OPE for Conformal Defects and Holography

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with M. Fukuda (UT) and N. Kobayashi (Kavli-IPMU)

Outline

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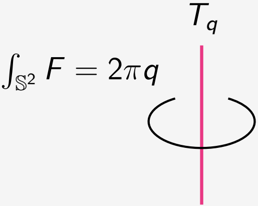
Introduction

Defects = Non-local objects in QFTs

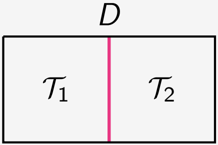
- Defined by boundary conditions around them



- Many examples:

1-dim : Line operators (Wilson-'t Hooft loops)

2-dim : Surface operators



Codim-1 : Domain walls and boundaries

Codim-2 : Entangling surface for entanglement entropy

Why defects?

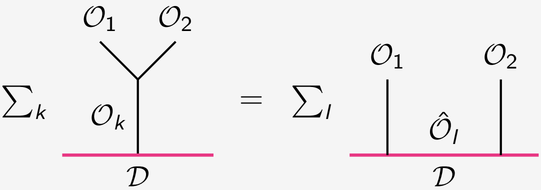
- Characterize phases of QFTs:

$$W\text{: Wilson loop}, \quad T\text{: 't Hooft loop}$$

$$\textcolor{blue}{\text{Confinement :}} \quad \langle W\rangle \sim e^{-\text{Area}}, \qquad \langle T\rangle \sim e^{-\text{Length}}$$

$$\textcolor{blue}{\text{Higgs :}} \quad \langle W\rangle \sim e^{-\text{Length}}, \quad \langle T\rangle \sim e^{-\text{Area}}$$

- Constrain bulk CFT data in defect CFT by conformal bootstrap
  [Liendo-Rastelli-van Rees 12]



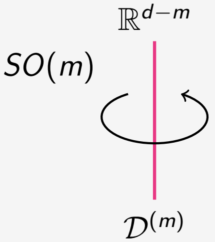
Conformal defects

- In Euclidean CFT_d, the conformal group is $SO(d+1,1)$

- Codimension-m conformal defects $\mathcal{D}^{(m)}$ are either flat or spherical, preserving

$SO(d+1-m,1)$: conformal symmetry on defects

$SO(m)$: rotation in the transverse direction

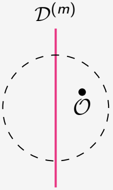


- Defects allow defect local operators $\hat{\mathcal{O}}_n$

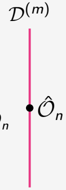
OPE for conformal defects

There are two types of OPE in DCFT

- Bulk-to-defect OPE : [Cardy 84, McAvity-Osborn 95]

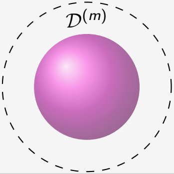


$$= \sum_n b^{(m)}_{\mathcal{O}\hat{\mathcal{O}}_n}$$



$$+ \ (\text{descendants})$$

- Defect OPE : [Berenstein-Corrado-Fischler-Maldacena 98, Gadde 16]



$$= \sum_n c^{(m)}_{\mathcal{O}_n} \ \mathcal{O}_n \ + \ (\text{descendants})$$

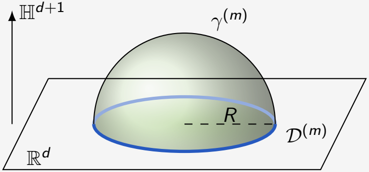
Goal of the talk

The defect OPE has been less understood than the bulk-to-defect OPE

1. To what extent are we able to determine the structure of the defect OPE by conformal symmetry?
   - Decomposition by the irreducible representations

$$\mathcal{D}^{(m)} = \sum_{n \,\in\, \text{primaries}} \mathcal{B}^{(m)}[\mathcal{O}_n]$$

2. Can we probe the bulk AdS information by conformal defects on the boundary?



Overview of our results

1. Give the integral representation of the defect OPE blocks

$$\mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \, \langle \mathcal{O}_n(x) \rangle_{\mathcal{D}^{(m)}} \tilde{\mathcal{O}}_n(x)$$

