Entanglement Spreading and Oscillation

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Introduction:

• EE is a candidate of an entropy in Non-equilibrium physics.

• In AdS/CFT correspondences, Entanglement in CFT living on the boundary is expected to be significantly related to Gravity in the bulk.
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- Late-time EE in a sudden quench is proportional to subsystem size (thermal-entropy-like behavior)
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• Late-time EE in a sudden quench is proportional to subsystem size (``thermalization”).
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• Late-time EE in a sudden quench is proportional to subsystem size (**"thermalization"**).

The dynamics of entanglement \(\leftrightarrow\) The dynamics of gravity
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The dynamics of entanglement The dynamics of gravity

Thermalization Black Hole Physics
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The dynamics of entanglement ↔ The dynamics of gravity

Thermalization

Creation of Black Hole
Introduction:

• EE is a candidate of \textit{an entropy in Non-equilibrium physics}.

\textbf{It is important to study the \textit{dynamical features} of Entanglement.}

The dynamics of entanglement \hfill The dynamics of gravity

\textit{Thermalization} \hfill \textit{Black Hole Physics
Motivation
Our Motivation

In the sudden global quenches where Hamiltonian suddenly change, in the late time, the change of entanglement entropy $(\Delta S_A(t) = S_A(t) - S_A(t_{initial}))$ is:

[Calabrese-Cardy, 06] [Hartman-Maldacena, 13] [Liu-Suh, 13]
Our Motivation

In the sudden global quenches where Hamiltonian suddenly change, in the late time, the change of entanglement entropy $(\Delta S_A(t) = S_A(t) - S_A(t_{initial}))$ is:

$$\Delta S_A \sim \text{Volume of Subsystem}$$

[Calabrese-Cardy, 06] [Hartman-Maldacena, 13] [Liu-Suh, 13]
Our Motivation

In the sudden global quenches where Hamiltonian suddenly change, in the late time, the change of entanglement entropy $(\Delta S_A(t) = S_A(t) - S_A(t_{\text{initial}}))$ is:

$$
\Delta S_A \rightarrow S_{\text{thermal}}
$$

[Calabrese-Cardy, 06] [Hartman-Maldacena, 13] [Liu-Suh, 13]
Our Motivation

Sudden Quenches

Thermalizes
Our Motivation

Sudden Quenches

Thermalizes

Is this unique behavior for sudden quenches?
Our Motivation
If the state is quenched gradually (smooth quenched),

*is subsystem thermalized?*

and

*how does entanglement time-evolve?*
What we have done

We have studied the time evolution of quantities (EE, LN, MI) in smooth quenches (ECP and CCP).

\[ \Delta S_A(t) = S_A(t) - S_A(t_{initial}) \]

\( EE \) for initial mass
Smooth Quenches

2d – Time-dependent Hamiltonian

\[ H(t) = \frac{1}{2} \int dx \left[ \Pi^2(x) + \partial_x \phi^2(x) + m^2(t) \phi^2(x) \right] \]

- **ECP:** \( m^2(t) = \frac{1}{\xi^2} \left( 1 - \tanh \left( \frac{t}{\delta t} \right) \right) \)

- **CCP:** \( m^2(t) = \frac{1}{\xi^2} \tanh^2 \left( \frac{t}{\delta t} \right) \)
Smooth Quenches

- **ECP:** \( m^2(t) = \frac{1}{\xi^2} \left( 1 - \tanh \left( \frac{t}{\delta t} \right) \right) \)

- **CCP:** \( m^2(t) = \frac{1}{\xi^2} \tanh^2 \left( \frac{t}{\delta t} \right) \)

[Das-Galante-Myers, 14]
Smooth Quenches

- ECP: \[ m^2(t) = \frac{1}{\xi^2} \left( 1 - \tanh \left( \frac{t}{\delta t} \right) \right) \]
- CCP: \[ m^2(t) = \frac{1}{\xi^2} \tanh^2 \left( \frac{t}{\delta t} \right) \]

Changing a ratio, \( \frac{\delta t}{\xi} \), we have studied time evolution of EE, LN and MI.
Smooth Quenches

We took *two extreme limits*:

**Fast limit:**

\[ \omega \ll 1 \ (\delta t \ll \xi) \]

**Slow limit:**

\[ \omega \gg 1 \ (\delta t \gg \xi) \]
Results (EE)

Late-time $\Delta S_A$ in ECP is proportional to subsystem size.

(Thermalized.)

Late-time $\Delta S_A$ in fast-CCP ($\xi \ll \delta t$) is proportional to subsystem size.
Assumptions: \( \frac{1}{m \cdot a} = \frac{\xi}{a} \gg 1 \), \( a \): is a lattice spacing.

- **ECP:**
  
  **Fast limit:** \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)
  
  \[ \Delta S_A \sim C_1 \frac{l}{\xi} \]

  **Slow limit:** \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1 \), \( E_{kz} = \frac{1}{\delta t} \ll 1 \)
  
  \[ \Delta S_A \sim C_2 E_{kz} \cdot l \]

- **CCP:**
  
  **Fast limit:** \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)
  
  \[ \Delta S_A \sim C_3(\omega) \frac{l}{\xi} \]

  **Slow limit:** \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1 \), \( \xi_{kz} = \sqrt{\frac{\delta t}{m}} = \sqrt{\xi \delta t} \gg 1 \)
Assumptions:

**ECP:**

- Fast limit: $\omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1$

  $\Delta S_A \sim C_1 \frac{l}{\xi}$

- Slow limit: $\omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1$, $E_{kz} = \frac{1}{\delta t} \ll 1$

  $\Delta S_A \sim C_2 E_{kz} \cdot l$

**CCP:**

- Fast limit: $\omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1$

  $\Delta S_A \sim C_3(\omega) \frac{l}{\xi}$

- Slow limit: $\omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1$, $\xi_{kz} = \sqrt{\frac{\delta t}{m}} = \sqrt{\xi \delta t} \gg 1$
Assumptions: \( \frac{1}{m \cdot a} = \frac{\xi}{a} \gg 1 \), \( a \): is a lattice spacing.

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  - Fast limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)
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\[ \Delta S_A \sim C_1 \]

- **CCP:**
  - Fast limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)
  - Slow limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1 \), \( \xi_{kz} = \sqrt{\frac{\delta t}{m}} = \sqrt{\xi \delta t} \gg 1 \)

\[ \Delta S_A \sim C_3(\omega) \frac{l}{\xi} \]

Depends on \( \omega \)
**Result 1**

**Assumptions:** \( \frac{1}{m \cdot a} = \frac{\xi}{a} \gg 1 \), \( a \): is a lattice spacing.

- **ECP:**
  
  Fast limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)  
  
  slow limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1 \)  
  
  \( E_{kz} = \frac{1}{\delta t} \ll 1 \)

  \[ \Delta S_A \sim C_1 \]

  **Depends on** \( \omega \)

  \[ \mathcal{C}_2 E_{kz} \cdot l \]

- **CCP:**
  
  Fast limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \ll 1 \)  
  
  slow limit: \( \omega = \delta t \cdot m = \frac{\delta t}{\xi} \gg 1 \), \( \xi_{kz} = \sqrt{\frac{\delta t}{m}} = \sqrt{\xi \delta t} \gg 1 \)

  \[ \Delta S_A \sim \mathcal{C}_3 (\omega) \left( \frac{l}{\xi} \right) \]
In fast limit, keeping $\xi$ constant and, $\omega$ decreases $\implies C_3(\omega)$ decreases.

