0}$$

# Deformations: Perturbative

We can compute the correction to the Liouville action using conformal perturbation with the (universal) correlators on UHP (wave function)

$$\langle O(x,z)\rangle = e^{-\Delta\phi(x,z)} \cdot \frac{1}{\left(z^2 + e^{-\phi(x,z)}\right)^{\Delta/2}}$$

$$\langle O(x_1,z_1)O(x_2,z_2)\rangle = \frac{e^{-\Delta\phi(x_1,z_1)-\Delta\phi(x_2,z_2)}}{\left(|x_{12}|^2 + |z_{12}|^2 + e^{-\phi(x_1,z_1)-\phi(x_2,z_2)}\right)^{\Delta}}$$

Position dependent cut-off

[Polyakov '81]

Keeping the leading terms in $\epsilon \to 0$

$$N[\phi,\lambda_\phi] = S_L[\phi] + \int dxdz \sum_{n=0}^{\infty} N_n (\lambda_0)^{n+2} e^{(2+(\Delta+2)(n+2))\phi}$$

# Deformations: Perturbative

After a careful treatment of on-point functions we derive a “universal” form of the Deformed Complexity Action

$$C[\lambda_0] = N[\phi, \lambda_\phi] + N_{1pt}[\phi, \lambda_\phi]$$

$$= \frac{c}{24\pi}\int dx dz\left[(\partial\phi)^2 + e^{2\phi} + \lambda_0^2 e^{(2\Delta-2)\phi}\right] - b\epsilon^{1-\Delta}\int dx \lambda_0 - \frac{1}{2}b^2\epsilon^{2-2\Delta}\left(\int dx\lambda_0\right)^2 \dots$$

From that, for constant coupling (neglecting derivatives) we can evaluate the corrections to the optimized metric

$$e^{\phi(z)} = z^{-1}\left(1 - \frac{\lambda_0^2}{2(5-2\Delta)}z^{-2\Delta+4} + \cdots\right)$$

What about geometry? Time slice of AdS/CFT geom?

# Deformations: Perturbative gravity

Consider the holographic setup

$$S = \frac{1}{2\kappa}\int d^3x\sqrt{-g}\left[R - \Lambda - \frac{1}{2}(\partial\Phi)^2 - \frac{1}{2}m^2\Phi^2\right]$$

With mass related to $\Delta = 1 + \sqrt{1+m^2}$ and $\Phi(z,x) = z^{d-\Delta}\lambda_0(x) + z^{\Delta}\langle O(x)\rangle$

To the leading order, we identify the scalar with our coupling

$$\Phi(z,x) \simeq \lambda_0(x) e^{(\Delta-2)\phi(z,x)} + O(\lambda_0^2)$$

and look for perturbative solutions [Hung,Myers,Smolkin'11]

$$ds^2 = \frac{1}{y^2}\left(dy^2 + f(z)(-dt^2 + dx^2)\right),$$

$$f(z) = 1 - \frac{\lambda_0^2}{4}z^{4-2\Delta} + \sum_{k=1}^{\infty} a_k\left(\lambda_0 z^{2-\Delta}\right)^{k+2}$$