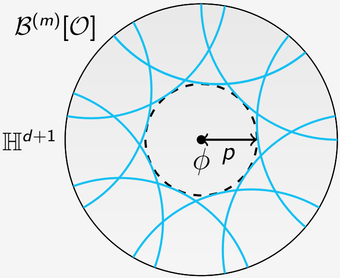
$\tilde{\mathcal{O}}$: shadow operator with $\tilde{\Delta} = d - \Delta$ for $\mathcal{O}$ with Δ

2. Reconstruct the AdS scalar field from the blocks
   - $\hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}]$:

     The Radon transform of the AdS scalar field ϕ when $\mathcal{O}$ scalar

   - Reproduce the (Euclidean) HKLL formula :

$$\phi(Y) = \phi(\hat{\phi}) = \int d^d x \, K(Y|x) \, \mathcal{O}(x)$$



Outline

Embedding space formalism

- Uplift $\mathbb{R}^d$ into the embedding space $\mathbb{R}^{d+1,1}$

$$x^\mu \ \rightarrow \ X^A \quad (A = 1, \cdots d+2)$$

- Choose the d-dim hypersurface (the projective null cone)

$$X \cdot X \equiv X^A X_A = 0 \ , \qquad X^A \sim \lambda \, X^A \quad (\lambda \in \mathbb{R})$$

- Realize the conformal symmetry $SO(d+1,1)$ linearly

$$X'^A = J^A_{\ B} X^B \ , \quad J^A_{\ B} \in so(d+1,1)$$

Correlation functions in embedding space

Construct the invariants under $SO(d+1,1)$ in the embedding space

- Two-point function of scalar fields Φ of dimension Δ:

$$\langle \Phi(X_1)\, \Phi(X_2) \rangle = \frac{1}{(X_1 \cdot X_2)^\Delta}$$

For a spin l operator $\mathcal{O}_{\Delta,l}(X)$, define the encoding polynomial by contracting the indices with an auxiliary vector Z^A:
[Costa-Penedones-Poland-Rychkov 11]

$$\mathcal{O}_\Delta(X,Z) \equiv \mathcal{O}_{\Delta,A_1\cdots A_l}(X)\, Z^{A_1}\cdots Z^{A_l} \ , \qquad X\cdot Z = Z^2 = 0$$

- Two-point functions are uniquely determined by the invariants:

$$\langle \mathcal{O}_{\Delta,l}(X_1,Z_1)\, \mathcal{O}_{\Delta,l}(X_2,Z_2) \rangle = \frac{1}{(X_{12})^\Delta}\left[\frac{(Z_1\cdot Z_2)(X_1\cdot X_2) - (Z_1\cdot X_2)(Z_2\cdot X_1)}{X_{12}}\right]^l \ .$$

Conformal defects in embedding space

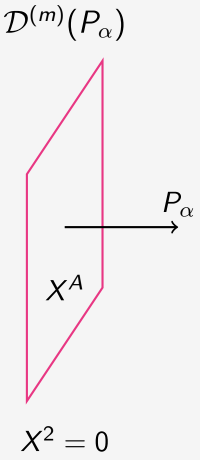
- Specify a codimension-m defect by the frame vectors P_α^A: [Gadde 16]

$$X \cdot P_\alpha = 0 \ , \quad (\alpha = 1, \cdots m)$$

- Choosing an orthonormal basis, $P_\alpha \cdot P_\beta = \delta_{\alpha\beta}$, the defect preserves the symmetry group $SO(d+1-m,1) \times SO(m)$

- Correlation function should be invariant under the frame choice ($P'_\alpha = g_\alpha{}^\beta P_\beta$, $g \in SO(m)$):

$$\langle \mathcal{D}^{(m)}(P_\alpha)\, \mathcal{O}_\Delta(X) \rangle \propto \frac{1}{[(P^\alpha \cdot X)(P_\alpha \cdot X)]^{\Delta/2}}$$



Defect OPE blocks

We expect the defect OPE of the form

$$\mathcal{D}^{(m)}(P_\alpha) = \langle \mathcal{D}^{(m)}(P_\alpha) \rangle \left[\sum_n c^{(m)}_{\mathcal{O}_n} R^{\Delta_n} \mathcal{O}_n(C) + (\text{descendants}) \right]$$