In slow limit, keeping $\xi$ constant and, $\omega$ increases $\implies C_3(\omega)$ decreases.
\( C_3(\omega) \)

In fast limit, keeping \( \xi \) constant and, \( \omega \) decreases.

**Consistent with a number operator in late time.**

In fast limit, keeping \( \xi \) constant and, \( \omega \) increases.

\[ C_3(\omega) \text{ decreases.} \]
How does the subsystem thermalize in slow ECP? (Quasi-particles are created?)

Quasi-particles are created when the adiabaticity breaks down! (Subsystems thermalize!)
Results in slow ECP

Quasi-particles are created when the adiabaticity breaks down!
(Subsystems thermalize!)

Quasi-particles are created at $t = t_{kz}$ and carry quantum entanglement.

Adiabatic

Thermalize around $t = t_{kz} + \frac{l}{2}$
Interpretation in ECP and fast CCP

• Time Evolution of $\Delta S_A(t)$

Propagation of Entangled particles
Results (EE)

Entanglement Oscillation

$\Delta S_A$

$(\xi, \delta t) = (5, 100)$

$\xi_{kz} = 100$
Results (EE)

Entanglement Oscillation

\[ (\xi, \delta t) = (5, 100) \]

The period of oscillation @ late time.

The periodicity of zero mode \( \sim \pi \xi \)
Results (EE)

Late-time $\Delta S_A$ in CCP oscillates with periodicity determined by a late-time mass.

$\Delta S_A$ in fast-CCP is minimized at $t = 2\xi$. 

$t = 2\xi (\xi, \delta t) = (5, 100)$
Results (EE)

Late-time $\Delta S_A$ in CCP oscillates with periodicity determined by a late-time mass.

$\Delta S_A$ in slow-CCP starts to oscillate after $t = 2\xi_{kz}$.

Adiabaticity breaks down.
Results (EE)

Late-time $\Delta S_A$ in CCP oscillates with periodicity determined by a late-time mass.

$t = 2\xi$ and $t = 2\xi_{kz}$ characterize time evolution.

Adiabaticity breaks down.

$t = 2\xi_{kz}$
Results (EE)

Time evolution is characterized by

\[ t \simeq 2\xi_{kz}, \quad 2\xi. \]

After \( t = 2\xi, \ 2\xi_{kz}, \ \Delta S_A \) oscillates.
Result(LN, MI (work in progress))

Their time-evolution in ECP can be interpreted in terms of *relativistic propagation of quasi-particles*.

- If two subsystem are well-separated, Late-time MI in fast-ECP *increases* (logarithmically?).

- If two subsystem are well-separated, Late-time LN in fast-ECP *decreases*.
Result (LN, MI (work in progress))

Their time-evolution in ECP can be interpreted in terms of *relativistic propagation of quasi-particles*.

- Late-time LN *weakly depends* on slow modes.

• If two subsystem are well-separated, Late-time LN in fast-ECP *decreases*.
Result (LN, MI (work in progress))

Time evolution of LN and MI in CCP strongly depends on $\Delta l$.

If $\Delta l < \xi$, LN and MI oscillate.

If $\Delta l \gg \xi$, oscillation is suppressed. Their time evolution can be interpreted in terms of relativistic propagation of quasiparticle.
The Contents of Talk

• Introduction
• Motivation
• Results(EE)
• Results(LN, MI)
• Setup
• Method
• ECP(EE)
• CCP(EE)
• MI and LN (work in progress)
• Summary and Future directions
Setup
Smooth Quenches

• These quenches are *more realistic*.

• Hamiltonian is not changed suddenly but is changed smoothly.

• We can excite the state *slowly or fast*.

• This is a kind of generalization of sudden quenches.
Our protocol (Smooth Quenches)

2d – Time-dependent Hamiltonian

\[ H(t) = \frac{1}{2} \int dx \left[ \Pi^2(x) + \partial_x \phi^2(x) + m^2(t)\phi^2(x) \right] \]

- **ECP:** \[ m^2(t) = \frac{1}{\xi^2} \left( 1 - \tanh \left( \frac{t}{\delta t} \right) \right) \]
- **CCP:** \[ m^2(t) = \frac{1}{\xi^2} \tanh^2 \left( \frac{t}{\delta t} \right) \]
Our protocol (Smooth Quenches) [Das-Galante-Myers, 14]

• **ECP:** $m^2(t) = \frac{1}{\xi^2} \left( 1 - \tanh \left( \frac{t}{\delta t} \right) \right)$

• **CCP:** $m^2(t) = \frac{1}{\xi^2} \tanh^2 \left( \frac{t}{\delta t} \right)$
Our protocol (Smooth Quenches)

- ECP: [Das-Galante-Myers, 14]
- CCP:

We have two tunable parameters. States are excited slowly and rapidly.

$\sim \delta t$

$\sim \frac{1}{\xi^2}$
Our setup

• Theory 2d Free scalar with time dependent mass $m(t)$.
• Put it on the lattice but take the *thermodynamic limit*.

Mass profile: $m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2$

At $t=0$, the theory is at critical point.
Our setup

- Theory 2d Free scalar with time dependent mass $m(t)$.
- Put it on the lattice but take the **thermodynamic limit**.

Mass profile: $m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2$

*Initial state: The Ground state for massive free scalar with mass $m^2$.*
Slow Quenches

Mass profile: \( m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2 \)
Slow Quenches

Mass profile: $m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2$

Very Early time: Observables can be computed adiabatically.
Adiabatic Expansion

\[ X_{ij} = X_{ij}^{(0)} + X_{ij}^{(1)} + \cdots \]

\[ P_{ij} = P_{ij}^{(0)} + P_{ij}^{(1)} + \cdots \]

\[ D_{ij} = D_{ij}^{(0)} + D_{ij}^{(1)} + \cdots \]

\[ \langle \phi_i \phi_j \rangle = X_{ij} \]

\[ \langle \dot{\phi}_i \dot{\phi}_j \rangle = P_{ij} \]

\[ \frac{1}{2} \left\langle \{ \phi_i, \dot{\phi}_j \} \right\rangle = D_{ij} \]

Higher orders has higher derivative with respect to \( t \).
Adiabatic Expansion

\[ X_{ij} = X_{ij}^{(0)} + X_{ij}^{(1)} + \cdots \]
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\[ \frac{1}{2} \langle \{ \phi_i, \dot{\phi}_j \} \rangle = D_{ij} \]

*Higher orders has higher derivative with respect to t.*

Landau Criteria

\[ \frac{1}{m^2(t)} \frac{dm(t)}{dt} \ll 1 \]

→ Adiabaticity holds
Slow Quenches

Mass profile: \( m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2 \)

Very Early time:
Observables \textit{can} be computed adiabatically.