Similar to the modified action…

# Deformations: Gravity

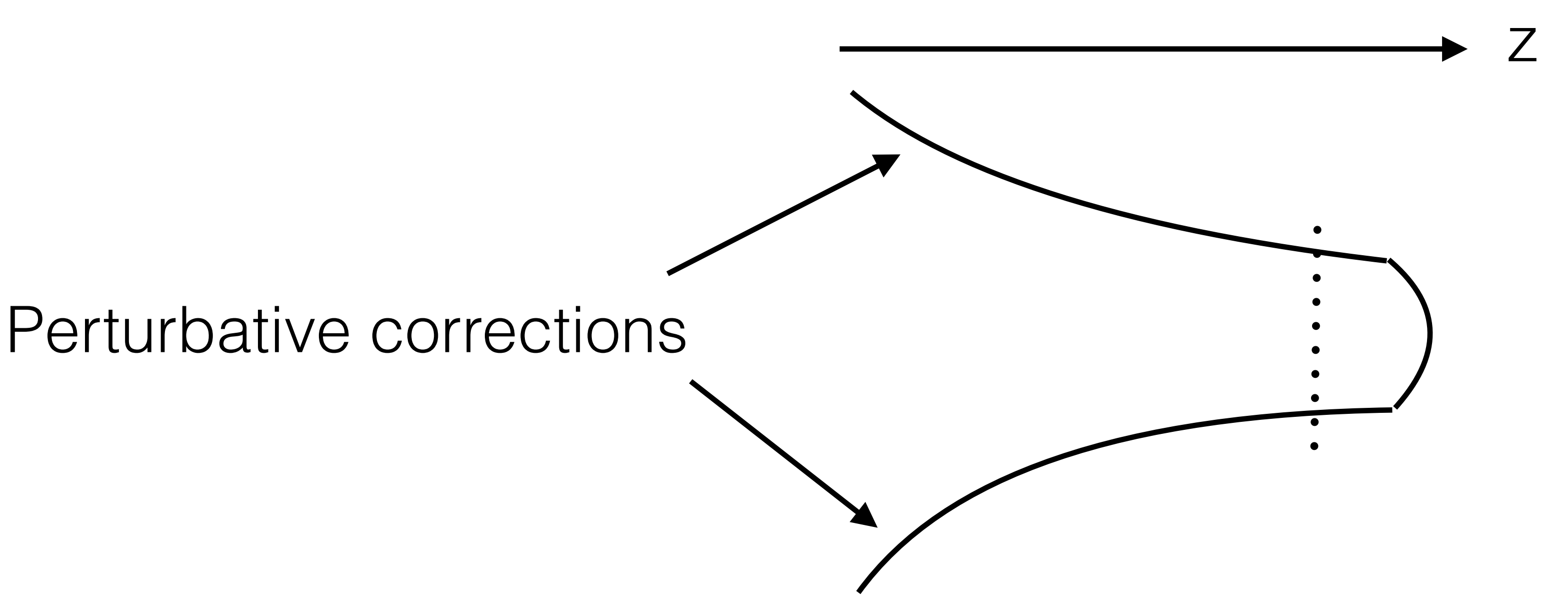
Taking the time-slice of the perturbative solution and an ansatz:

$$ds^2 = e^{2\phi(z)}(dz^2 + dx^2)$$

we find again that metric becomes

$$e^{\phi(z)} \simeq z^{-1}\left(1 - \frac{2-\Delta}{5-2\Delta}\lambda_0^2 z^{4-2\Delta}\right)$$

So this agrees with our perturbative computation (up to numerical factor dep. on reg etc).




We expect the geometry to be capped at IR (non-perturbative, kinetic term..)

# Generalizations II: Symmetries

# Extended Symmetries

How to generalise our prescription to CFTs with extended symmetries (Potts model, W algebras, etc.) ?

$$H = -J\sum_a \left(\hat{\sigma}^\dagger_{a+1}\hat{\sigma}_a + \hat{\sigma}^\dagger_a\hat{\sigma}_{a+1}\right) - f\sum_a \left(\hat{\tau}^\dagger_a + \hat{\tau}_a\right)$$

At each site we have 3 possible states A,B,C ( $(\mathbb{C}^3)^{\otimes L}$) and

$$\hat{\sigma} = |A\rangle\langle A| + \omega|B\rangle\langle B| + \omega^2|C\rangle\langle C| = \begin{pmatrix} 1 & & \\ & \omega & \\ & & \omega^2 \end{pmatrix} \qquad \omega = e^{2\pi i/3}$$

$$\hat{\tau} = |B\rangle\langle A| + |C\rangle\langle B| + |A\rangle\langle C| = \begin{pmatrix} & & 1 \\ 1 & & \\ & 1 & \end{pmatrix}.$$

$$\hat{\sigma}_a^3 = 1\ , \qquad \hat{\tau}_a^3 = 1\ , \qquad \hat{\sigma}_a\hat{\tau}_a = \omega\hat{\tau}_a\hat{\sigma}_a\ , \qquad \sigma_a\tau_b = \tau_b\sigma_a \text{ for } a \neq b.$$

