OPE for Conformal Defects and Holography

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Outline

1. Introduction

2. The structure of defect OPE blocks

3. Reconstruction of AdS scalar fields from conformal defects
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1 Introduction

2 The structure of defect OPE blocks

3 Reconstruction of AdS scalar fields from conformal defects
Defects = Non-local objects in QFTs

- Defined by boundary conditions around them
- Many examples:
  - 1-dim: Line operators (Wilson-'t Hooft loops)
  - 2-dim: Surface operators
  - Codim-1: Domain walls and boundaries
  - Codim-2: Entangling surface for entanglement entropy
Why defects?

- Characterize phases of QFTs:
  \[ W: \text{Wilson loop, } T: \text{'t Hooft loop} \]

Confinement: \[ \langle W \rangle \sim e^{-\text{Area}}, \quad \langle T \rangle \sim e^{-\text{Length}} \]

Higgs: \[ \langle W \rangle \sim e^{-\text{Length}}, \quad \langle T \rangle \sim e^{-\text{Area}} \]

- Constrain bulk CFT data in defect CFT by conformal bootstrap
  [Liendo-Rastelli-van Rees 12]
Conformal defects

- In Euclidean $\text{CFT}_d$, the conformal group is $SO(d + 1, 1)$

- Codimension-$m$ conformal defects $\mathcal{D}^{(m)}$ are either flat or spherical, preserving $SO(d + 1 - m, 1)$: conformal symmetry on defects $SO(m)$: rotation in the transverse direction

- Defects allow defect local operators $\hat{O}_n$
OPE for conformal defects

There are two types of OPE in DCFT

- **Bulk-to-defect OPE**: [Cardy 84, McAvity-Osborn 95]
  
  \[
  \mathcal{D}^{(m)} = \sum_n b^{(m)}_{\mathcal{O}} \mathcal{O}_n + \text{(descendants)}
  \]

- **Defect OPE**: [Berenstein-Corrado-Fischler-Maldacena 98, Gadde 16]
  
  \[
  \mathcal{D}^{(m)} = \sum_n c^{(m)}_{\mathcal{O}_n} \mathcal{O}_n + \text{(descendants)}
  \]
Goal of the talk

The **defect OPE** has been less understood than the bulk-to-defect OPE

1. **To what extent are we able to determine the structure of the defect OPE by conformal symmetry?**
   - Decomposition by the irreducible representations
     \[
     D^{(m)} = \sum_{n \in \text{primaries}} B^{(m)}[\mathcal{O}_n]
     \]

2. **Can we probe the bulk AdS information by conformal defects on the boundary?**
Overview of our results

1. Give the integral representation of the defect OPE blocks

\[ \mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \langle \mathcal{O}_n(x) \rangle D^{(m)} \tilde{\mathcal{O}}_n(x) \]

\( \tilde{\mathcal{O}} : \) shadow operator with \( \tilde{\Delta} = d - \Delta \) for \( \mathcal{O} \) with \( \Delta \)

2. Reconstruct the AdS scalar field from the blocks

- \( \hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}] : \) The Radon transform of the AdS scalar field \( \phi \) when \( \mathcal{O} \) scalar

- Reproduce the (Euclidean) HKLL formula:

\[ \phi(Y) = \phi(\hat{\phi}) = \int d^d x K(Y|x) \mathcal{O}(x) \]
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Embedding space formalism

- Uplift $\mathbb{R}^d$ into the embedding space $\mathbb{R}^{d+1,1}$
  \[ x^\mu \rightarrow X^A \ (A = 1, \cdots, d+2) \]

- Choose the $d$-dim hypersurface (the projective null cone)
  \[ X \cdot X \equiv X^A X_A = 0 \ , \quad X^A \sim \lambda X^A \ (\lambda \in \mathbb{R}) \]

- Realize the conformal symmetry $SO(d+1,1)$ linearly
  \[ X'{}^A = J^A_B \ X^B \ , \quad J^A_B \in so(d+1,1) \]
Correlation functions in embedding space

Construct the invariants under $SO(d + 1, 1)$ in the embedding space

- Two-point function of scalar fields $\Phi$ of dimension $\Delta$:

$$\langle \Phi(X_1) \Phi(X_2) \rangle = \frac{1}{(X_1 \cdot X_2)^\Delta}$$

For a spin $l$ operator $O_{\Delta,l}(X)$, define the encoding polynomial by contracting the indices with an auxiliary vector $Z^A$:

[Costa-Penedones-Poland-Rychkov 11]

$$O_{\Delta}(X, Z) \equiv O_{\Delta,A_1\cdots A_l}(X) Z^{A_1} \cdots Z^{A_l}, \quad X \cdot Z = Z^2 = 0$$

- Two-point functions are uniquely determined by the invariants:

$$\langle O_{\Delta,l}(X_1, Z_1) O_{\Delta,l}(X_2, Z_2) \rangle = \frac{1}{(X_{12})^\Delta} \left[ \frac{(Z_1 \cdot Z_2)(X_1 \cdot X_2) - (Z_1 \cdot X_2)(Z_2 \cdot X_1)}{X_{12}} \right]^l.$$
Specify a codimension-\(m\) defect by the frame vectors \(P^A_\alpha\): [Gadde 16]

\[
X \cdot P_\alpha = 0 \quad (\alpha = 1, \cdots m)
\]

Choosing an orthonormal basis, \(P_\alpha \cdot P_\beta = \delta_{\alpha\beta}\), the defect preserves the symmetry group \(SO(d + 1 - m, 1) \times SO(m)\)

Correlation function should be invariant under the frame choice \((P'_\alpha = g^{\alpha\beta} P_\beta, g \in SO(m))\):

\[
\langle \mathcal{D}^{(m)}(P_\alpha) \mathcal{O}_\Delta(X) \rangle \propto \frac{1}{[(P_\alpha \cdot X)(P_\alpha \cdot X)]^{\Delta/2}}
\]
Defect OPE blocks

We expect the defect OPE of the form

$$D^{(m)}(P_\alpha) = \langle D^{(m)}(P_\alpha) \rangle \left[ \sum_n c^{(m)}_{\mathcal{O}_n} R^{\Delta_n} \mathcal{O}_n(C) + \text{(descendants)} \right]$$

(R : radius, C : center vector)

- The descendant terms are fixed by the primary $\mathcal{O}_n$ and the conformal symmetry,

$$D^{(m)}(P_\alpha) = \sum_n \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n]$$

- The defect OPE block is in the irreducible representation of $\mathcal{O}_n$:

$$\langle \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n] \mathcal{O}_n(X) \rangle = \langle D^{(m)}(P_\alpha) \mathcal{O}_n(X) \rangle.$$
Projectors and shadows

- Want to characterize the defect OPE blocks by their irreps

- Spectral decomposition by the irreps of the conformal group:

\[ 1 = \sum_n |\mathcal{O}_n| \]

- \( |\mathcal{O}_n| \): Projector onto the conformal multiplet of the primary \( \mathcal{O}_n \)

[Ferrara-Grillo-Parisi-Gatto 72, · · ·, Simmons-Duffin 12]

For a spin \( l \) operator,

\[ |\mathcal{O}_{\Delta,l}| = \frac{1}{\mathcal{N}_{\Delta,l}} \int D^dX |\mathcal{O}_{\Delta,l}(X, DZ)||\tilde{\mathcal{O}}_{d-\Delta,l}(X, Z)| \]

\( \tilde{\mathcal{O}}_{d-\Delta,l} \): the shadow operator of \( \mathcal{O}_{\Delta,l} \)

\( D_Z \): Todorov’s differential operator \( (\sim \partial_Z) \)
Integral representation of defect OPE blocks

- Expand the defect by the projectors:

\[
\langle D^{(m)}(P_{\alpha}) \cdots \rangle = \sum_{\Delta,l} \langle D^{(m)}(P_{\alpha})|O_{\Delta,l}| \cdots \rangle + \text{(other irrep.)}
\]

\[
= \sum_{\Delta,l} \frac{1}{N_{\Delta,l}} \int D^dX \langle D^{(m)}(P_{\alpha})O_{\Delta,l}(X, DZ)\rangle \langle \tilde{O}_{d-\Delta,l}(X, Z) \cdots \rangle + \text{(other irrep.)}
\]

- Can read off the block contribution:

The integral rep of the defect OPE block

\[
B^{(m)}[P_{\alpha}, O_{\Delta,l}] = \frac{1}{N_{\Delta,l}} \int D^dX \tilde{O}_{d-\Delta,l}(X, DZ) \langle D^{(m)}(P_{\alpha})O_{\Delta,l}(X, Z)\rangle
\]
Alternative representation and monodromy condition

- The integral rep is **invariant under the exchange** $\mathcal{O} \leftrightarrow \tilde{\mathcal{O}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = \frac{1}{\mathcal{N}_{d-\Delta,l}} \int D^d X \mathcal{O}_{\Delta,l}(X, DZ) \langle \mathcal{D}^{(m)}(P_\alpha) \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z) \rangle$$

- The block includes the **shadow contribution** $g_{\tilde{\mathcal{O}}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = g_{\mathcal{O}} + K g_{\tilde{\mathcal{O}}}$$

- To extract the shadow contribution, impose the **monodromy condition**

$$g_{\mathcal{O}} \rightarrow e^{2\pi i \Delta} g_{\mathcal{O}}$$

$$g_{\tilde{\mathcal{O}}} \rightarrow e^{2\pi i (d-\Delta)} g_{\tilde{\mathcal{O}}}$$

under the phase rotation $P_\alpha \rightarrow e^{-2\pi i} P_\alpha$
Constraint equations

There are two types of equations the defect OPE block satisfies

The conformal Casimir equation

\[(L^2(P_\alpha) + C_{\Delta,l}) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0\]

- \(L^2(P_\alpha) \equiv \frac{1}{2} L_{AB}(P_\alpha) L^{AB}(P_\alpha)\) : quadratic Casimir operator
- \(C_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)\) : the eigenvalue

“Trivial” equations for scalar primaries \(\mathcal{O}_\Delta\)

\[C_{ABCD}(P_\alpha) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_\Delta] = 0\]

- \(C_{ABCD}(P_\alpha) \equiv \frac{1}{2} L_{[AB}(P_\alpha)L_{CD]}(P_\alpha)\): \(\frac{d(d^2-1)(d+2)}{24}\) quadratic operators
Moduli space of conformal defects

- The moduli space has a coset structure:

\[ \mathcal{M}^{(d,m)} = \frac{SO(d + 1, 1)}{SO(m) \times SO(d + 1 - m, 1)} \]

- The quadratic Casimir operator is the Laplacian on \( \mathcal{M}^{(d,m)} \)

\[ -L^2(P_\alpha) = \Box \mathcal{M}^{(d,m)} \]

- The defect OPE block is a scalar field propagating on \( \mathcal{M}^{(d,m)} \)

Klein-Gordon equation on \( \mathcal{M}^{(d,m)} \)

\[ (\Box \mathcal{M}^{(d,m)} - M^2) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0 \, , \quad M^2 = C_{\Delta,l} \]
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Conformal defects and submanifolds in AdS

- Associated to a given defect $\mathcal{D}^{(m)}$ is a unique submanifold $\gamma^{(m)}$ in AdS s.t. $\partial \gamma^{(m)} = \mathcal{D}^{(m)}$

- Their moduli spaces are equivalent:

$$\mathcal{M}^{(d,m)} = \frac{\text{Isom}(\text{AdS}_{d+1})}{\text{Stab}(\gamma^{(m)} \in \text{AdS}_{d+1})}$$

- What is the dual description of the defect OPE block in AdS?
**Moduli space and AdS**

- The defect OPE block $B^{(m)}$ is a scalar field not on $\mathbb{H}^{d+1}(=\text{Euclidean AdS}_{d+1})$, but on $\mathcal{M}^{(d,m)}$

- The two spaces have similar coset structures:

  $$\mathcal{M}^{(d,m)} = \frac{SO(d + 1, 1)}{SO(m) \times SO(d + 1 - m, 1)}, \quad \mathbb{H}^{d+1} = \frac{SO(d + 1, 1)}{SO(d)}$$

- In this case, the Radon transform maps a function from one to the other [e.g. Helgason 10]
Radon transform

From $\mathbb{H}^{d+1}$ to $\mathcal{M}^{(d,m)}$:

$$\hat{\phi}(\xi) = \int_{x \in \xi} d\nu(x) \phi(x)$$

- $\xi$: a codim-$m$ submanifold in $\mathbb{H}^{d+1}$
- $\phi(x)$: a function on $\mathbb{H}^{d+1}$

