# A VIEW OF THE BULK FROM THE WORLDLINE

[1712.00885]

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AdS/CFT: UV complete, nonperturbative quantum gravity

Problem: describe bulk physics in CFT language

Bulk EFT $\leftrightsquigarrow$ CFT data & bootstrap

Local bulk geometry $\Leftrightarrow$ large $N$ and gap?

Main idea: quantise a particle rather than a field

Integrate over field configurations $\longrightarrow$ particle trajectories

$$\underbrace{\int}_{\text{Field configurations}} \mathcal{D}\phi \, e^{-\int d^{d+1}x \left(\frac{1}{2}(\partial\phi)^2 + \frac{1}{2}m^2\phi^2 + \cdots\right)} \longrightarrow \underbrace{\int}_{\text{Worldlines}} \mathcal{D}x \, e^{-mL[x]}$$

Worldline action is the length of path.

Why worldline formulation?

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**Geodesic approximation** is worldline classical limit ($\hbar \sim m^{-1}$)

- Geodesics, corrected by $m^{-1}$ worldline loops
- Natural local bulk probes: $\ell_{AdS}^{-1} \ll m \ll \ell_{pl}^{-1}$

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- Vertex operator correlators in worldline QM

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$$\int \mathcal{D}g e^{-\frac{1}{G_N}S_{\mathrm{EH}}[g]} \underbrace{\int \mathcal{D}x\, e^{-mL[x,g]}}_{\text{Worldlines}}$$

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# MAIN RESULTS

- Naturally isolate conformal blocks
- Multi-graviton exchange $\longrightarrow$ Virasoro blocks (in $AdS_3$)
- Systematic bulk computation
- Complementary to Witten diagrams

# THE WORLDLINE QUANTUM MECHANICS

Path integral for worldlines between points $x_0, x_1$:

$$G(x_0, x_1) = \frac{1}{2m} \int_0^\infty dT \, e^{-\frac{mT}{2}} \int_{x(0)=x_0, x(T)=x_1} \mathcal{D}x \, e^{-\frac{m}{2} \int_0^T ds \, g_{ab}(x) \dot{x}^a \dot{x}^b}$$

Compare: worldsheet string theory

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Quantum mechanical sigma-model: $H = -\frac{1}{2m} \nabla^2$

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$\longrightarrow$ scalar propagator

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$\longrightarrow$ scalar propagator

Allow vertices $\longrightarrow$ perturbative QFT

Integral between points in $AdS_{d+1}$, at proper distance $L$:

$$e^{-W_L(m,T)} = \int \mathcal{D}x \; e^{-\frac{m}{2}\int_0^T ds \; g_{ab}(x)\dot{x}^a\dot{x}^b}$$

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$\mathrm{AdS}_{d+1}$: $$ds^2 = (1+q^2)dt^2 + dq^2 - \frac{(q.dq)^2}{1+q^2} \qquad q{=}(q^1,\ldots,q^d)$$

Write $t(s) = s + u(s)$, with $u(0) = u(L) = q(0) = q(L) = 0$:

$$S[q,u] = \frac{L}{2\lambda} + \frac{1}{2\lambda}\int_0^L ds \left(\dot{u}^2 + \dot{q}^2 + q^2 + 2\dot{u}q^2 + \dot{u}^2 q^2 - \frac{(q\cdot\dot{q})^2}{1+q^2}\right)$$

Leading order: $d$ harmonic oscillators $q$ of frequency $\ell_{AdS}^{-1}$

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Correlation functions: points $\to$ boundary, $L \to \infty$

$W_L(\lambda) \sim L\mathcal{E}(\lambda) \Rightarrow$ modulus integral $\int d\lambda \, e^{-\frac{m^2\lambda}{2}L - W_L(\lambda)}$ localises.

$$m^2 = -2\mathcal{E}'(\lambda), \quad \text{physical parameter } \Delta = \mathcal{E}(\lambda) - \lambda\mathcal{E}'(\lambda)$$