$\mathbb{Z}_3$ symmetry   we can cyclically permute A->B->C at each site

Critical point (J=f) described by the simplest CFT with spin 3 currents W, $\bar{\text{W}}$

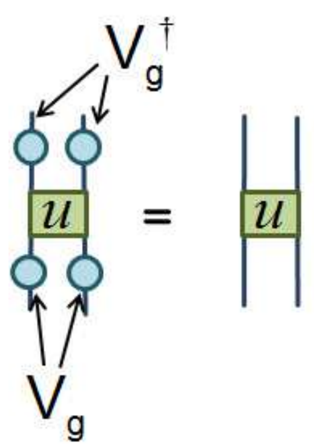
# Extended Symmetries

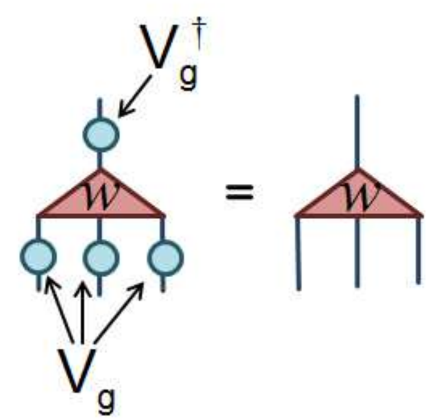
1) MERA with symmetries $\Gamma_g \; H \; \Gamma_g^{\dagger} = H, \quad \forall g \in \mathcal{G}$ [Vidal, Singh, Pfeifer.'11]

$\Gamma_g \equiv \cdots V_g \otimes V_g \otimes V_g \cdots$ unitary reps. $V_g$

States are invariant: $\Gamma_g |\psi\rangle \; = \; |\psi\rangle$

Tensors satisfy extra constraints (block diagonal in a certain basis)




More constraints=Lower computational cost!

Important question: States with non-trivial properties under the symmetry action?
Primary operators with W-charge and TN?

Hints for PI optimization.

# Step back: Liouville optimization

$$4\partial_w\partial_{\bar w}\phi(w,\bar w) = e^{2\phi(w,\bar w)} \qquad e^{2\phi} = \frac{4A'(w)B'(\bar w)}{(1-A(w)B(\bar w))^2} \qquad e^{2\phi(z=\epsilon,x)} = 1/\epsilon^2$$

Equivalently we can write this equation as

$$\begin{aligned}\partial_w^2 e^{-\phi(w,\bar w)} &= T(w)e^{-\phi(w,\bar w)}\\ \partial_{\bar w}^2 e^{-\phi(w,\bar w)} &= \bar T(\bar w)e^{-\phi(w,\bar w)}\end{aligned}$$

Two-holomorphic functions

$$\partial_w \bar T(\bar w) = 0, \qquad \partial_{\bar w} T(w) = 0$$

Given by

$$\begin{aligned}T(w) &= \frac{1}{2}\left[\partial_w^2\phi - (\partial_w\phi)^2\right] = \{A(w), w\}\\ \bar T(\bar w) &= \frac{1}{2}\left[\partial_{\bar w}^2\phi - (\partial_{\bar w}\phi)^2\right] = \{B(\bar w), \bar w\}\end{aligned}$$

Mathematically: Specifying them + bdr. cond.=Our optimized background

# A natural candidate: Toda action

$$\mathcal{A}_{TFT} = \int \left( \frac{1}{8\pi} \hat{g}^{ab} (\partial_a \varphi, \partial_b \varphi) + \frac{(Q,\varphi)}{4\pi} \hat{R} + \mu \sum_{k=1}^{n-1} e^{b(e_k,\varphi)} \right) \sqrt{\hat{g}}\, d^2x$$

$\varphi = (\varphi_1 \ldots \varphi_{n-1})$ $\qquad K_{ij} = (e_i, e_j)$ (Cartan matrix)

