Mutual Information in Conformal Field Theories

Jiang Long

based on arXiv: 1611.02485, 1612.00114,1704.03692

Université Libre de Bruxelles and International Solvay Institutes

March 15, 2018

- Question: Mutual information between two spacelike regions in CFTs?
- $I_{A,B} = S_A + S_B - S_{A \cup B}$

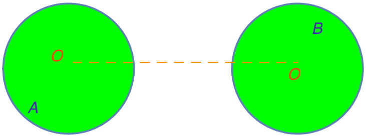

Figure: Entanglement of two spacelike regions in QFTs.

- $d = 2$ $\quad \surd$
- $d > 2$ $\quad ?$

Twist operator $\mathcal{T}_n$

- From replica trick, Rényi entropy is equivalent to the expectation value of a twist operator inserted along the entangle surface A.

$$
\begin{aligned}
tr_A \rho_A^n &= \frac{1}{Z(M_1)^n} \int_{M(n)} \mathcal{D}\phi e^{-S(\phi)} \\
&= \frac{1}{Z(M_1)^n} \prod_{j=1}^{n} \int_{M_j} \mathcal{D}\phi_j \mathcal{T}_n e^{-\sum_{j=1}^{n} S(\phi_j)} \\
&= < \mathcal{T}(A)_n > . \qquad (1)
\end{aligned}
$$

- Rényi entropy

$$
S_A^{(n)} = \frac{\log < \mathcal{T}(A)_n >}{1-n} . \qquad (2)
$$

- Mutual Rényi information of two disjoint regions:

$$
I_{A,B}^{(n)} = S_A^{(n)} + S_B^{(n)} - S_{A\cup B}^{(n)} = \frac{1}{1-n} \log \frac{< \mathcal{T}(A)_n >< \mathcal{T}(B)_n >}{< \mathcal{T}(A)_n \mathcal{T}(B)_n >} \qquad (3)
$$

Twist operator $\mathcal{T}_n$

- Twist operator $\mathcal{T}_n$ is a codimension 2 surface operator
- Key point: Operator product expansion (OPE) of twist operator
- In general, it is hard to find exact result because OPE is shape dependent
- Focus on planer or spherical twist operator

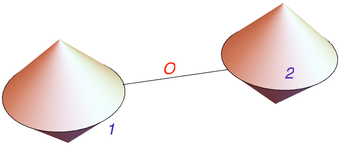

Figure: Correlation function of two disjoint twist operators.

- Twist operator breaks conformal symmetry

$$so(d,2) \to so(d-1,1) \times so(1,1)$$

- Use residue symmetry to constrain the OPE of a twist operator $\mathcal{T}_n$.
- Constraints: one point function

$$< O(x) >_n = \frac{< O(x)\mathcal{T}_n >}{< \mathcal{T}_n >} = a_{\Delta,J} \frac{T_J}{|x|^\Delta}$$

is fixed by symmetry up to one coefficient.

- Infinite many of constraints, there should be contribution from each primary operator $O_{\Delta,J}$ family.

OPE of Twist Operator $\mathcal{T}_n$(Long, 2016)

- Proposal: flat or spherical twist operator $\mathcal{T}_n$ in any parity invariant CFTs can be expanded as

$$\mathcal{T}_n =< \mathcal{T}_n > \sum_{O_{\Delta,J}} c_{O_{\Delta,J}} Q[O_{\Delta,J}].$$

  - $Q[O_{\Delta,J}] = \int_{\mathcal{D}(A)} d^d x K^{\mu_1} \cdots K^{\mu_J} |K|^{\Delta-J-d} O_{\mu_1 \cdots \mu_J}$
  - K^{μ} is the modular hamiltonian generator of region A.
  - $\mathcal{D}(A)$ is the causal development of the defect operator.
  - $c_{\Delta,J}$ is determined by one point function coefficient $a_{\Delta,J}$.
  - $c_1 = 1$.
- More generally, this is also the OPE of a codimension 2 flat or spherical defect operator $\mathcal{D}$