In $\mathrm{AdS}_{d+1}$,
$\mathcal{E}(\lambda) = \frac{1}{2\lambda} + \frac{d}{2} + \frac{d^2}{8} \implies \Delta = \frac{1}{\lambda} + \frac{d}{2}, \quad m^2 = \frac{1}{\lambda^2} - \frac{d^2}{4} = \Delta(\Delta - d).$

Couple to gravity: perturb metric ($g \longrightarrow g_{\text{AdS}} + h$)

Deform by vertex operator $\mathcal{V}_h = \frac{1}{2\lambda} \int ds\, h_{ab}(x(s)) \dot{x}^a(s) \dot{x}^b(s)$

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$\longrightarrow$ correlation functions of $\mathcal{V}_h$

$$\langle \mathcal{V}_h \rangle = \int ds\, \left( \frac{1}{2\lambda} h_{00} + \frac{1}{8} \partial_i \partial_i h_{00} - \frac{1}{4} h_{ii} + O(\lambda) \right)$$

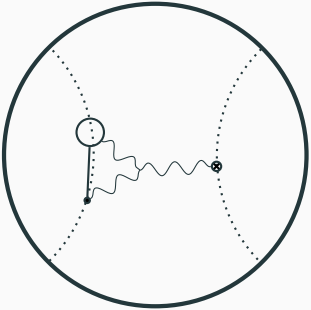
Higher order in $h$: $\langle \mathcal{V}_h \mathcal{V}_h \rangle$, $\langle \mathcal{V}_h \mathcal{V}_h \mathcal{V}_h \rangle$, ...

Higher order in $\lambda$: transverse derivatives

Systematic expansion in powers of $h$, $\lambda$

# A FOUR-POINT FUNCTION

Four-point function of scalars, quantised in worldline formalism, coupled to second-quantised ‘light’ bulk fields (gravity)



$$\langle\phi\phi\phi\phi\rangle = \int \mathcal{D}g\mathcal{D}x\ \exp\left[-\frac{1}{G_N}S_{EH}[g] - mL[x,g]\right]$$

Worldline quantum mechanics coupled to bulk theory

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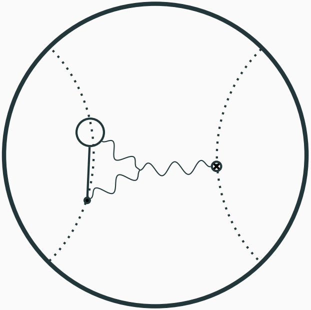

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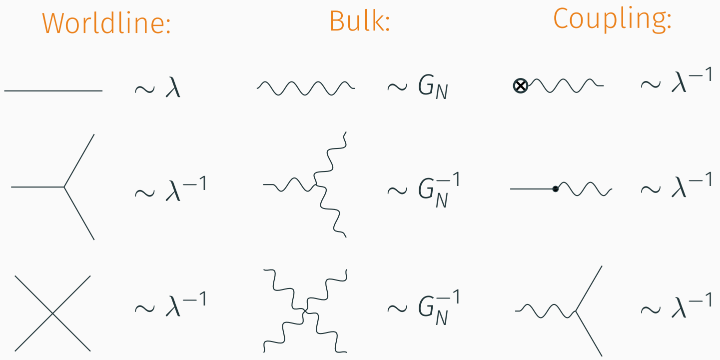
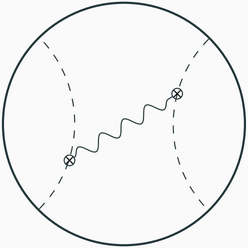
Perturbation theory in $\lambda \sim m^{-1}$ (loops) and $\lambda^{-1}G_N$ (gravity)

$$\frac{\langle\phi\phi\phi\phi\rangle}{\langle\phi\phi\rangle\langle\phi\phi\rangle} = \exp\left[\sum(\text{connected diagrams, coupling worldlines})\right]$$



Order $m^2 G_N$:



$$= \frac{1}{2}\langle\langle\mathcal{V}_h^{(1)}\rangle_{\text{tree}}^{(1)}\langle\mathcal{V}_h^{(2)}\rangle_{\text{tree}}^{(2)}\rangle_{\text{gravity}}$$

$$= \frac{1}{8\lambda_1\lambda_2}\int ds_1\, ds_2 \langle h_{00}(s_1) h_{00}(s_2)\rangle$$

