Holography inspired numerical experiments on exactly solvable tensor networks.

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Based on: arXiv:1711.03109
Holography and criticality in matchgate tensor networks
with: Alexander Jahn, Marek Gluza & Jens Eisert

& Based on: unpublished work
Stabilizer code synthesis from tensor network geometries
with: Arkin Tikku

Freie Universität Berlin

Alexander von Humboldt Stiftung/Foundation

Holography & TN

Entanglement structure

compressed description

Holographic duality

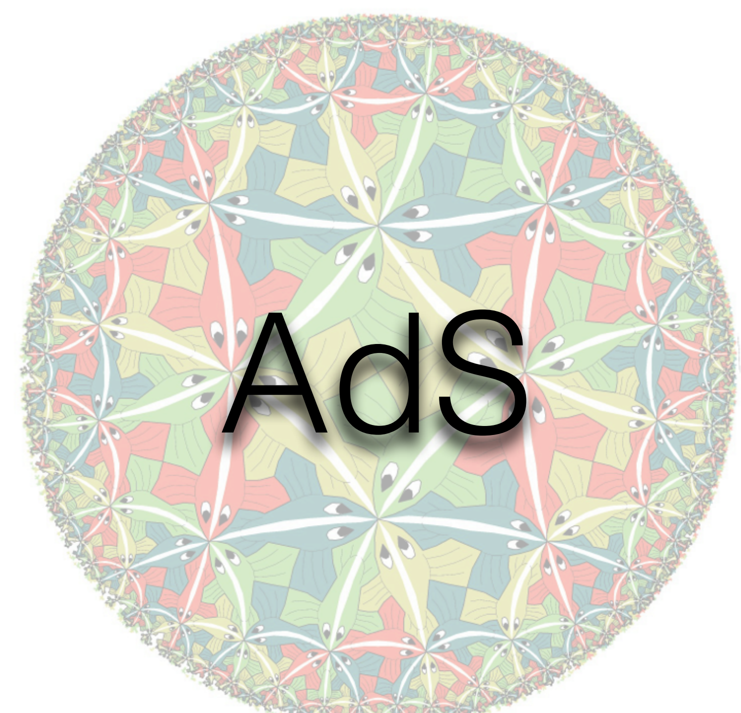
Tensor network geometry

graph discretization of geometry

Holographic geometry

based on example:

CFT



Motivation

1. Build "synthetic" entanglement structure from tensor network geometries?
   - Construct quantum error-correcting codes
   - Understand holography beyond CFTs (H_2 geom.)

2. Elucidate necessary assumptions for holography
   - Are randomness assumptions in TN necessary ?
   - Is a strongly interacting CFT necessary ?
   - How does integrability obstruct holography ?

Outline

1. Tensor network review.
   - States, expectation values & (computer) complexity

2. Match-gate tensor network (free-fermions)
   - Majorana operators,
   - Covariance matrix, Wick's theorem & n-point functions.
   - Parent Hamiltonian & nullifiers
   - Jordan-Wigner transformation, 5qubit code states
   - Efficient contraction.
   - Numerical results, interpretation

3. Stabilizer tensor networks
   - Generalized Pauli group, 5qubit code states
   - Stabilizer states, Calculating n-point functions
   - Efficient contraction
   - Numerical results