Some examples

$$
\begin{aligned}
&< O(x) >_{\mathcal{D}} = a_{\Delta,0}\frac{1}{|x|^{\Delta}},\\
&< O_a >_{\mathcal{D}} = a_{\Delta,1}\frac{\epsilon_{ab}n^b}{|x|^{\Delta}},\\
&< O_{ab} >_{\mathcal{D}} = -a_{\Delta,2}\frac{(d-1)\delta_{ab} - dn_a n_b}{|x|^{\Delta}}, < O_{ij} >_{\mathcal{D}} = a_{\Delta,2}\frac{\delta_{ij}}{|x|^{\Delta}},\\
&\ldots
\end{aligned}
$$

with $x^{\mu} = (x^a, y^i), n^a = \frac{x^a}{|x|}$.

Mutual Information

- For two spherical region, (Long, 2016)

$$
\begin{aligned}
I_{A,B}^{(n)} &= -\frac{1}{1-n} \log \frac{< \mathcal{T}_n(A)\mathcal{T}_n(B) >}{< \mathcal{T}_n(A) >< \mathcal{T}_n(B) >} \\
&= \frac{1}{n-1} \log(1 + \sum_{O_{\Delta,J}} s_{\Delta,J} G_{\Delta,J}(u,v)).
\end{aligned}
$$

where $s_{\Delta,J} \sim c_{\Delta,J}^2 \sim \frac{a_{\Delta,J}^2}{N_{\Delta,J}}$, the summation is over all possible primary operators in n-replicated CFT.

- $n \to 1$ limit, conformal block expansion of mutual information Chen & Long(2016)

$$
I_{A,B} = \sum_{O_{\Delta,J}} b_{\Delta,J} G_{\Delta,J}
$$

where $b_{\Delta,J} = s'_{\Delta,J}(1)$.

Mutual Information

- Kinematic Information $G(u, v)$, conformal block, fixed by symmetry.
- Dynamical Information $a_{\Delta,J}$ & $b_{\Delta,J}$.
- Compute Mutual Rényi Information
  - Classify all possible primary operators in n-replicated CFT.
  - Determine one point function $< O_{\Delta,J} >_n \to a_{\Delta,J}$ in n-sheets manifold M_n and normalization $N_{\Delta,J}$(Mutual R'enyi Information)

Mutual Information

- Previous discussion depends on the knowledge ($a_{\Delta,J}$) of CFTs in manifold M_n. Usually, it is a difficult task (theory dependent).
- By definition, mutual information has nothing to do with M_n.
- There should be another way to determine mutual information, given the knowledge of a CFT(spectrum, 3pt).
- We need to study ($a_{\Delta,J}$) and correlation functions on M_n more carefully, especially constraints from
  - Periodic boundary condition: $\theta \sim \theta + 2\pi n$.
  - $\mathcal{Z}(n)$ Symmetry.
  - $so(2)$ residue symmetry
  - Analytic property of $a_{\Delta,J}$ around $n = 1$.

Constraints for $a_{\Delta,J}(n)$(Chen, Chen, Hao & Long, 2017)

.

- Single sheet operator O^j.
  - $a^j_{\Delta,J}$ is independent of j. ($\mathcal{Z}(n)$ Symmetry)
  - $a^j_{\Delta,J} = \mathcal{O}(n-1)$. ($a_{\Delta,J}(1) = 0$, well defined Taylor expansion).
  - No contribution to mutual information.
    Faulkner, Lewkowycz & Maldacena, 2013
- Multi-sheet operator $O^{j_1 \cdots j_k}$.
  - $k \geq 2$, composite operators, such as $\phi^{j_1}\phi^{j_2}$, $J^{j_1 j_2}_{\mu}$.
  - Their coefficients a can be obtained from

$$G_n(x_1, \cdots, x_k) =< O_1(x_1)O_2(x_2)\cdots O_k(x_k) >_n \qquad (4)$$

  - $so(2)$ symmetry, rotation.

$$G_n(x_1, \cdots, x_k) = G_n(\theta_{j_1 j_2}, \cdots, \theta_{j_{k-1} j_k}),\ \theta_{ij} = \theta_i - \theta_j. \qquad (5)$$

$1/n$ prescription (Chen, Chen, Hao & Long, 2017)