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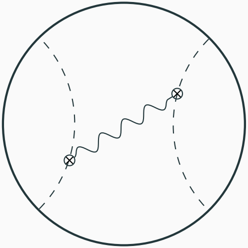

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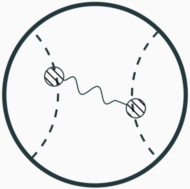
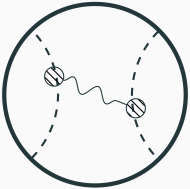
Geodesic Witten diagram *[Hijano,Kraus,Perlmutter,Snively]*

Conformal block $\mathcal{F}_T$



Expand $\langle \mathcal{V}_h \rangle$ in $\lambda$: $\langle \mathcal{V}_h \rangle = \int ds \, \left( \frac{1}{2\lambda} h_{00} + \frac{1}{8} \partial_i \partial_i h_{00} - \frac{1}{4} h_{ii} + O(\lambda) \right)$



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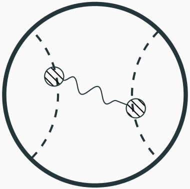
Use EOM: $= \left( \frac{1}{2\lambda} + \frac{d}{4} + \cdots \right) \int ds \, h_{00}(s)$



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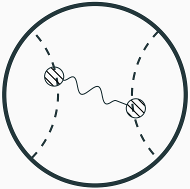
Correct normalisation: $\lambda = \frac{1}{\Delta - \frac{d}{2}} \longrightarrow C_{T\mathcal{O}\mathcal{O}} \propto \sqrt{G_N}\Delta$



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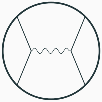
Not equal to Witten diagram  !

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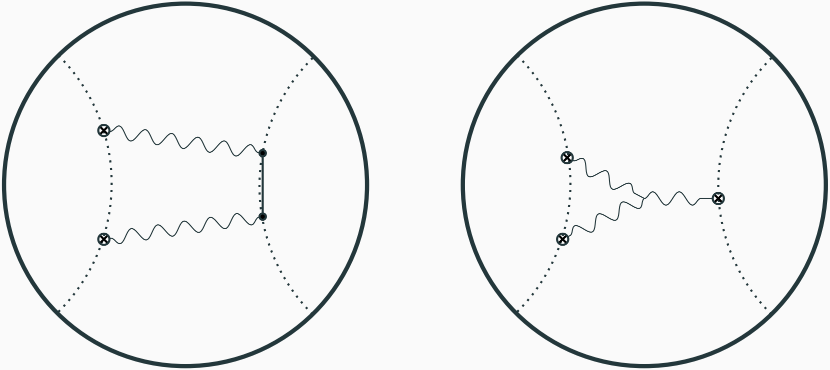

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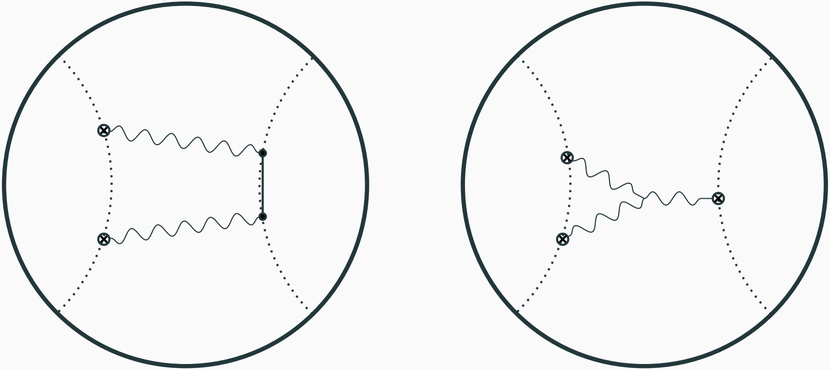


Order $\lambda^{-n-1}G_N^n$ (tree): solves classical physics of particles

# CONFORMAL BLOCKS

Important properties of conformal field theories:

**Radial quantisation:** Use dilatation as 'Hamiltonian'

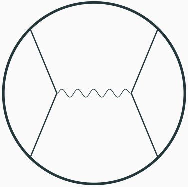
**State-operator:** Local operators $\longleftrightarrow$ states on $S^{d-1}$