(R : radius, C : center vector)

- The descendant terms are fixed by the primary $\mathcal{O}_n$ and the conformal symmetry,

$$\mathcal{D}^{(m)}(P_\alpha) = \sum_n \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n]$$

- The defect OPE block is in the irreducible representation of $\mathcal{O}_n$:

$$\langle \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_n] \, \mathcal{O}_n(X) \rangle = \langle \mathcal{D}^{(m)}(P_\alpha) \, \mathcal{O}_n(X) \rangle \ .$$

Projectors and shadows

- Want to characterize the defect OPE blocks by their irreps

- Spectral decomposition by the irreps of the conformal group:

$$\mathbf{1} = \sum_n |\mathcal{O}_n|$$

- $|\mathcal{O}_n|$: Projector onto the conformal multiplet of the primary $\mathcal{O}_n$
  [Ferrara-Grillo-Parisi-Gatto 72, $\cdots$, Simmons-Duffin 12]

  For a spin l operator,

$$|\mathcal{O}_{\Delta,l}| = \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, |\mathcal{O}_{\Delta,l}(X, D_Z)\rangle \langle \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z)|$$

  $\tilde{\mathcal{O}}_{d-\Delta,l}$: the shadow operator of $\mathcal{O}_{\Delta,l}$
  D_Z: Todorov's differential operator ($\sim \partial_Z$)

Integral representation of defect OPE blocks

- Expand the defect by the projectors:

$$\langle \mathcal{D}^{(m)}(P_\alpha)\cdots\rangle = \sum_{\Delta,l} \langle \mathcal{D}^{(m)}(P_\alpha)|\mathcal{O}_{\Delta,l}|\cdots\rangle + (\text{other irrep.})$$
$$= \sum_{\Delta,l} \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, \langle \mathcal{D}^{(m)}(P_\alpha)\, \mathcal{O}_{\Delta,l}(X, D_Z)\rangle \, \langle \tilde{\mathcal{O}}_{d-\Delta,l}(X,Z)\cdots\rangle$$
$$+ (\text{other irrep.})$$

- Can read off the block contribution:

The integral rep of the defect OPE block

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = \frac{1}{\mathcal{N}_{\Delta,l}} \int D^d X \, \tilde{\mathcal{O}}_{d-\Delta,l}(X, D_Z)\, \langle \mathcal{D}^{(m)}(P_\alpha)\, \mathcal{O}_{\Delta,l}(X,Z)\rangle$$

Alternative representation and monodromy condition

- The integral rep is invariant under the exchange $\mathcal{O} \leftrightarrow \tilde{\mathcal{O}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = \frac{1}{\mathcal{N}_{d-\Delta,l}} \int D^d X \, \mathcal{O}_{\Delta,l}(X, D_Z) \, \langle \mathcal{D}^{(m)}(P_\alpha) \, \tilde{\mathcal{O}}_{d-\Delta,l}(X, Z) \rangle$$

- The block includes the shadow contribution $g_{\tilde{\mathcal{O}}}$

$$\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = g_{\mathcal{O}} + K \, g_{\tilde{\mathcal{O}}}$$

- To extract the shadow contribution, impose the monodromy condition

$$\begin{aligned} g_{\mathcal{O}} \quad &\to \quad e^{2\pi i \Delta} \, g_{\mathcal{O}} \\ g_{\tilde{\mathcal{O}}} \quad &\to \quad e^{2\pi i (d-\Delta)} \, g_{\tilde{\mathcal{O}}} \end{aligned}$$

under the phase rotation $P_\alpha \to e^{-2\pi i} P_\alpha$

Constraint equations

There are two types of equations the defect OPE block satisfies

The conformal Casimir equation

$$\left(L^2(P_\alpha) + \mathcal{C}_{\Delta,l}\right) \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0$$

- $L^2(P_\alpha) \equiv \frac{1}{2} L_{AB}(P_\alpha)\, L^{AB}(P_\alpha)$: quadratic Casimir operator
- $\mathcal{C}_{\Delta,l} = \Delta(\Delta - d) + l(l + d - 2)$: the eigenvalue