- Leading contribution, bilinear operators $O^{j_1} O^{j_2}$,

$$a^{j_1 j_2}(n) \sim G_n(\theta_{j_1 j_2}). \tag{6}$$

- Assume periodic boundary condition, $1/n$ prescription

$$G_n(\theta_j, 0) = G_1(\frac{\theta_j}{n}) + \mathcal{O}(n-1) \tag{7}$$

- Identity

$$b \sim \lim_{n\to 1} \sum_{j=1}^{n-1} \frac{G_n^2(j)}{n-1} = \lim_{n\to 1} \sum_{j=1}^{n-1} \frac{G_1^2(\frac{\theta_j}{n})}{n-1} \tag{8}$$

- $G_1(x_1, x_2)$ is fixed by conformal symmetry.
- Bilinear operators contribute universally to mutual information.

Conclusion

- General formulas
  - OPE of codimension two flat or spherical twist (defect) operators

$$\mathcal{D} = <\mathcal{D}> \sum_{O_{\Delta,J}} c_{O_{\Delta,J}} Q[O_{\Delta,J}].$$

  - Mutual Rényi Information from Conformal Block.

$$I_{A,B}^{(n)} = \frac{1}{n-1} \log(1 + \sum_{O_{\Delta,J}} s_{\Delta,J} G_{\Delta,J}(u,v))$$

with $s_{\Delta,J} \sim \sum_{O_{\Delta,J}} \frac{a_{\Delta,J}^2}{N_{\Delta,J}}$.

  - Mutual Information from Conformal Block.

$$I_{A,B} = \sum_{O_{\Delta,J}} b_{\Delta,J} G_{\Delta,J}(u,v).$$

with $b_{\Delta,J} = s'_{\Delta,J}(1)$.

Conclusion

- Develop $1/n$ prescription to determine $b_{\Delta,J}$.
- Mutual Information are contributed from multi-replica operators, no contribution from single replica operators.
- Universal behaviour from bilinear operators
- Leading order contribution should be universal.
- Leading order contribution is from a spin 1 bilinear operator for fermionic theory.
- Holographic duality: $b \sim \mathcal{O}(1)$, quantum correction of mutual information
- Higher spin operator is important for higher order correction
- Mutual information analytic results fits well with lattice computation for free boson and free fermion.
- May still be valid for large cross ratio (small distance).
- Multi regions can be discussed in parallel.

Open issues

- Gauge theory?
- Mutual Rényi information ($n \neq 1$)
  - we computed the first few orders for free boson, and reproduced Cardy's result at large distances (leading order).
  - however, leading order result does not fit well with lattice computation. Three possible resolutions:
    - Check the OPE for general CFTs (any loop hole)?
    - Check the computation of a_n (Green's function, conical singularity correction)?
    - Improve the precision of lattice computation?

Thanks for your attention!

Numerical analysis

- For free boson and free fermion, one can use lattice method to compute Entanglement entropy for any region.
- The framework is well established, I just report the numerical result and compare it with analytic result.
- Two ball region:
  - $3d$ free boson,

$$I_{A,B} = \frac{1}{12} G_{1,0}^{d=3}(z) - \frac{1}{120} G_{2,1}^{d=3}(z) + \left(\frac{1}{360} + \frac{1}{12\pi^2}\right) G_{2,0}^{d=3}(z) + \cdots$$

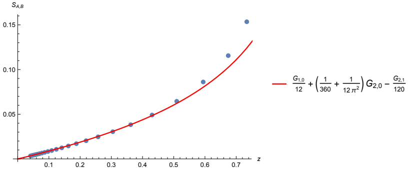

Figure: Mutual information of two disks for $3d$ free boson ($0 < z < 0.75$).

Numerical analysis for 3d free boson

- Fits very well for small z, large distance.
- Even though z is not too small, matches well.
- $z > 0.55$, discrepancy due to the contribution of other conformal blocks.
- Supports the proposal that mutual information can be expanded as summation of conformal blocks.
- Shed light on analytic computation of mutual information.