Around critical point:
Observables \textit{can not} be computed adiabatically.
Adiabatic Expansion

\[ X_{ij} = X_{ij}^{(0)} + X_{ij}^{(1)} + \cdots \]

\[ P_{ij} = P_{ij}^{(0)} + P_{ij}^{(1)} + \cdots \]

\[ D_{ij} = D_{ij}^{(0)} + D_{ij}^{(1)} + \cdots \]

Higher orders has higher derivative with respect to t.

Landau Criteria

\[ \frac{1}{m^2(t)} \frac{dm(t)}{dt} \sim 1 \]

Adiabaticity breaks down.
Adiabatic Expansion

\[ X_{ij} = X_{ij}^{(0)} + X_{ij}^{(1)} + \cdots \]
\[ P_{ij} = P_{ij}^{(0)} + P_{ij}^{(1)} + \cdots \]
\[ D_{ij} = D_{ij}^{(0)} + D_{ij}^{(1)} + \cdots \]

Higher orders has higher derivative with respect to \( t \).

Landau Criteria \[ \frac{1}{m^2(t)} \frac{dm(t)}{dt} \sim 1 \rightarrow \text{Adiabaticity breaks down.} \]

The time when adiabaticity breaks down is called Kibble-Zurek Time \( t_{kz} \).
Slow Quenches

Mass profile: \( m^2(t) = m^2 \tanh \left( \frac{t}{\delta t} \right)^2 \)

Very Early time:
Observables \textbf{can} be computed adiabatically.

Around critical point:
We assume that at \( t = -t_{kz} \), adiabaticity breaks down.

(Around \( t = t_{kz} \), time process becomes adiabatic again.)
If $t_{kz}$ is so small, **the most of whole time evolution** is adiabatic.

More precisely, \( \frac{t_{kz}}{\delta t} \ll 1 \)
If $t_{kz}$ is so small, the most of whole time evolution is adiabatic.

More precisely, $\frac{t_{kz}}{\delta t} \ll 1$

$\omega = m \delta t \gg 1$
If $t_{kz}$ is so small, the most of whole time evolution is adiabatic.

More precisely,

$$\frac{t_{kz}}{\delta t} \ll 1$$

$$\omega = m\delta t \gg 1$$

$$t_{kz} = \frac{1}{m}(-t_{kz}) = \xi_{kz} = \sqrt{\frac{\delta t}{m}}$$
If $t_{kz}$ is so small, the most of whole time evolution is adiabatic.

More precisely, \[ \frac{t_{kz}}{\delta t} \ll 1 \]

In slow quenches, Kibble-Zurek time is small.

\[ t_{kz} = m(-t_{kz}) = \xi_{kz} = \sqrt{\frac{\delta t}{m}} \]
• ECP: \[ m^2(t) = \frac{1}{\xi^2} \left(1 - \tanh\left(\frac{t}{\delta t}\right)\right) \]

• CCP: \[ m^2(t) = \frac{1}{\xi^2} \tanh^2\left(\frac{t}{\delta t}\right) \]

Fast Quench limit: \( \omega \ll 1 \quad (\delta t \ll \xi) \)

Slow Quench limit: \( \omega \gg 1 \quad (\delta t \gg \xi) \)
• ECP: \[ m^2(t) = \frac{1}{\xi^2} \left(1 - \tanh \left(\frac{t}{\delta t}\right)\right) \]

• CCP: \[ m^2(t) = \frac{1}{\xi^2} \tanh^2 \left(\frac{t}{\delta t}\right) \]

Parameters

\begin{align*}
\xi_{kz} &= 1/E_{kz} = \delta t \\
t_{kz} &= \delta t \log \omega \\
\xi_{kz} &= t_{kz} = \sqrt{\delta t \cdot \xi}
\end{align*}
Method
Discretize

• We put our theory on the lattice so that we compute $\Delta S'_A$ by the correlator method.

Correlator method

• This is a method to compute $\Delta S'_A$ by using the correlation functions.

Conditions: 1. State is a Gaussian state.
2. Local observables can be computed by Wick theorem.
Correlator Method

• If an initial state $|\Psi\rangle$ is given by a gaussian state:
  For example, $|\Psi\rangle \left( a_k |\Psi\rangle = 0 \right)$
Correlator Method

• If an initial state $|\Psi\rangle$ is given by a gaussian state:
  For example, $|\Psi\rangle$ ( $a_k |\Psi\rangle = 0$)

We assume that a reduced density matrix is given by

$$\rho_A = tr_B \rho \sim e^{-\sum \gamma_k b_k^\dagger b_k}$$

If $\phi_i, \phi_j$ are included in A,

$$\langle \phi_i \phi_j \rangle = tr \left( \rho \phi_i \phi_j \right) = tr_A \left( \rho_A \phi_i \phi_j \right) = \langle \phi_i \phi_j \rangle_A$$
Correlator Method

• If an initial state $|\Psi\rangle$ is given by a gaussian state:
  For example, $|\Psi\rangle (a_k |\Psi\rangle = 0)$

We assume that a reduced density matrix is given by

$$
\rho_A = tr_B \rho \sim e^{- \sum \gamma_k b_k^\dagger b_k}
$$

If $\phi_i, \phi_j$ are included in $A$, 

$$
\langle \phi_i \phi_j \rangle = tr (\rho \phi_i \phi_j) = tr_A (\rho_A \phi_i \phi_j) = \langle \phi_i \phi_j \rangle_A
$$
Correlator Method

\[ \langle \phi_i \phi_j \rangle = \text{tr} (\rho \phi_i \phi_j) = \text{tr}_A (\rho_A \phi_i \phi_j) = \langle \phi_i \phi_j \rangle_A \]

Determined by E.O.M and so on.
Correlator Method

\[ \langle \phi_i \phi_j \rangle = \text{tr} \left( \rho \phi_i \phi_j \right) = \text{tr}_A \left( \rho_A \phi_i \phi_j \right) = \langle \phi_i \phi_j \rangle_A \]

Determined by E.O.M and so on.