4. Conclusions and further work.

Tensor network review

Tensors

Scalars
c-numbers

$$c$$

Ket
pure state

$$\sum_j \psi_j |j\rangle$$

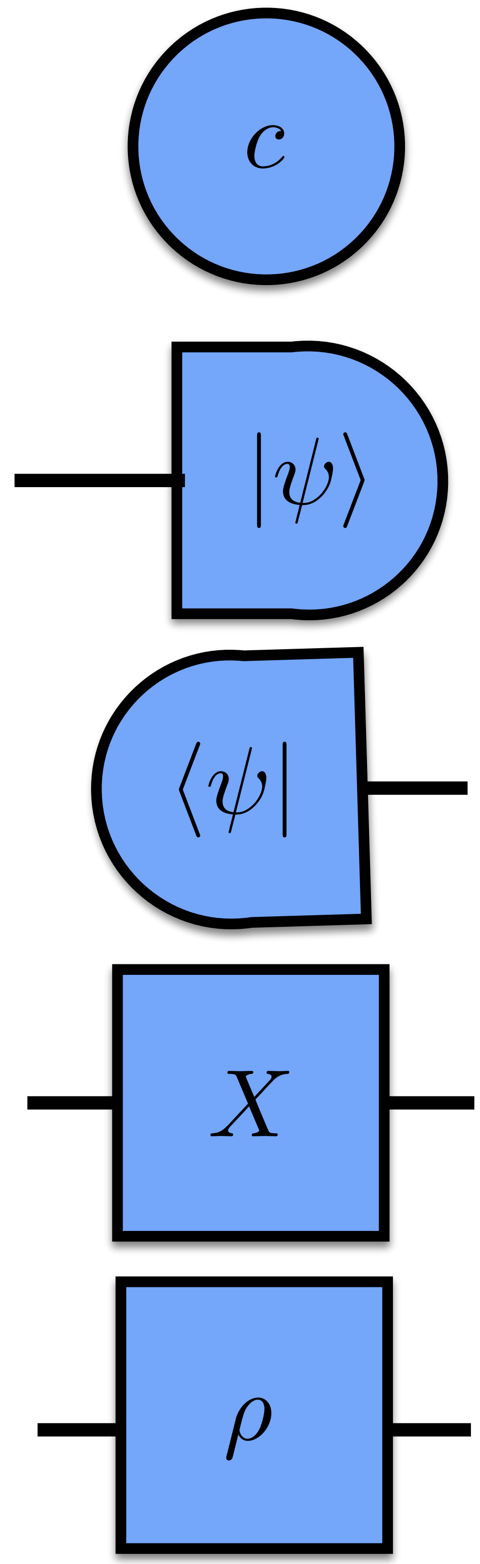
Bra
pure state

$$\sum_j \bar{\psi}_j \langle j|$$

Operator:
i.e. observable,
unitary,
density matrix

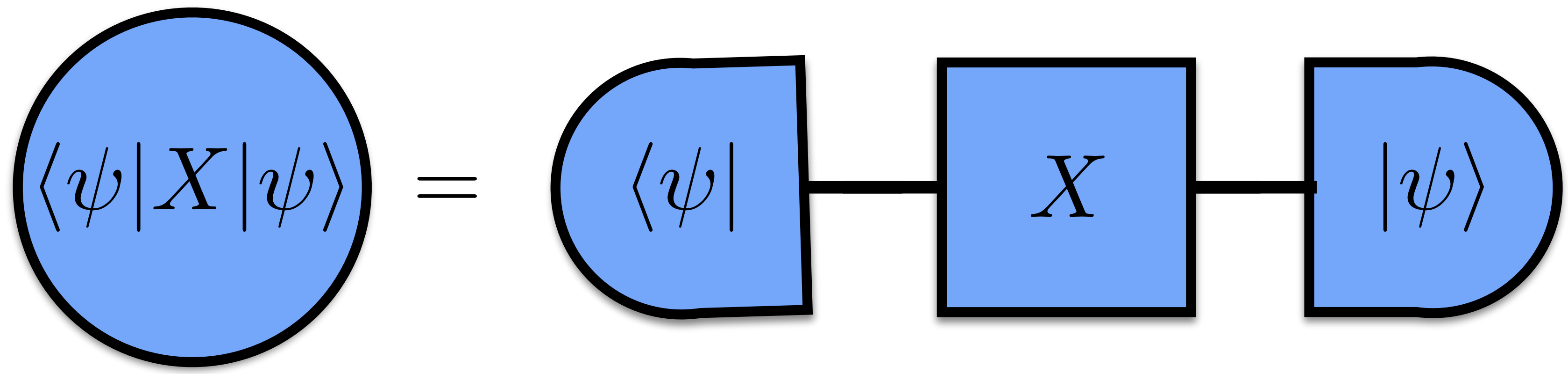
$$\sum_{i,j} X_{i,j} |i\rangle\langle j|$$