From $\mathcal{M}^{(d,m)}$ to $\mathbb{H}^{d+1}$:

$$\check{f}(x) = \int_{x \in \xi} d\mu(\xi) f(\xi)$$

- $\xi$: a codim-$m$ submanifold through $x$
- $f(\xi)$: a function on $\mathcal{M}^{(d,m)}$

Intertwining property

$$(\Box_{\mathbb{H}^{d+1}} - M^2) \phi = 0 \iff (\Box_{\mathcal{M}^{(d,m)}} - M^2) \hat{\phi} = 0$$
Defect OPE blocks as Radon transformed fields

- We may regard the defect OPE block

\[ \hat{\phi} = B^{(m)} \]

as the Radon transform of an AdS scalar field \( \phi \).

- The Radon transform maps \( \phi \) into the manifold with larger dimensions

\[ \dim \mathcal{M}^{(d,m)} - \dim \mathbb{H}^{d+1} = (m - 1)(d + 1 - m) \geq 0 \]

Hence there should be constraints for \( \hat{\phi} \), which are known as the range characterization [Ishikawa 97]

\[ C_{ABCD} \hat{\phi} = 0 \]

- This is what the block \( B^{(m)}[\mathcal{O}_\Delta] \) satisfies for a scalar primary \( \mathcal{O}_\Delta \).
## Dictionary for the Radon transform

In summary, the defect OPE block of a scalar primary is identified with an AdS scalar field

- $\mathcal{M}^{(d,m)}$
- $\mathcal{H}^{d+1}$
- scalar block: $\mathcal{B}^{(m)}[O_\Delta]$
- AdS scalar field: $\phi$

### Quadratic Casimir equation

$$(\Box_{\mathcal{M}^{(d,m)}} - M^2) \mathcal{B}^{(m)}[O_\Delta] = 0$$

### Constraint equations

$$C_{ABCD} \mathcal{B}^{(m)}[O_\Delta] = 0$$

### Klein-Gordon equation

$$(\Box_{\mathcal{H}^{d+1}} - M^2) \phi = 0$$

### Range characterization

$$C_{ABCD} \hat{\phi} = 0$$
Construction of AdS scalar fields from defects

- Can construct an AdS scalar $\phi$ from the scalar block $\hat{\phi} = B^{(m)}$ by the dual Radon transform $f \rightarrow \hat{f}$?

- The inversion formula allows such a construction (note $(\hat{\phi}) \neq \phi$)

$$\phi = \mathcal{I} \circ \hat{\phi} = \mathcal{I} \circ B^{(m)}[O_\Delta]$$

$\mathcal{I}$: an integral transformation [Helgason 10]

- Equivalent to the bulk reconstruction formula

$$\phi(Y) = \int D^dX K_\Delta(Y|X) O_\Delta(X)$$

with the Euclidean version of the HKLL kernel [Hamilton-Kabat-Lifschytz-Lowe 06]

$$(\Box_{\mathbb{H}^{d+1}} - M^2) K_\Delta(Y|X) = 0, \quad M^2 = \Delta(d - \Delta)$$
Summary

1. Give the integral representation of the defect OPE blocks

\[ B^{(m)}[O_n] = \int d^d x \langle O_n(x) \rangle_{D^{(m)}} \tilde{O}_n(x) \]

\( \tilde{O} \): shadow operator with \( \tilde{\Delta} = d - \Delta \) for \( O \) with \( \Delta \)

2. Reconstruct the AdS scalar field from the blocks

- \( \hat{\phi} = B^{(m)}[O] \):
  
  The Radon transform of the AdS scalar field \( \phi \) when \( O \) scalar

- Reproduce the (Euclidean) HKLL formula:

\[ \phi(Y) = \phi(\hat{\phi}) = \int d^d x K(Y|x) O(x) \]
Future direction

- Constraints on the bulk OPE data from the block?
  - Need the relation to the bulk-to-defect OPE

- Can extend the construction to higher spin fields in AdS?
  - No known Radon transform beyond a scalar field in AdS
  - Spinning defects to incorporate spins ⇒ [Nozomu’s talk]

- Generalized notion of entanglement entropy?
  - Codim-2 deficit angle ⇒ codim-$m$ deficit ??
  - Analytic continuation to Lorentzian signature?