$e_i$ simple roots of sl(n)

For sl(3) we can write the equations in a form

$$\left( -\partial^3 + \frac{1}{2} \partial \mathsf{T} + \mathsf{T} \partial + \mathsf{W} \right) e^{-\Phi_1} = 0$$
$$\left( -\bar{\partial}^3 + \frac{1}{2} \bar{\partial} \bar{\mathsf{T}} + \bar{\mathsf{T}} \bar{\partial} + \bar{\mathsf{W}} \right) e^{-\Phi_1} = 0$$

T,W
determine the opt.
geometry

Similar for the second field.

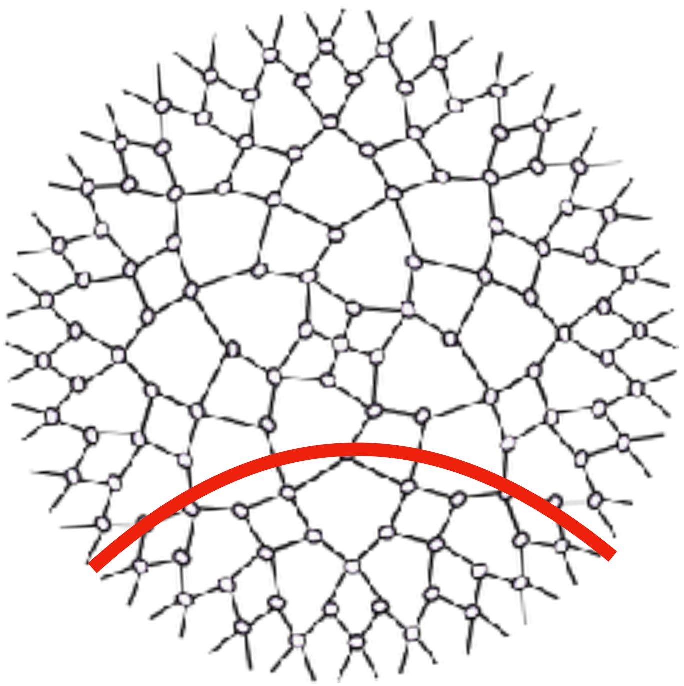
Our setup? B.C.?
Classical W-geom: Embedding into to CP^n

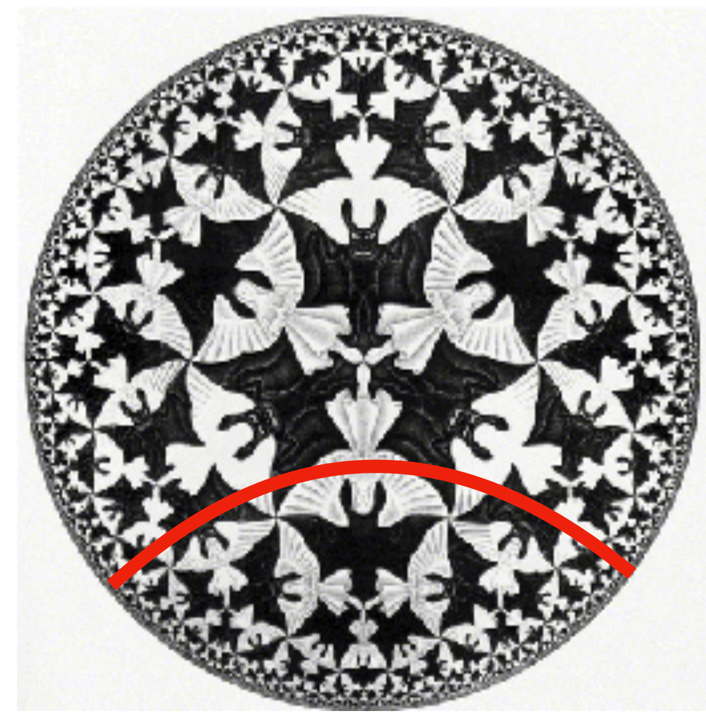
[Gervais & Matsuo….]