Two point functions \( \gamma_k \)
Correlator Method

\[ \langle \phi_i \phi_j \rangle = tr (\rho \phi_i \phi_j) = tr_A (\rho_A \phi_i \phi_j) = \langle \phi_i \phi_j \rangle_A \]

Determined by E.O.M and so on.

Two point functions

\[ \gamma_k \]

\[ \rho_A \sim e^{-\sum \gamma_k b_k^\dagger b_k} \]
Correlator Method

\[ \langle \phi_i \phi_j \rangle = tr \left( \rho \phi_i \phi_j \right) = tr_A \left( \rho_A \phi_i \phi_j \right) = \langle \phi_i \phi_j \rangle_A \]

- Two point functions
- \( \gamma_k \)
- \( \rho_A \sim e^{- \sum \gamma_k b_k^\dagger b_k} \)

\( S_A \) is determined by two point functions.
Correlator method

Entanglement Entropy:

\[ S_A = \sum_{k=1}^{l} s_A(\gamma_k) \]

\[ s_A(\gamma_k) = \left( \frac{1}{2} + \gamma_k \right) \log \left( \frac{1}{2} + \gamma_k \right) - \left( -\frac{1}{2} + \gamma_k \right) \log \left( -\frac{1}{2} + \gamma_k \right) \]

\[ \Gamma = \begin{pmatrix} X_{ij} & \frac{1}{2} D_{ij} \\ \frac{1}{2} D_{ji} & P_{ij} \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ X_{ij} = \langle \phi_i \phi_j \rangle \quad P_{ij} = \langle \pi_i \pi_j \rangle \quad D_{ij} = \langle \{\phi, \pi_j\} \rangle \]

\[ M = i J \Gamma \] has eigenvalues \( \pm \gamma_k \).
Correlator method

Entanglement Entropy:

\[ S_A = \sum_{k=1}^{l} s_A(\gamma_k) \]

\[ s_A(\gamma_k) = \left( \frac{1}{2} + \gamma_k \right) \log \left( \frac{1}{2} + \gamma_k \right) - \left( -\frac{1}{2} + \gamma_k \right) \log \left( -\frac{1}{2} + \gamma_k \right) \]

\[ \gamma = \begin{pmatrix} X_{ij} & \frac{1}{2} D_{ij} \\ \frac{1}{2} D_{ji} & P_{ij} \end{pmatrix} \]

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ X_{ij} = \langle \phi_i \phi_j \rangle \quad P_{ij} = \langle \pi_i \pi_j \rangle \quad D_{ij} = \langle \{\phi_i, \pi_j\} \rangle \]

\[ M = iJ \Gamma \] has eigenvalues ±γ_k.
Correlator method

Entanglement Entropy:

\[ S_A = \sum_{k=1}^{l} s_A(\gamma_k) \]

\[ s_A(\gamma_k) = \left( \frac{1}{2} + \gamma_k \right) \log \left( \frac{1}{2} + \gamma_k \right) - \left( -\frac{1}{2} + \gamma_k \right) \log \left( -\frac{1}{2} + \gamma_k \right) \]

The subsystem size = \( l \)

By evaluating \( M \), we can compute \( S_A \).

\[ X_{ij} = \langle \phi_i | \phi_j \rangle \quad Y_{ij} = \langle \pi_i | \pi_j \rangle \quad D_{ij} = \langle \{ \phi_i, \pi_j \} \rangle \]

\[ M = iJ \Gamma \] has eigenvalues \( \pm \gamma_k \).
EE in ECP
EE in fast ECP
Plot of EE in Fast ECP

Orange Curve: \( l=2000 \), Green Curve: \( l=1000 \), Pink Curve: \( l=500 \),
Blue Curve: \( l=100 \), Purple Curve: \( l=10 \), Red Curve: \( l=5 \)

If $l$ is sufficiently larger than $\xi$ and $\xi \ll t \leq l/2$,

$\Delta S_A$ does not depend on $l$ and linearly increases with time.

If $l$ is sufficiently larger than $\xi$ and $\xi \ll t \leq l/2$,

$$\Delta S_A(t) \sim 0.57 \times \frac{t}{\xi}$$

If $t$ is sufficiently larger than $l/2$ ($t \gg l/2$),

$$\Delta S_A(l) \sim 0.28 \times \frac{l}{\xi}$$

Thermalize

Slowly increase
Quasi-Particle Interpretation

• As in sudden quenches, around \( t=0 \), entangled quasi-particle are created everywhere.
• Entangled pair is constructed two particles. They propagate in the opposite directions with $\mathcal{U}$:
• Entangled pair is constructed two particles. They propagate in the opposite directions with $\mathcal{U}$:

If one of them is included in $A$ and the other is out of $A$, *Entangled pair can contribute to entanglement entropy.*
At $l/2 > t > 1/m$, the distance between entangled particles is given $2vt$: $2vt$
At \( \frac{l}{2} > t > \frac{1}{m} \), the distance between entangled particles is given as \( 2vt \): 

\[ 2vt \]

The particle created at the boundary at \( t=0 \) is at \( x = vt \) or \( x = -l - vt \).
At $l/2 > t > 1/m$, the distance between entangled particles is given $2vt$:

The entangled pairs in the blue region can contribute to $S_A$. 
At $l/2 > t > 1/m$, the distance between entangled particles is given $2vt$:

The entangled pairs in the blue region can contribute to $S_A$. 

\[ \text{# of entangled pair} \sim t \]
At $l/2 > t > 1/m$, the distance between entangled particles is given $2vt$:

The entangled pairs in the blue region can contribute to $S_A$. 

$\# \text{ of entangled pair} \sim t$
At $l/2 > t > 1/m$, the distance between entangled particles is given by $2vt$:

The entangled pairs in the blue region can contribute to $S_A$.

The number of entangled pairs $\sim t$ and $S_A$ linearly grows with $t$. 
At $t = l/2v$, the distance between quasi-particles is the subsystem size.
At $t=\frac{l}{2v}$, all entangled pairs in the region $A$ can contribute to...
At $t=l/2v$, all entangled pairs in the region A can contribute to $l$.

# of entangled pair $\sim l$
At $t = l/2v$, all entangled pairs in the region A can contribute to \( \# \text{ of entangled pair} \sim l \). The region A has volume law.
Quasi-Particle Interpretation

• As in sudden quenches, around \( t=0 \), entangled quasi-particle are created everywhere.

• Their speed is given by the group velocity at \( t=0 \)

\[
\nu_k = \frac{d\omega_k(t)}{dk}, \quad \omega_k(t) = \sqrt{4 \sin^2 \left( \frac{k}{2} \right) + m^2(t)}.
\]

[Jordan-Mark-Mark-Mark, 16] [Andrea-Erik-Pasquale, 16]
Quasi-Particle Interpretation

• As in sudden quenches, around $t=0$, entangled quasi-particle are created everywhere.