$$\sum_{i,j} \rho_{i,j} |i\rangle\langle j|$$

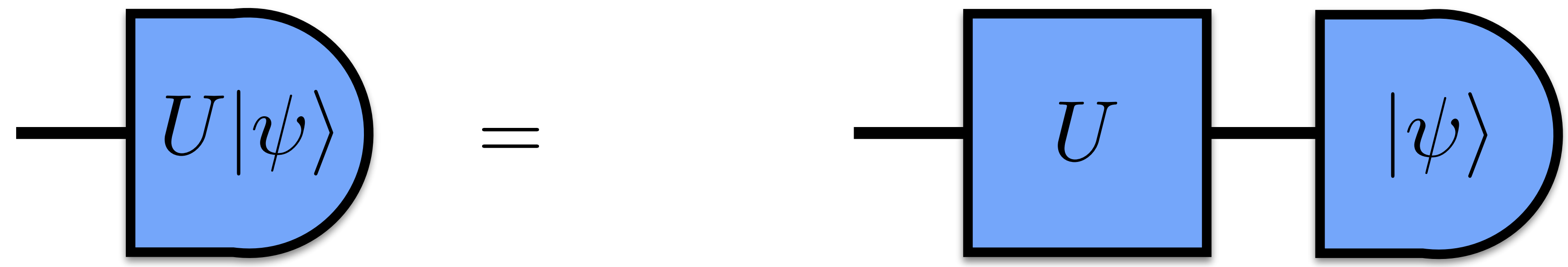


Contraction

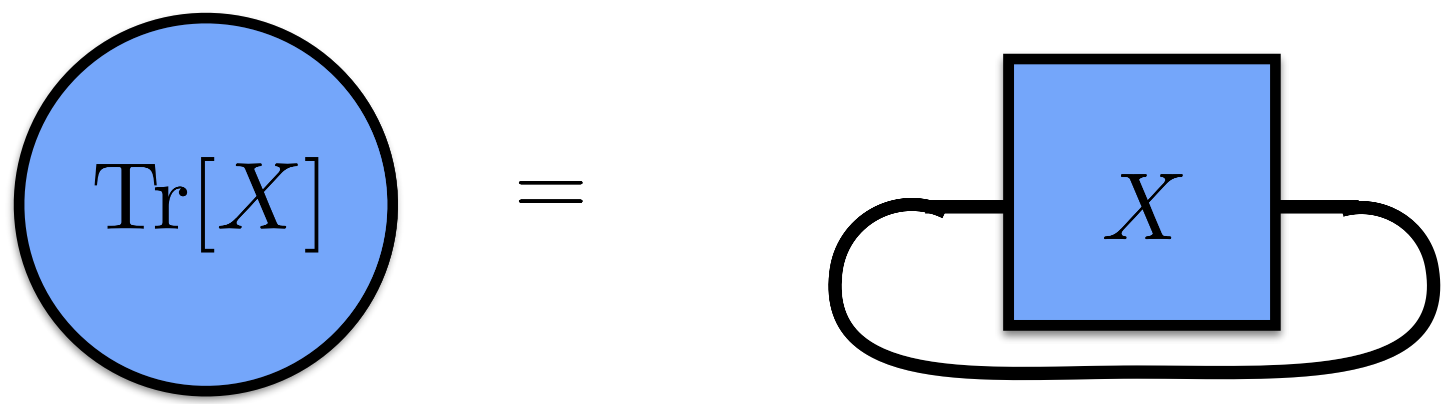
Expectation value $X_i^j \bar{\psi}^i \psi_j$