**OPE:** $\mathcal{O}_1(0)\mathcal{O}_2(x) = \sum_i C_{12i}|x|^{\Delta_i - \Delta_1 - \Delta_2}\mathcal{O}_i$

Leads to conformal block decomposition

$$\langle \mathcal{O}_1\mathcal{O}_2\mathcal{O}_3\mathcal{O}_4\rangle = \sum_p C_{\mathcal{O}_1\mathcal{O}_2 p} C_{\mathcal{O}_3\mathcal{O}_4 p}\mathcal{F}_p$$

Decompose a Witten diagram as conformal blocks:



$$= C_{12T}C_{34T}\mathcal{F}_T + \sum_{n,l} a_{n,l}\mathcal{F}_{[12]_{n,l}} + \sum_{n,l} b_{n,l}\mathcal{F}_{[34]_{n,l}}$$

'Double trace' operators $[\mathcal{O}_1\mathcal{O}_2]_{n,l} \sim :\mathcal{O}_1\partial^l\Box^n\mathcal{O}_2:$

Spin $l$, dimension $\Delta_{n,l} = 2\Delta + 2n + l$

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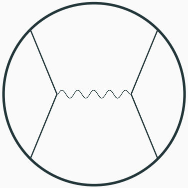

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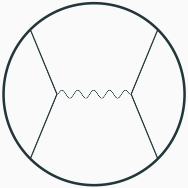

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DT non-perturbative in $\lambda$: second saddle point, same topology

All order perturbation theory: all multi-trace $T$ exchanges

In $CFT_2$, determined by kinematics: Virasoro block

Not so universal for $d > 2$: higher curvature

Amplitude is exponential of connected diagrams

Consequence: single-trace blocks exponentiate!

$$\langle \mathcal{O}\mathcal{O}\mathcal{O}\mathcal{O} \rangle = \exp\left( G_N \Delta_1 \Delta_2 \mathcal{F}_T + O(G_N^2 \Delta^3) \right)$$

Seen in Virasoro/kinematic limit *[Fitzpatrick,Kaplan,Walters,Wang]*

A piece of Witten diagrams at all loop orders

# OUTLOOK

# SUMMARY

- New approach to compute correlation functions in AdS, using worldline formalism
- Systematic perturbative expansion
- Complementary to Witten diagrams
- Bulk computation of Virasoro conformal blocks
- General conformal blocks (higher point, thermal, higher genus)

# OUTLOOK

Plenty to explore:

- $AdS_3$:
  - Computing Virasoro blocks (heavy particles)
  - Connect to Chern-Simons formalism *[Fitzpatrick,Kaplan,Li,Wang]*
  - One-loop exact (localisation) in $AdS_3$ *[Duistermaat-Heckman]*
- Higher-point functions
- Other backgrounds *[Kraus,Maloney,HM,Ng,Wu][Anous,Hatman,Rovai,Sonner]*
- Fermions ($\mathcal{N} = 1$ worldline supersymmetry)
- Diff invariant bulk observables*[Czech,Lamprou,McCandlinsh,Mosk,Sully]*
- Lorentzian signature, shockwaves, ANEC
- Bootstrap $\Delta \gg 1$

ありがとうございました

# EXTRA SLIDES

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- Integrate over functions $x(s)$ from an interval into spacetime.

$$G(x_0, x_1) = \int_{x(0)=x_0, x(T)=x_1} \frac{\mathcal{D}x}{V_{\text{diffs}}}\, e^{-m \int_0^T ds\, \sqrt{g_{ab}(x(s))\dot{x}^a(s)\dot{x}^b(s)}}$$

# WORLDLINE PATH INTEGRAL

Want $G(x_0, x_1) = \int_{\text{Worldlines}} \mathcal{D}x \, e^{-mL[x,g]}$ over worldlines between $x_0, x_1$.

- Integrate over functions $x(s)$ from an interval into spacetime.
- Introduce worldline auxiliary metric ($e(s) = \sqrt{g_{ss}(s)}$).

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Compare: worldsheet string theory