• Their speed is given by the group velocity at $t=0$

$$v_k = \frac{d\omega_k(t)}{dk}, \quad \omega_k(t) = \sqrt{4\sin^2\left(\frac{k}{2}\right) + m^2(t)}.$$

$$|v_{max}| \sim 1 \quad \text{Around } t = l/2, \text{ the time evolution of } \Delta S_A \text{ changes.}$$

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Slow mode ( ~ zero mode and large \( k \) mode)

Slowly increases in the late time

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Slow mode ( ~ zero mode and large k mode)

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[Jordan-Mark-Mark-Mark, 16]  [Andrea-Erik-Pasquale, 16]
EE in slow ECP
Plot of EE in Slow ECP

Orange Curve: $l=2000$, Gray Curve: $l=1000$, Brown Curve: $l=800$,
Green Curve: $l=600$, Pink Curve: $l=500$, Blue Curve: $l=100$, Purple Curve: $l=10$

$(\xi, \delta t) = (1, 100)$
Plot of EE in Slow ECP

Orange Curve: $l=2000$, Gray Curve: $l=1000$, Brown Curve: $l=800$,
Green Curve: $l=600$, Pink Curve: $l=500$, Blue Curve: $l=100$, Purple Curve: $l=10$

\[ t_{kz} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}} \]

\[ \Delta S_A \text{ does not depend on } l. \]
Plot of EE in Slow ECP

Orange Curve: $l=2000$, Gray Curve: $l=1000$, Brown Curve: $l=800$,
Green Curve: $l=600$, Pink Curve: $l=500$, Blue Curve: $l=100$, Purple Curve: $l=10$

\[ t_{kz} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}} \]

\[ \Delta S_A \sim \frac{1}{3} E_{kz} \cdot t \]

$(\xi, \delta t) = (1, 100)$
Plot of EE in Slow ECP

Orange Curve: \( l = 2000 \), Gray Curve: \( l = 1000 \), Brown Curve: \( l = 800 \), Green Curve: \( l = 600 \), Pink Curve: \( l = 500 \), Blue Curve: \( l = 100 \), Purple Curve: \( l = 10 \)

Adiabaticity breaks down.

\[
\frac{t_{kz}}{2} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}},
\]

\[
\Delta S_A \sim \frac{1}{3} E_{kz} \cdot t
\]

\[
\frac{1}{\xi} \rightarrow E_{kz}
\]

\[
(\xi, \delta t) = (1, 100)
\]
Plot of EE in Slow ECP


After $t = t_{kz} + \frac{l}{2}$,

$\Delta S_A$ depends on $l$. 

$(\xi, \delta t) = (1, 100)$
Plot of EE in Slow ECP


$t \gg t_{kz} + \frac{l}{2}$,

$\Delta S_A \sim \frac{1}{6} E_{kz} \cdot l$

$\frac{1}{\xi} \rightarrow E_{kz}$

$(\xi, \delta t) = (1, 100)$
Quasi-Particle Interpretation

\[
t_{kz} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}}, \quad \Delta S_A \sim \frac{1}{3} E_{kz} \cdot t
\]

\[
t \gg t_{kz} + \frac{l}{2}, \quad \Delta S_A \sim \frac{1}{6} E_{kz} \cdot l
\]
Quasi-Particle Interpretation

\[ t_{kz} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}}, \quad \Delta S_A \sim \frac{1}{3} E_{kz} \cdot t \]

\[ t \gg t_{kz} + \frac{l}{2}, \quad \Delta S_A \sim \frac{1}{6} E_{kz} \cdot l \]

Quasi-particles are created at \( t = t_{kz} \).
Quasi-Particle Interpretation

\[ t_{kz} + \frac{l}{2} = \delta t \log \omega + \frac{l}{2} > t \gg \frac{1}{E_{kz}}, \quad \Delta S_A \sim \frac{1}{3} E_{kz} \cdot t \]

\[ t \gg t_{kz} + \frac{l}{2}, \quad \Delta S_A \sim \frac{1}{6} E_{kz} \cdot l \]

Quasi-particles are created at \( t = t_{kz} \).

When adiabaticity breaks down, quasi-particles are created.
Proportionality Coefficient

The proportionality coefficient of $l$ or $t$ is set by

*an initial correlation length* $\xi$ in the fast limit,

*a scale generated when adiabaticity breaks down*, $E_{kz}$, in the slow limit.
EE in fast CCP
Plot of EE in fast CCP


$\Delta S_A$

$((\xi, \delta t) = (5, 100))$
Plot of EE in fast CCP
Orange Curve: $l=2000$, Green Curve: $l=1000$, Pink Curve: $l=500$,
Blue Curve: $l=100$, Purple Curve: $l=10$, Red Curve: $l=5$

• If $\xi \ll t \leq l/2$

$\Delta S_A$ doesn’t depend on $l$. 

$(\xi, \delta t) = (5, 100)$
Plot of EE in fast CCP


- If $t \gg l/2$,

$$\Delta S_A \sim -\omega^2 \log(\omega) \times \frac{l}{\xi}$$

$(\xi, \delta t) = (5, 100)$
Variation of EE in fast CCP


• If $t \gg l/2$,

$$\Delta S_A \sim -\omega^2 \log(\omega) \times \frac{l}{\xi}$$

**Entangled particles are created around $t=0$.**

$(\xi, \delta t) = (5, 100)$
Plot of EE in fast CCP

Orange Curve: \( l = 2000 \), Green Curve: \( l = 1000 \), Pink Curve: \( l = 500 \),
Blue Curve: \( l = 100 \), Purple Curve: \( l = 10 \), Red Curve: \( l = 5 \)

- If \( t \gg \frac{l}{2} \),
  \[
  \Delta S_A \sim -\omega^2 \log(\omega) \times \frac{l}{\xi}
  \]

If \( \omega \) decreases (\( \xi \) is fixed),

\(-\omega^2 \log(\omega)\) decreases. \( (\xi, \delta t) = (5, 100) \)
Plot of EE in fast CCP

Orange Curve: $l=2000$, Green Curve: $l=1000$, Pink Curve: $l=500$,
Blue Curve: $l=100$, Purple Curve: $l=10$, Red Curve: $l=5$

$\Delta S_A$ is oscillating
Plot of EE in fast CCP

Orange Curve: \( l = 2000 \), Green Curve: \( l = 1000 \), Pink Curve: \( l = 500 \), Blue Curve: \( l = 100 \), Purple Curve: \( l = 10 \), Red Curve: \( l = 5 \)

- \( \Delta S_A \) is oscillating

- Frequency is determined by final mass.

\[
\text{periodicity} \sim \pi \xi
\]

\( (\xi, \delta t) = (5, 100) \)
Minimum Value

- Minimum value of $\Delta S_A$ is at $t = 2\xi$.

$(\xi, \delta t) = (5, 100)$
Minimum Value

• Minimum value of $\Delta S_A$ is at $t = 2\xi$.

