Mutual Information in Conformal Field Theories

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Question: **Mutual information** between two spacelike regions in CFTs?

\[ I_{A,B} = S_A + S_B - S_{A \cup B} \]

**Figure:** Entanglement of two spacelike regions in QFTs.

- \( d = 2 \) \( \checkmark \)
- \( d > 2 \) ?
From replica trick, Rényi entropy is equivalent to the expectation value of a twist operator inserted along the entangle surface $A$.

$$\text{tr}_{\mathcal{A}} \rho_{\mathcal{A}}^n = \frac{1}{Z(M_1)^n} \int_{M(n)} D\phi e^{-S(\phi)}$$

$$= \frac{1}{Z(M_1)^n} \prod_{j=1}^{n} \int_{M_j} D\phi_j T_n e^{-\sum_{j=1}^{n} S(\phi_j)}$$

$$= \langle T(A)_n \rangle .$$ (1)

- Rényi entropy

$$S_{A}^{(n)} = \log \frac{\langle T(A)_n \rangle}{1 - n} .$$ (2)

- Mutual Rényi information of two disjoint regions:

$$I_{A,B}^{(n)} = S_{A}^{(n)} + S_{B}^{(n)} - S_{A\cup B}^{(n)} = \frac{1}{1 - n} \log \frac{\langle T(A)_n \rangle \langle T(B)_n \rangle}{\langle T(A)_n T(B)_n \rangle} .$$ (3)
Twist operator $\mathcal{T}_n$

- Twist operator $\mathcal{T}_n$ is a **codimension 2** surface operator.
- Key point: Operator product expansion (OPE) of twist operator.
- In general, it is hard to find exact result because OPE is shape dependent.
- Focus on planer or spherical twist operator.

**Figure:** Correlation function of two disjoint twist operators.
• Twist operator breaks conformal symmetry

\[ so(d, 2) \rightarrow so(d - 1, 1) \times so(1, 1) \]

• Use **residue symmetry** to constrain the OPE of a twist operator \( \mathcal{T}_n \).

• **Constraints**: one point function

\[
\langle O(x) \rangle_n = \frac{\langle O(x) \mathcal{T}_n \rangle}{\langle \mathcal{T}_n \rangle} = a_{\Delta,J} \frac{T_J}{|x|^\Delta}
\]

is fixed by symmetry up to one coefficient.

• **Infinite** many of constraints, there should be contribution from each primary operator \( O_{\Delta,J} \) family.
Proposal: flat or spherical twist operator $\mathcal{T}_n$ in any parity invariant CFTs can be expanded as

$$\mathcal{T}_n = \langle \mathcal{T}_n \rangle \sum_{\Delta, J} c_{\Delta, J} Q[O_{\Delta, J}].$$

- $Q[O_{\Delta, J}] = \int_{D(A)} d^d x K^\mu_1 \cdots K^\mu_J |K|^{\Delta - J - d} O_{\mu_1 \cdots \mu_J}$
- $K^\mu$ is the modular hamiltonian generator of region $A$.
- $D(A)$ is the causal development of the defect operator.
- $c_{\Delta, J}$ is determined by one point function coefficient $a_{\Delta, J}$.
- $c_1 = 1$.

More generally, this is also the OPE of a codimension 2 flat or spherical defect operator $D$.
Some examples

\[ < O(x) >_D = a_{\Delta,0} \frac{1}{|x|^{\Delta}}, \]
\[ < O_a >_D = a_{\Delta,1} \frac{\epsilon_{ab} n^b}{|x|^{\Delta}}, \]
\[ < O_{ab} >_D = -a_{\Delta,2} \frac{(d - 1) \delta_{ab} - d n_a n_b}{|x|^{\Delta}}, < O_{ij} >_D = a_{\Delta,2} \frac{\delta_{ij}}{|x|^{\Delta}}, \]
\[ \ldots \]

with \( x^\mu = (x^a, y^i) \), \( n^a = \frac{x^a}{|x|} \).
For two spherical region, (Long, 2016)

\[ I_{A,B}^{(n)} = -\frac{1}{1-n} \log \frac{\langle T_n(A)T_n(B) \rangle}{\langle T_n(A) \rangle \langle T_n(B) \rangle} = \frac{1}{n-1} \log(1 + \sum_{O_{\Delta,J}} s_{\Delta,J} G_{\Delta,J}(u, v)). \]

where \( s_{\Delta,J} \sim c_{\Delta,J}^2, J \sim \frac{a_{\Delta,J}^2}{N_{\Delta,J}}, \) the summation is over all possible primary operators in n-replicated CFT.