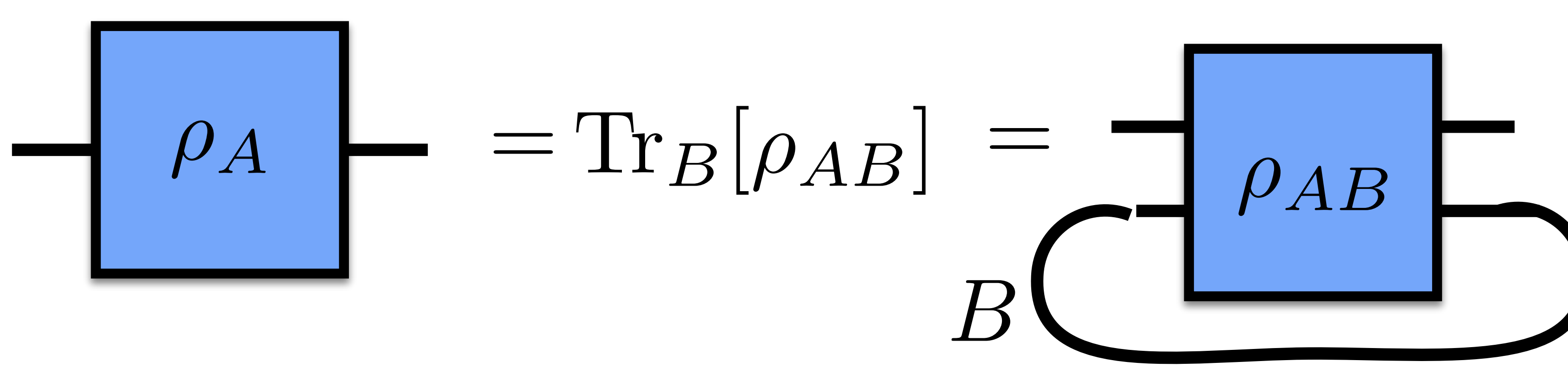
Unitary evolution $U_i^j \psi_j$



Trace $X_i^j \delta_j^i$



Partial trace $\rho_{i_a, i_b}^{j_a, j_b} \delta_{j_b}^{i_b}$

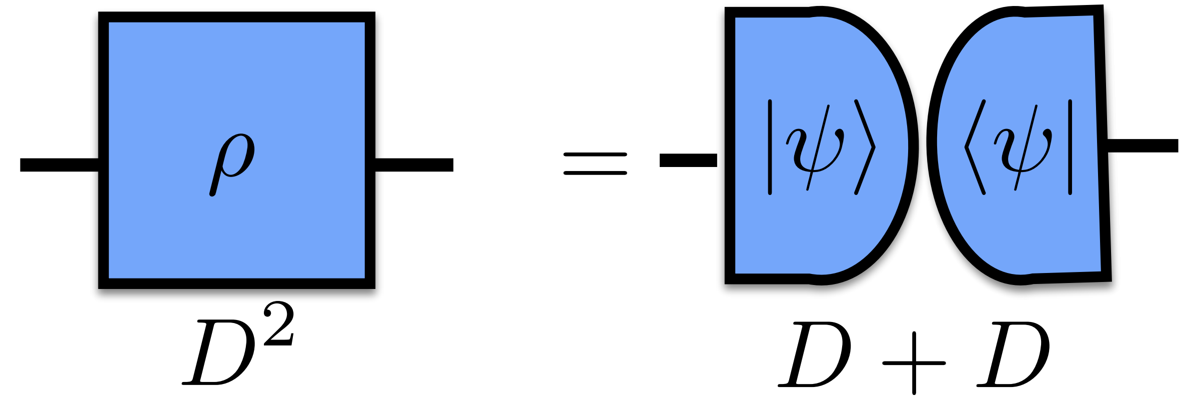


Factorization(s)