The plot for $l$-dependence of $\Delta S_A$ at $t = 2\xi$. 
Minimum Value

- Minimum value of $\Delta S_A$ is at $t = 2\xi$.
- Around $l = \xi$, $\Delta S_A$ is minimized.
- Around $l = 4\xi$, $\Delta S_A$ is constant.

The plot for $l$-dependence of $\Delta S_A$ at $t = 2\xi$
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The plot for $l$-dependence of $\Delta S_A$ at $t = 2\xi$
Minimum Value

- Around $l = 4\xi$, $\Delta S_A$ is constant.

Initially, the blue region of the subsystem $A$ is entangled with the complemental region.

$l > \xi$, $S_A$ is constant.
Minimum Value

- Around \( l = 4\xi \), \( \Delta S_A \) is constant.

Initially, the blue region of the subsystem \( A \) is entangled with the complemental region.

For \( l > \xi \), \( S_A \) is constant.

The plot for \( l \)-dependence of \( \Delta S_A \) at \( t = 2\xi \):

\[
S_A \sim K \log (\xi)
\]
Minimum Value

- Around $l = 4\xi$, $\Delta S_A$ is constant.

*Entangled particle picture*

At $t = 2\xi$, the blue region of the subsystem $A$ is entangled with the complemental region.

$l > 4\xi$, $\Delta S_A$ is constant ($<0$).
Minimum Value

• Around \( l = 4\xi \), \( \Delta S_A \) is constant.

Entangled particle picture

At \( t = 2\xi \), the blue region of the subsystem \( A \) is entangled with the complemental region.

\[ l > 4\xi, \quad \Delta S_A \text{ is constant (<0)}. \]

The plot for \( l\)-dependence of \( \Delta S_A \) at \( t = 2\xi \)

\[ \Delta S_A \sim K \log \left( \xi_{\text{effective}} \right) - K \log (\xi) \]

related with a distance between entangled pair.
Minimum Value

• Around \( l = 4\xi \), \( \Delta S_A \) is constant.

**Entangled particle picture**

At \( t = 2\xi \), the blue region of the subsystem \( A \) is entangled with the complemental region.

\[ l > 4\xi, \quad \Delta S_A \text{ is constant (<0)}. \]

\[ \Delta S_A \sim K \log(\xi_{effective}) - K \log(\xi) \]

\( \xi_{effective} < \xi \)
EE in slow CCP
Plot of EE in Slow CCP

Green Curve: $l=3000$, Orange Curve: $l=2500$, Black Curve: $l=1000$,
Blue Curve: $l=500$, Purple Curve: $l=100$, Red Curve: $l=10$
Plot of EE in Slow CCP
Green Curve: \( l = 3000 \), Orange Curve: \( l = 2500 \), Black Curve: \( l = 1000 \), Blue Curve: \( l = 500 \), Purple Curve: \( l = 100 \), Red Curve: \( l = 10 \)

After \( t = 2\xi_{kz} \), \( \Delta S_A \) starts to oscillate.
Plot of EE in Slow CCP

Green Curve: \( l=3000 \), Orange Curve: \( l=2500 \), Black Curve: \( l=1000 \),
Blue Curve: \( l=500 \), Purple Curve: \( l=100 \), Red Curve: \( l=10 \)

After \( t = 2\xi_{kz} \),
\( \Delta S_A \) starts to oscillate.

Similar to the results in fast quenches.
Plot of EE in Slow CCP
Green Curve: $l=3000$, Orange Curve: $l=2500$, Black Curve: $l=1000$,
Blue Curve: $l=500$, Purple Curve: $l=100$, Red Curve: $l=10$

Periodicity of $\Delta S_A$ at late time

$\sim$ Periodicity of zero mode $\sim \pi \xi$
\[ \Delta S_A (t = 2 \xi_{kz}) \cdot l > 6 \xi_{kz} \]

\[ \Delta S_A \text{ is a constant (>0).} \]

Red: \((\omega, \xi_{kz}) = (100, 100)\)

Blue: \((\omega, \xi_{kz}) = (100, 200)\)

Green: \((\omega, \xi_{kz}) = (400, 200)\)
\[ \Delta S_A(t = 2\xi_{kz}) \]
\[ \cdot l > 6\xi_{kz} \]

\[ \Delta S_A \] is a constant (>0).

- Entangled particle interpretation

Adiabaticity breaks down.

\[ @ \quad t \sim -t_{kz} \]

Entangled particles are created.

Red: \((\omega, \xi_{kz}) = (100, 100)\)

Blue: \((\omega, \xi_{kz}) = (100, 200)\)

Green: \((\omega, \xi_{kz}) = (400, 200)\)
\[ \Delta S_A(t = 2 \xi_{kz}) \cdot l > 6 \xi_{kz} \]

\[ \Delta S_A \] is a constant (>0).

- Entangled particle interpretation
  
  *Adiabaticity breaks down.*

@ \( t \sim -t_{kz} \)

Entangled particles are created.

\[ \Delta S_A \sim \xi_{\text{effective}} > \xi \]

\[ K \log (\xi_{\text{effective}}) - K \log (\xi) \]

Red: \((\omega, \xi_{kz}) = (100, 100)\)

Blue: \((\omega, \xi_{kz}) = (100, 200)\)

Green: \((\omega, \xi_{kz}) = (400, 200)\)
Late time in CCP

\( (\xi, t) = (10, 1000000) \)

\[
\frac{\delta t}{C_3(\omega)}
\]

\( \Delta S_A \) is fitted by

\[
\Delta S_A \sim \frac{C_3(\omega)}{\xi} l + B
\]
Late time in CCP

$$\begin{align*}
(\xi, t) &= (10, 1000000) \\
\delta t C_3(\omega) \\
\end{align*}$$

If $\omega$ increases (\xi is fixed), $C_3(\omega)$ decreases.

$\Delta S_A$ is fitted by $\Delta S_A \sim \frac{C_3(\omega)}{\xi} l + B$.
Volume law in Fast and slow limits

- In the fast limit, the fitting function:

- In the slow limit, the fitting function:

\[ \Delta S_A \sim -\omega^2 \log(\omega) \times \frac{l}{\xi} \]

\[ \Delta S_A \sim \frac{C_3(\omega)}{\xi} l + B \]

\[ \xi = 100 \]

If \( \omega \) decreases (\( \xi \) is fixed), 
\[-\omega^2 \log(\omega) \text{ decreases.} \]

If \( \omega \) increases (\( \xi \) is fixed), 
\[C_3(\omega) \text{ decreases.} \]

\( N \) is the number operator/Volume at late time.
Volume law in Fast and slow limits

- In the fast limit, Fitting function:
- In the slow limit, Fitting function:

\[
\Delta S_A \sim -\omega^2 \log(\omega) \times \frac{l}{\xi} \\
\Delta S_A \sim \frac{C_3(\omega)}{\xi} l + B
\]

\( \xi = 100 \)

If \( \omega \) decreases (\( \xi \) is fixed),

The behavior of entanglement entropy at late time
is consistent with the behavior of number operator at late time.