"Trivial" equations for scalar primaries $\mathcal{O}_\Delta$

$$C_{ABCD}(P_\alpha)\, \mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_\Delta] = 0$$

- $C_{ABCD}(P_\alpha) \equiv \frac{1}{2} L_{A[B}(P_\alpha) L_{CD]}(P_\alpha)$: $\frac{d(d^2-1)(d+2)}{24}$ quadratic operators

Moduli space of conformal defects

- The moduli space has a coset structure:

$$\mathcal{M}^{(d,m)} = \frac{SO(d+1,1)}{SO(m)\times SO(d+1-m,1)}$$

- The quadratic Casimir operator is the Laplacian on $\mathcal{M}^{(d,m)}$

$$-L^2(P_\alpha) = \Box_{\mathcal{M}^{(d,m)}}$$

- The defect OPE block is a scalar field propagating on $\mathcal{M}^{(d,m)}$

Klein-Gordon equation on $\mathcal{M}^{(d,m)}$

$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right)\mathcal{B}^{(m)}[P_\alpha, \mathcal{O}_{\Delta,l}] = 0\ , \quad M^2 = \mathcal{C}_{\Delta,l}$$

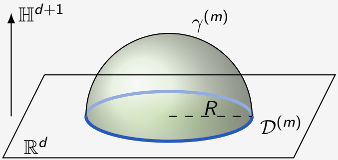
Outline

Conformal defects and submanifolds in AdS

- Associated to a given defect $\mathcal{D}^{(m)}$ is a unique submanifold $\gamma^{(m)}$ in AdS s.t. $\partial\gamma^{(m)} = \mathcal{D}^{(m)}$

- Their moduli spaces are equivalent:

$$\mathcal{M}^{(d,m)} = \frac{\mathrm{Isom}(\mathrm{AdS}_{d+1})}{\mathrm{Stab}(\gamma^{(m)} \in \mathrm{AdS}_{d+1})}$$



- What is the dual description of the defect OPE block in AdS?

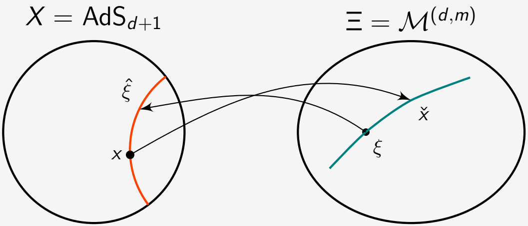
Moduli space and AdS

- The defect OPE block $\mathcal{B}^{(m)}$ is a scalar field not on $\mathbb{H}^{d+1}$(= Euclidean AdS$_{d+1}$), but on $\mathcal{M}^{(d,m)}$

- The two spaces have similar coset structures:

$$\mathcal{M}^{(d,m)} = \frac{SO(d+1,1)}{SO(m)\times SO(d+1-m,1)}\,, \qquad \mathbb{H}^{d+1} = \frac{SO(d+1,1)}{SO(d)}$$

- In this case, the Radon transform maps a function from one to the other [e.g. Helgason 10]



Radon transform

From $\mathbb{H}^{d+1}$ to $\mathcal{M}^{(d,m)}$:

$$\hat{\phi}(\xi) = \int_{x\in\xi} d\nu(x)\, \phi(x)$$

- ξ : a codim-m submanifold in $\mathbb{H}^{d+1}$
- $\phi(x)$: a function on $\mathbb{H}^{d+1}$

From $\mathcal{M}^{(d,m)}$ to $\mathbb{H}^{d+1}$:

$$\check{f}(x) = \int_{x\in\xi} d\mu(\xi)\, f(\xi)$$

- ξ : a codim-m submanifold through x
- $f(\xi)$: a function on $\mathcal{M}^{(d,m)}$

Intertwining property

$$\left(\Box_{\mathbb{H}^{d+1}} - M^2\right)\, \phi = 0 \quad \Leftrightarrow \quad \left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right)\, \hat{\phi} = 0$$