\( n \to 1 \) limit, conformal block expansion of mutual information

Chen & Long(2016)

\[ I_{A,B} = \sum_{O_{\Delta,J}} b_{\Delta,J} G_{\Delta,J} \]

where \( b_{\Delta,J} = s'_{\Delta,J}(1) \).
Mutual Information

- Kinematic Information $G(u, v)$, conformal block, fixed by symmetry.
- Dynamical Information $a_{\Delta,J}$ & $b_{\Delta,J}$.
- Compute Mutual Rényi Information
  - Classify all possible primary operators in n-replicated CFT.
  - Determine one point function $< O_{\Delta,J} >_n \rightarrow a_{\Delta,J}$ in n-sheets manifold $M_n$ and normalization $N_{\Delta,J}$(Mutual Rényi Information)
Previous discussion depends on the knowledge \((a_{\Delta,J})\) of CFTs in manifold \(M_n\). Usually, it is a difficult task (theory dependent).

By definition, mutual information has nothing to do with \(M_n\).

There should be another way to determine mutual information, given the knowledge of a CFT (spectrum, 3pt).

We need to study \((a_{\Delta,J})\) and correlation functions on \(M_n\) more carefully, especially constraints from

- Periodic boundary condition: \(\theta \sim \theta + 2\pi n\).
- \(\mathbb{Z}(n)\) Symmetry.
- so(2) residue symmetry
- Analytic property of \(a_{\Delta,J}\) around \(n = 1\).
Constraints for $a_{\Delta,j}(n)$ (Chen, Chen, Hao & Long, 2017)

- Single sheet operator $O^j_i$.
  - $a^j_{\Delta,j}$ is independent of $j$. ($\mathbb{Z}(n)$ Symmetry)
  - $a^j_{\Delta,j} = \mathcal{O}(n-1)$. ($a_{\Delta,j}(1) = 0$, well defined Taylor expansion).
  - No contribution to mutual information.
    Faulkner, Lewkowycz & Maldacena, 2013

- Multi-sheet operator $O^{j_1\cdots j_k}$.
  - $k \geq 2$, composite operators, such as $\phi^{j_1} \phi^{j_2}$, $J^{j_1j_2}$.
  - Their coefficients $a$ can be obtained from
    \[
    G_n(x_1, \cdots, x_k) = \langle O_1(x_1)O_2(x_2)\cdots O_k(x_k) \rangle_n
    \]  
    (4)

- $so(2)$ symmetry, rotation.
  \[
  G_n(x_1, \cdots, x_k) = G_n(\theta_{j_1j_2}, \cdots, \theta_{j_{k-1}j_k}), \theta_{ij} = \theta_i - \theta_j.
  \]  
  (5)
Leading contribution, bilinear operators $O^{j_1} O^{j_2}$,

$$a^{j_1 j_2} (n) \sim G_n(\theta_{j_1 j_2}). \quad (6)$$

Assume periodic boundary condition, $1/n$ prescription

$$G_n(\theta_j, 0) = G_1\left(\frac{\theta_j}{n}\right) + O(n - 1) \quad (7)$$

Identity

$$b \sim \lim_{n \to 1} \sum_{j=1}^{n-1} \frac{G_n^2(j)}{n - 1} = \lim_{n \to 1} \sum_{j=1}^{n-1} \frac{G_1^2\left(\frac{\theta_j}{n}\right)}{n - 1} \quad (8)$$

$G_1(x_1, x_2)$ is fixed by conformal symmetry.

Bilinear operators contribute universally to mutual information.
General formulas

- OPE of codimension two flat or spherical twist (defect) operators

\[ \mathcal{D} = \langle \mathcal{D} \rangle = \sum_{O_{\Delta, J}} c_{O_{\Delta, J}} Q[O_{\Delta, J}] \]

- Mutual Rényi Information from Conformal Block.

\[ I_{A,B}^{(n)} = \frac{1}{n-1} \log(1 + \sum_{O_{\Delta, J}} s_{\Delta, J} G_{\Delta, J}(u, v)) \]

with \( s_{\Delta, J} \sim \sum_{O_{\Delta, J}} \frac{a_{\Delta, J}^2}{N_{\Delta, J}} \).