If \( \omega \) increases (\( \xi \) is fixed),

\( C_3(\omega) \) decreases.

N is the number operator/Volume at late time.
Oscillation (naive)

• In the late time, the mass profile **slowly changes**.

• Physical quantities can be computed **adiabatically**.
Oscillation (naive)

• In the late time, the mass profile slowly changes.

• Physical quantities can be computed adiabatically.

\[ \nu_k = \partial_k \omega_k \quad \omega_k = \sqrt{4 \sin^2 \left( \frac{k}{2} \right) + m_j^2} \]
Oscillation (naive)

- In the late time, the mass profile **slowly changes**.
- Physical quantities can be computed **adiabatically**.

As in ECP, in the late time, slow mode (zero mode) contribute to $\Delta S_A$.

**zero mode:** $e^{-i\omega_k t} \sim e^{-im_f t}$
Oscillation

Entanglement Entropy:

\[ S_A = \sum_{k=1}^{l} s_A(\gamma_k) \]

\[ s_A(\gamma_k) = \left( \frac{1}{2} + \gamma_k \right) \log \left( \frac{1}{2} + \gamma_k \right) - \left( -\frac{1}{2} + \gamma_k \right) \log \left( -\frac{1}{2} + \gamma_k \right) \]

\[ \Gamma = \begin{pmatrix} X_{ij} & \frac{1}{2} D_{ij} \\ \frac{1}{2} D_{ji} & P_{ij} \end{pmatrix} \]

\[ J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

\[ X_{ij} = \langle \phi_i \phi_j \rangle \quad P_{ij} = \langle \pi_i \pi_j \rangle \quad D_{ij} = \langle \{\phi, \pi_j\} \rangle \]

\[ M = i J \Gamma \] has eigenvalues \( \pm \gamma_k \).
Oscillation

Entanglement Entropy:

\[ S_A = \sum_{k=1}^{l} s_A(\gamma_k) \]

\[ s_A(\gamma_k) = \left( \frac{1}{2} + \gamma_k \right) \log \left( \frac{1}{2} + \gamma_k \right) - \left( -\frac{1}{2} + \gamma_k \right) \log \left( -\frac{1}{2} + \gamma_k \right) \]

\[ \Gamma = \begin{pmatrix} X_{ij} & \frac{1}{2} D_{ij} \\ \frac{1}{2} D_{ji} & P_{ij} \end{pmatrix} \quad J = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \]

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- \( M = iJ\Gamma \) has eigenvalues \( \pm \gamma_k \).
Oscillation

\[ X_{ab}(t) = \langle X_a(t)X_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_k \cos (k |a - b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos (k |a - b|), \]

\[ P_{ab}(t) = \langle P_a(t)P_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} P_k \cos (k |a - b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k'(t)|^2 \cos (k |a - b|), \]

\[ D_{ab}(t) = \frac{1}{2} \langle \{X_a(t), P_b(t)\} \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} D_k \cos (k |a - b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \text{Re} \left[ f_k^*(t)f_k(t) \right] \cos (k |a - b|) \]

In the late limit,

\[ f_k(t) \simeq A_k e^{i\omega_k t} + B_k e^{-i\omega_k t} \]

Left moving mode + Right moving mode
Oscillation

\[ X_{ab}(t) = \langle X_a(t)X_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos(k|a-b|), \]

\[ P_{ab}(t) = \langle P_a(t)P_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} P_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left| \dot{f}_k(t) \right|^2 \cos(k|a-b|), \]

\[ D_{ab}(t) = \frac{1}{2} \langle \{X_a(t), P_b(t)\} \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} D_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \text{Re} \left[ \dot{f}_k^*(t)f_k(t) \right] \cos(k|a-b|) \]

In the late limit,

\[ X_k, P_k \simeq C_k^{x,p} + D_k^{x,p} \cos(2\omega_0 t + \Theta_k^{x,p}), \]

\[ D_k \simeq D_k^d \cos(2\omega_0 t + \Theta_k^d), \quad \omega_0 = \sqrt{k^2 + m^2} \]
Oscillation

\[ X_{ab}(t) = \langle X_a(t) X_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos(k|a-b|), \]

\[ P_{ab}(t) = \langle P_a(t) P_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} P_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \left| \dot{f}_k(t) \right|^2 \cos(k|a-b|), \]

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slow mode (physical) = zero mode (k=0)
Oscillation

\[ X_{ab}(t) = \langle X_a(t)X_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_k \cos (k |a - b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos (k |a - b|), \]

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\[ D_{ab}(t) = \frac{1}{2} \langle \{X_a(t), P_b(t)\} \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} D_k \cos (k |a - b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} \Re \left[ \dot{f}_k^*(t)f_k(t) \right] \cos (k |a - b|) \]

In the late limit,

\[ X_k, P_k \simeq C_{k}^{x,p} + D_{k}^{x,p} \cos (2\omega_0 t + \Theta_k^{x,p}), \]

\[ D_k \simeq D_{k}^{d} \cos (2\omega_0 t + \Theta_k^{d}), \quad \omega_0 = m \]

slow mode (physical) = zero mode (k=0)
Oscillation

\[ X_{ab}(t) = \langle X_a(t) X_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} X_k \cos(k|a-b|) = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos(k|a-b|), \]

\[ P_{ab}(t) = \langle P_a(t) P_b(t) \rangle = \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos(k|a-b|), \]

\[ D_{ab}(t) = \frac{1}{2} \int_{-\pi}^{\pi} \frac{dk}{2\pi} |f_k(t)|^2 \cos(k|a-b|), \]

\[ X_0, P_0, D_0 \text{ oscillate with } \pi \xi. \]

\[ \Delta S_A \text{ oscillate with } \pi \xi. \]

\[ X_k, P_k, D_k = C_k + D_k \cos(2\omega_0 t + \Theta_k), \]

\[ D_k \sim D_k^d \cos(2\omega_0 t + \Theta_k^d), \quad \omega_0 = m. \]

slow mode (physical) = zero mode (k=0)
Oscillation

• In the late time, the mass profile slowly changes.

• Physical quantities can be computed adiabatically.

As in ECP, in the late time, slow mode (zero mode) contribute to $\Delta S_A$.