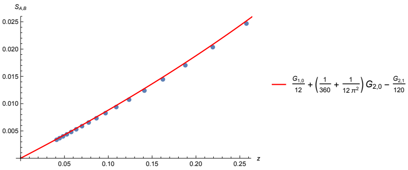

Figure: Mutual information of two disks for $3d$ free boson ($0 < z < 0.25$).

Numerical analysis for 4d free boson

Analytic results:

$$I_{A,B}^{d=4} = \frac{1}{60} G_{2,0}^{d=4} - \frac{1}{420} G_{3,1}^{d=4} + \frac{1}{840} G_{4,2}^{d=4} + \frac{1}{840} G_{4,0}^{d=4} + \cdots$$

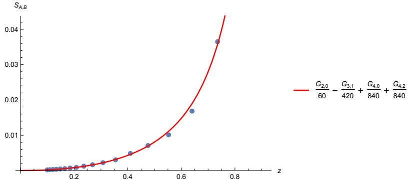

Figure: Mutual information of two balls for $4d$ free boson ($0 < z < 0.8$).

Numerical analysis for 4d free boson

- Fits very well for a large range of z.

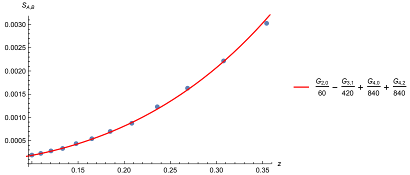

Figure: Mutual information of two balls for $4d$ free boson $(0 < z < 0.4)$.

- The prediction for $d > 4$ dimension

$$
\begin{aligned}
I_{A,B}^{d>4} &= \frac{4^{1-d}\sqrt{\pi}\Gamma[d-1]}{\Gamma[-\frac{1}{2}+d]} G_{d-2,0}^{d>4}(z) - \frac{4^{-d}(d-2)\sqrt{\pi}\Gamma[d-1]}{\Gamma[\frac{1}{2}+d]} G_{d-}^{d>} \\
&+ \frac{2^{-3-2d}(4-4d+3d^2)\sqrt{\pi}\Gamma[d-1]}{\Gamma[\frac{3}{2}+d]} G_{d,2}^{d>4}(z) + \cdots
\end{aligned}
$$

Numerical analysis for 3d free fermion

Chen, Chen, Hao & Long, 2017

$$I_{d=3} = \frac{1}{15} G_{2,1}^{(d=3)}(z) + \cdots \qquad (10)$$

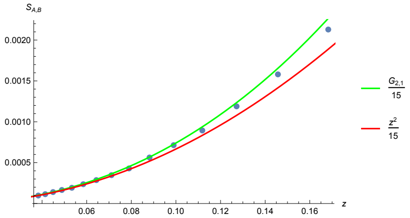

Figure: Mutual information of two disks for 3*d* free Dirac fermion. The red line is the leading order contribution in z, while the green line is the leading order conformal block.

Chen, Chen, Hao & Long, 2017

$$I_{d=4} = \frac{1}{35} G_{3,1}^{(d=4)}(z) + \cdots$$

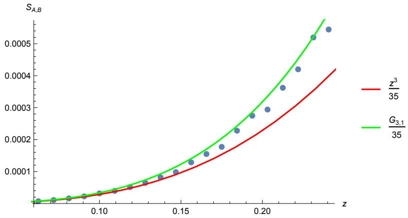

Figure: Mutual information of two balls for 4d free Dirac fermion. The red line is the leading order contribution in z, while the green line is the leading order conformal block.

- First few orders

$$
\begin{aligned}
I_{A,B}^{(n)} \quad = \quad & \frac{1}{n-1}\log(1+\frac{n^4-1}{240n^3}G_{2,0}(z)+\frac{n^6-21n^2+20}{15120n^5}G_{3,1}(z) \\
& +\frac{-420+7n+400n^2-14n^5+20n^8+7n^9}{806400n^7}G_{4,0}(z) \\
& +\frac{-21+20n^2+n^8}{40320n^7}G_{4,2}(z)+\cdots)
\end{aligned}
$$

- Leading order at large distance

$$
I_{A,B}^{(n)}=\frac{(n+1)(n^2+1)}{240n^3}z^2+\cdots \tag{11}
$$