Pure state density matrix

$$\rho = |\psi\rangle\langle\psi|$$

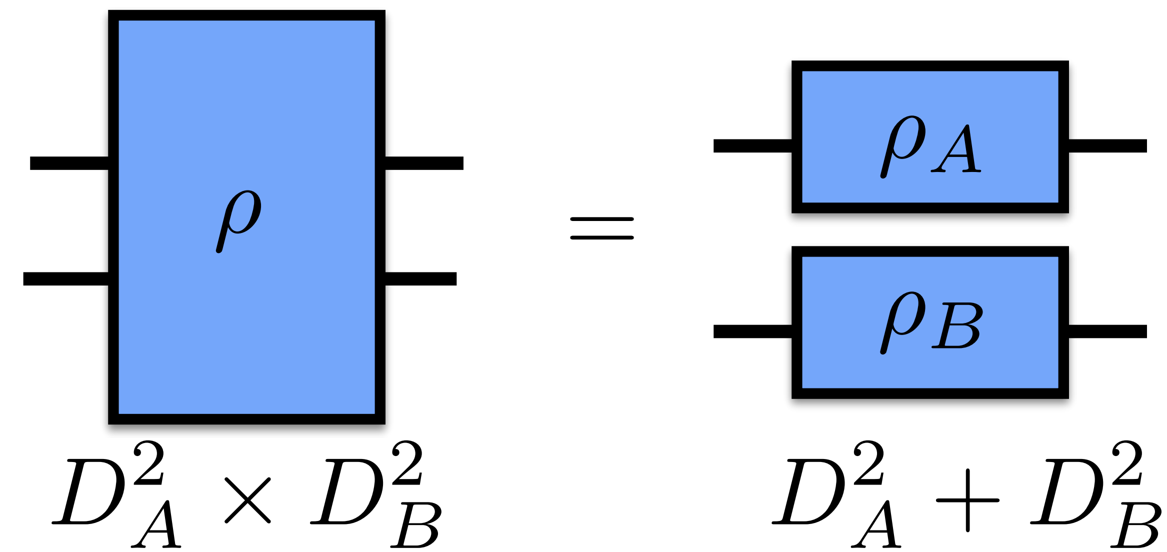
$$\rho_i^j = \psi_i \bar{\psi}^j$$



Product state (similarly op.)

$$\rho = \rho_A \otimes \rho_B$$

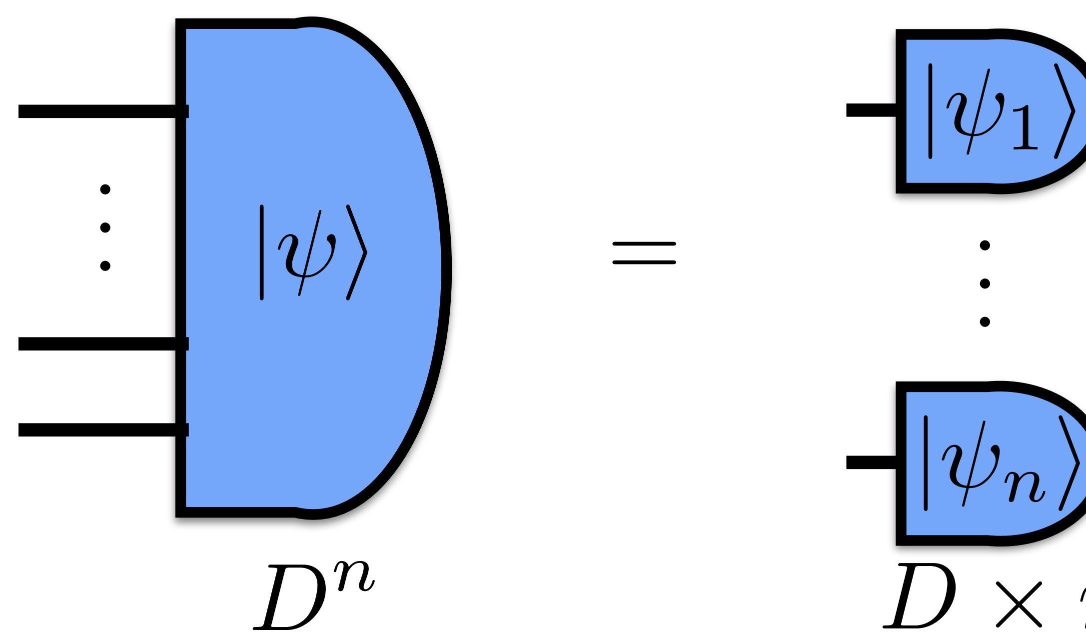
$$\rho_{i_a,i_b}^{j_a,j_b} = \rho_{i_a}^{j_a} \rho_{i_b}^{j_b}$$