**Zero mode:** $e^{-i\omega_k t} \sim e^{-im_f t}$
MI and LN
Time evolution of MI and LN

• Mutual Information:

\[ I_{A,B} = S_A + S_B - S_{A\cup B} \]

• Logarithmic Negativity:

\[ \mathcal{E} = \log \| \rho_{A\cup B}^{T_B} \| = \log \sum_i |\lambda_i| \]

\[ \rho_{A\cup B} = \rho_{ij,kl} |i\rangle \langle j|_A \otimes |k\rangle \langle l|_B, \quad \rho_{A\cup B}^{T_B} = \rho_{ij,lk} |i\rangle \langle j|_A \otimes |k\rangle \langle l|_B \]
Time evolution of MI and LN

• **Mutual Information:**

\[ I_{A,B} = S_A + S_B - S_{A \cup B} \]

• **Logarithmic Negativity:**

\[ \mathcal{E} = \log ||\rho_{A \cup B}^T|| = \log \sum_i |\lambda_i| \]

\[ \rho_{A \cup B} = \rho_{ij,kl} |i\rangle \langle j|_A \otimes |k\rangle \langle l|_B, \quad \rho_{A \cup B}^T = \rho_{ij,lk} |i\rangle \langle j|_A \otimes |k\rangle \langle l|_B \]

\[ \text{size} \simeq l_a \quad \text{size} \simeq l_b \]

\[ \Delta l \]

A \hspace{5cm} B
What we are studying

Change of MI and LN:

$$\Delta I_{A,B}(t) = I_{A,B}(t) - I_{A,B}(t_{in})$$

$$\Delta \mathcal{E}(t) = \mathcal{E} - \mathcal{E}(t_{in})$$
What we are studying

Change of MI and LN:

$$\Delta I_{A,B}(t) = I_{A,B}(t) - I_{A,B}(t_{in})$$

$$\Delta \mathcal{E}(t) = \mathcal{E} - \mathcal{E}(t_{in})$$

MI and LN for initial mass
Time evolution of MI and LN in Fast-ECP

Parameter: \((\xi, \delta t, l_a, l_b) = (100, 5, 1000, 1000)\)

\(\Delta l = 0\) : Red
\(\Delta l = 100\) : Blue
\(\Delta l = 800\) : Pink
\(\Delta I_{A,B}(t)\)

\(\Delta l = 10\) : Purple
\(\Delta l = 500\) : Green
\(\Delta l = 1500\) : Black
\(\Delta \mathcal{E}(t)\)
Time evolution of MI and LN in Fast-ECP

Parameter: \((\xi, \delta t, l_a, l_b) = (100, 5, 1000, 1000)\)

- \(\Delta l = 0\): Red
- \(\Delta l = 100\): Blue
- \(\Delta l = 800\): Pink
- \(\Delta l = 1500\): Black

\(\Delta \mathcal{E}(t)\):
- Slightly decreases

\(\Delta I_{A,B}(t)\):
- Increases (logarithmically)
Time evolution of MI and LN in Fast-ECP

Parameter: \((\xi, \delta t, l_a, l_b) = (100, 5, 1000, 1000)\)

- \(\Delta l = 0\): Red
- \(\Delta l = 10\): Purple
- \(\Delta l = 100\): Blue
- \(\Delta l = 500\): Green

Slow mode contributes to late-time behavior.

\(\Delta \mathcal{E}(t)\)

Increases (logarithmically)

Slightly decreases
Time evolution of MI and LN in Fast-ECP

Parameter: \((\xi, \delta t, l_a, l_b) = (100, 5, 1000, 1000)\)

\[ \Delta l = 0 : \quad \text{Red} \quad \Delta l = 10 : \quad \text{Purple} \]

\[ \Delta l = 100 : \quad \text{Blue} \quad \Delta l = 500 : \quad \text{Green} \]

\[ \Delta E(t) \]

\[ \Delta I_{A,B}(t) \]

**LN is independent of slow mode.**

Slightly decreases

Increases (logarithmically)
Time evolution of MI and LN in Fast-ECP

Parameter: \((\xi, \delta t, l_a, l_b) = (100, 5, 1000, 1000)\)

\[ \Delta \mathcal{E}(t) \]

\[ \Delta I_{A,B}(t) \]

\(\Delta \mathcal{E}(t)\) and \(\Delta I_{A,B}(t)\) for \(\Delta l \gg \xi\) **increase** after \(t = \frac{\Delta}{2}\).

The width of wavelet is \(\omega = l_a\).
Quasi-particle picture

$\Delta I_{A,B} = \Delta \varepsilon = 0$

$t = 0$
Quasi-particle picture

\[ \Delta I_{A,B} = \Delta \mathcal{E} = 0 \]

\[ \Delta I_{A,B} = \Delta \mathcal{E} \neq 0 \]
Quasi-particle picture

$\Delta I_{A,B} = \Delta \mathcal{E} = 0$

$\Delta I_{A,B} = \Delta \mathcal{E} \neq 0$

$\Delta \mathcal{E} = 0$
Time evolution of MI and LN in Fast-CCP

Parameter: \((\xi, \delta t, l_a, l_b) = (200, 5, 600, 600)\)

\[\Delta l = 0: \quad \text{Red}\]

\[\Delta l = 100: \quad \text{Brown}\]

\[\Delta l = 800: \quad \text{Black}\]

\[\Delta l = 1500: \quad \text{Cray}\]

\[\Delta l = 500: \quad \text{Magenta}\]
Time evolution of MI and LN in Fast-CCP

Parameter: \((\xi, \delta t, l_a, l_b) = (200, 5, 600, 600)\)

\[
\Delta l = 0
\]
\[
\Delta l = 100
\]
\[
\Delta l = 800
\]

\[
\Delta \xi(t)
\]
\[
\Delta I_{A,B}(t)
\]

Periodicity = \(\pi \frac{\xi}{\xi}\)
Time evolution of MI and LN in Fast-CCP

Parameter: \((\xi, \delta t, l_a, l_b) = (200, 5, 600, 600)\)

- \(\Delta l = 0\): \text{Red}  
- \(\Delta l = 10\): \text{Cyan} 
- \(\Delta l = 100\) 
- \(\Delta l = 80\)

\(\Delta l \gg \xi\), *oscillation vanishes.*
Time evolution of MI and LN in Fast-CCP

Parameter: \((\xi, \delta t, l_a, l_b) = (200, 5, 600, 600)\)

\[\Delta l = 0: \text{Red} \quad \Delta l = 10: \text{Cyan}\]

Short-range entanglement \(\rightarrow\) Oscillation

\[\Delta E(t) \quad \Delta I_{A,B}(t)\]
Summary and
Future directions
Summary

• We study what makes quasi-particles.

  In slow ECP, adiabaticity-breaking plays an important role.

• Scaling of EE depend on scales when adiabaticity breaks down.

• Late time behavior depends on slow mode (zero mode).
Future directions

• Why does change of EE oscillate after $t = 2\xi_kz$, $t = 2\xi$?

• Interacting theories

• Holographic Dual

• Floquet type potential
Thank you for your attention