# TN and RT

[Swingle ’12]

Time slice of AdS?





RT formula in tensor networks!?

Driving force behind AdS/TN developments

[Harlow-Pastawski-Preskill-Yoshida,Hayden-Nezami-Qi-Thomas-Walter-Yang, Pastawski-Preskill, Evenbly, Czech….]

# Entanglement Entropy and cTN

Entanglement entropy is reproduced from the following formula (~RT on cTN)

$$S_A = \frac{c}{6}\int_{\partial\Sigma_+} e^{\phi} ds \qquad\qquad ds^2 = e^{2\phi(w,\bar{w})} dw d\bar{w}$$

Integral is a “Radon transform” and for a general solution yields (single interval)

$$S_l = \frac{c}{12}\log\left(\frac{(f(w_1)-f(w_2))^2}{f'(w_1)f'(w_2)\epsilon^2}\right) + \frac{\bar{c}}{12}\log\left(\frac{(g(\bar{w}_1)-g(\bar{w}_2))^2}{g'(\bar{w}_1)g'(\bar{w}_2)\epsilon^2}\right)$$

After setting w=iu, w=iv, A=B and EE satisfies (continued) Liouville equation

$$\partial_u\partial_v\left(-\frac{6}{c}S_A\right) = -\frac{1}{\epsilon^2}e^{2\left(-\frac{6}{c}S_A\right)}$$

[de Boer,Heller,Hael,Myers’16]
[Czech’17]

# Higher spin entanglement entropies

[de Boer Jottar; Ammon,Castro,Iqbal'13]

From the generalization of RT to Chern-Simons with higher SL(n,R)xSL(n,R)

$$S_R(P,Q) = c_R \log \mathrm{Tr}_R(\bar{\mathcal{W}}(Q,P)\,\mathcal{W}(P,Q))$$

n=3, principal embedding

$$A = g^{-1}dg = \begin{pmatrix} 0 & 1 & 0 \\ T_1 & 0 & 1 \\ W & T_2 & 0\,, \end{pmatrix}$$

Define spin 2 and spin 3 EE

[Kraus et.al…..]

$$\begin{aligned} S^{(2)}_{\mathrm{EE}} &= \mathcal{S}_{\mathrm{adj}}\,, \\ S^{(3)}_{\mathrm{EE}} &= \mathcal{S}_{\mathrm{fun}} - \frac{1}{2}\mathcal{S}_{\mathrm{adj}} \end{aligned}$$

Then one can show SL(3,R) Toda [de Boer,Heller,Hael,Myers'16]

$$\epsilon_{\rm fun}\,\frac{\partial S^{(2)}_{\rm EE}}{\partial u\partial v} = 2\,c_{\rm fun}\,e^{-S^{(2)}_{\rm EE}/(2c_{\rm fun})}\cosh\left(3S^{(3)}_{\rm EE}/c_{\rm fun}\right)$$
$$\epsilon_{\rm fun}\,\frac{\partial S^{(3)}_{\rm EE}}{\partial u\partial v} = -c_{\rm fun}\,e^{-S^{(2)}_{\rm EE}/(2c_{\rm fun})}\,\sinh\left(3S^{(3)}_{\rm EE}/c_{\rm fun}\right)$$

This "supports" the story with Toda and PI optimization and continuation.

How do we compute h.s. EE from the Toda field?

## Conclusions

- A new proposal for AdS/(c)TN and “PI Complexity”
- Classical geometries from Minimization of PI Complexity.
- In CFT our proposal applies to arbitrary central charge!
- Progress for deformations and massive theories
- Perturbative computations on the gravity side reproduce optimized metrics
- Complexity <=> Dynamics of Geometry (Gravity)
- W-symmetry and Toda action (still a lot to explore…)

## Open Questions

- Beyond Universality? Is our approach useful for many-body problems? (Fernando's Talk)
- Free CFTs and relation to cMERA? Path Integral vs "Unitary Gates"
- Geometry of networks with W-symmetry?
- Kinematic space with W-symmetry?
- Time dependent states !?

Thank You!