- Mutual Information from Conformal Block.

\[ I_{A,B} = \sum_{O_{\Delta, J}} b_{\Delta, J} G_{\Delta, J}(u, v) \]

with \( b_{\Delta, J} = s'_{\Delta, J}(1) \).
Conclusion

- Develop $1/n$ prescription to determine $b_{\Delta,J}$.
- Mutual Information are contributed from multi-replica operators, no contribution from single replica operators.
- Universal behaviour from bilinear operators
- Leading order contribution should be universal.
- Leading order contribution is from a spin 1 bilinear operator for fermionic theory.
- Holographic duality: $b \sim \mathcal{O}(1)$, quantum correction of mutual information
- Higher spin operator is important for higher order correction
- Mutual information analytic results fits well with lattice computation for free boson and free fermion.
- May still be valid for large cross ratio (small distance).
- Multi regions can be discussed in parallel.
Open issues

- **Gauge theory**?
- **Mutual Rényi information** \( (n \neq 1) \)
  - we computed the first few orders for free boson, and reproduced Cardy’s result at large distances (leading order).
  - however, leading order result does **not** fit well with lattice computation. Three possible resolutions:
    - Check the OPE for general CFTs (any loop hole)?
    - Check the computation of \( a_n \) (Green's function, conical singularity correction)?
    - Improve the precision of lattice computation?
Thanks for your attention!
For free boson and free fermion, one can use lattice method to compute Entanglement entropy for any region. The framework is well established, I just report the numerical result and compare it with analytic result.

Two ball region:
- 3d free boson,

\[ I_{A,B} = \frac{1}{12} G_{1,0}^{d=3}(z) - \frac{1}{120} G_{2,1}^{d=3}(z) + \left( \frac{1}{360} + \frac{1}{12\pi^2} \right) G_{2,0}^{d=3}(z) + \cdots \]

Figure: Mutual information of two disks for 3d free boson (0 < z < 0.75).
Numerical analysis for 3d free boson

- Fits very well for small \( z \), large distance.
- Even though \( z \) is not too small, matches well.
- \( z > 0.55 \), discrepancy due to the contribution of other conformal blocks.
- Supports the proposal that mutual information can be expanded as summation of conformal blocks.
- Shed light on analytic computation of mutual information.

Figure: Mutual information of two disks for 3d free boson (\( 0 < z < 0.25 \)).
Analytic results:

\[ I_{A,B}^{d=4} = \frac{1}{60} G_{2,0}^{d=4} - \frac{1}{420} G_{3,1}^{d=4} + \frac{1}{840} G_{4,2}^{d=4} + \frac{1}{840} G_{4,0}^{d=4} + \cdots \]

**Figure:** Mutual information of two balls for 4d free boson (0 < z < 0.8).
Numerical analysis for 4d free boson

- Fits very well for a large range of $z$.

Figure: Mutual information of two balls for 4d free boson ($0 < z < 0.4$).

- The prediction for $d > 4$ dimension

$$I_{A,B}^{d>4} = \frac{4^{1-d} \sqrt{\pi} \Gamma [d-1]}{\Gamma [-\frac{1}{2} + d]} G_{d-2,0}^{d>4}(z) - \frac{4^{-d}(d-2) \sqrt{\pi} \Gamma [d-1]}{\Gamma [\frac{1}{2} + d]} G_{d-1}^{d>4} + 2^{-3-2d}(4-4d+3d^2)\sqrt{\pi} \Gamma [d-1] G_{d,2}^{d>4}(z) + \cdots$$
Numerical analysis for 3d free fermion

Chen, Chen, Hao & Long, 2017

\[ l_{d=3} = \frac{1}{15} G_{2,1}^{(d=3)}(z) + \cdots \]  \hspace{1cm} (10)

**Figure:** Mutual information of two disks for 3d free Dirac fermion. The red line is the leading order contribution in \( z \), while the green line is the leading order conformal block.
Numerical analysis for 4d free fermion

Chen, Chen, Hao & Long, 2017

\[ I_{d=4} = \frac{1}{35} G_{3,1}^{(d=4)}(z) + \cdots \]

Figure: Mutual information of two balls for 4d free Dirac fermion. The red line is the leading order contribution in \( z \), while the green line is the leading order conformal block.
First few orders

\[ I^{(n)}_{A,B} = \frac{1}{n - 1} \log(1 + \frac{n^4 - 1}{240n^3} G_{2,0}(z) + \frac{n^6 - 21n^2 + 20}{15120n^5} G_{3,1}(z) \]
\[ + \frac{-420 + 7n + 400n^2 - 14n^5 + 20n^8 + 7n^9}{806400n^7} G_{4,0}(z) \]
\[ + \frac{-21 + 20n^2 + n^8}{40320n^7} G_{4,2}(z) + \cdots \]

Leading order at large distance

\[ I^{(n)}_{A,B} = \frac{(n + 1)(n^2 + 1)}{240n^3} z^2 + \cdots \] (11)