Defect OPE blocks as Radon transformed fields

- We may regard the defect OPE block

$$\hat{\phi} = \mathcal{B}^{(m)}$$

as the Radon transform of an AdS scalar field ϕ

- The Radon transform maps ϕ into the manifold with larger dimensions

$$\dim \mathcal{M}^{(d,m)} - \dim \mathbb{H}^{d+1} = (m-1)(d+1-m) \geq 0$$

Hence there should be constraints for $\hat{\phi}$, which are known as the range characterization [Ishikawa 97]

$$C_{ABCD}\, \hat{\phi} = 0$$

- This is what the block $\mathcal{B}^{(m)}[\mathcal{O}_\Delta]$ satisfies for a scalar primary $\mathcal{O}_\Delta$

Dictionary for the Radon transform

In summary, the defect OPE block of a scalar primary is identified with an AdS scalar field

- $\mathcal{M}^{(d,m)}$
- scalar block: $\mathcal{B}^{(m)}[\mathcal{O}_\Delta]$

quadratic Casimir equation

$$\left(\Box_{\mathcal{M}^{(d,m)}} - M^2\right) \mathcal{B}^{(m)}[\mathcal{O}_\Delta] = 0$$

constraint equations

$$C_{ABCD}\, \mathcal{B}^{(m)}[\mathcal{O}_\Delta] = 0$$

- $\mathbb{H}^{d+1}$
- AdS scalar field: ϕ

Klein-Gordon equation

$$\left(\Box_{\mathbb{H}^{d+1}} - M^2\right) \phi = 0$$

range characterization

$$C_{ABCD}\, \hat{\phi} = 0$$

Construction of AdS scalar fields from defects

- Can construct an AdS scalar ϕ from the scalar block $\hat{\phi} = \mathcal{B}^{(m)}$ by the dual Radon transform $f \to \check{f}$?

- The inversion formula allows such a construction (note $(\hat{\phi})\check{} \neq \phi$)

$$\phi = \mathcal{I} \circ \hat{\phi} = \mathcal{I} \circ \mathcal{B}^{(m)}[\mathcal{O}_\Delta]$$

  $\mathcal{I}$: an integral transformation [Helgason 10]

- Equivalent to the bulk reconstruction formula

$$\phi(Y) = \int D^d X \, K_\Delta(Y|X) \, \mathcal{O}_\Delta(X)$$

  with the Euclidean version of the HKLL kernel [Hamilton-Kabat-Lifschytz-Lowe 06]

$$(\Box_{\mathbb{H}^{d+1}} - M^2) \, K_\Delta(Y|X) = 0 \,, \quad M^2 = \Delta(d - \Delta)$$

Summary

1. Give the integral representation of the defect OPE blocks

$$\mathcal{B}^{(m)}[\mathcal{O}_n] = \int d^d x \, \langle \mathcal{O}_n(x) \rangle_{\mathcal{D}^{(m)}} \tilde{\mathcal{O}}_n(x)$$

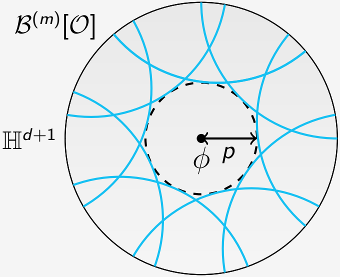
$\tilde{\mathcal{O}}$: shadow operator with $\tilde{\Delta} = d - \Delta$ for $\mathcal{O}$ with Δ

2. Reconstruct the AdS scalar field from the blocks
   - $\hat{\phi} = \mathcal{B}^{(m)}[\mathcal{O}]$:

     The Radon transform of the AdS scalar field ϕ when $\mathcal{O}$ scalar

   - Reproduce the (Euclidean) HKLL formula :

$$\phi(Y) = \phi(\hat{\phi}) = \int d^d x \, K(Y|x) \, \mathcal{O}(x)$$



Future direction

- Constraints on the bulk OPE data from the block?
  - Need the relation to the bulk-to-defect OPE
- Can extend the construction to higher spin fields in AdS?
  - No known Radon transform beyond a scalar field in AdS
  - Spinning defects to incorporate spins $\Rightarrow$ [Nozomu's talk]
- Generalized notion of entanglement entropy?
  - Codim-2 deficit angle $\Rightarrow$ codim-m deficit ??
  - Analytic continuation to Lorentzian signature?