Pure product state (ket)

$$|\psi\rangle = |\psi_1\rangle \otimes \ldots \otimes |\psi_n\rangle$$

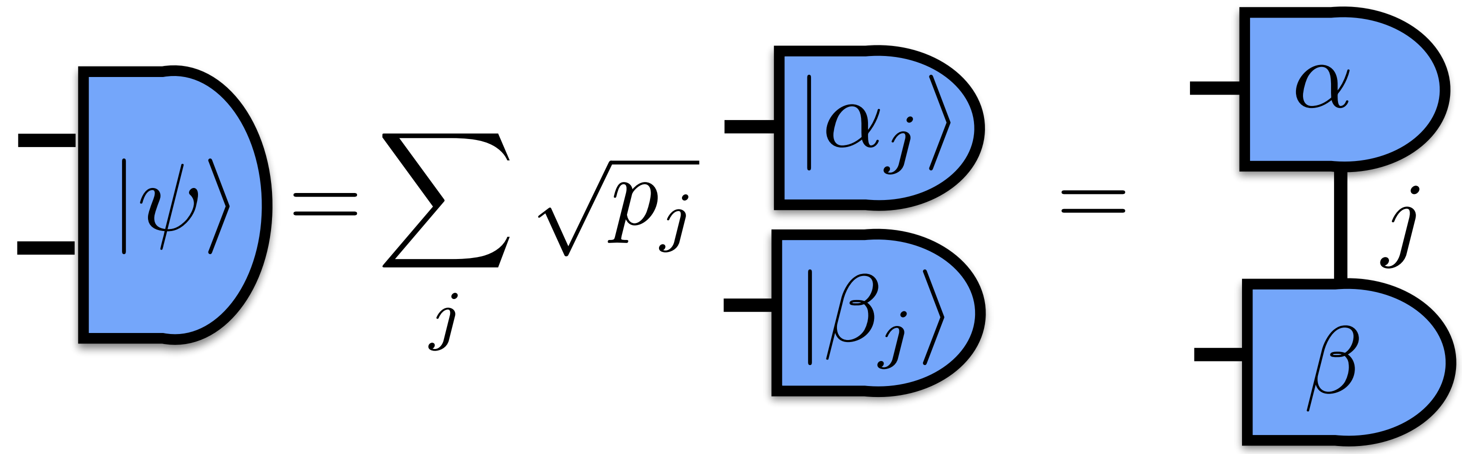
$$\psi_{\vec{i}} = \psi_{i_1} \ldots \psi_{i_n}$$



Schmidt decomposition

Entangled state

$$|\psi\rangle = \sum_j \sqrt{p_j} |\alpha_j\rangle \otimes |\beta_j\rangle$$



$D_A \times D_B$ $\quad$ $s \times (D_A + D_B)$

$$\psi_{i_a, i_b} = \alpha_{i_a, s_a} \beta_{i_b, s_b} \delta^{s_a, s_b}$$

low entanglement states well approximated by **low Schmidt** rank *s*.

$+$

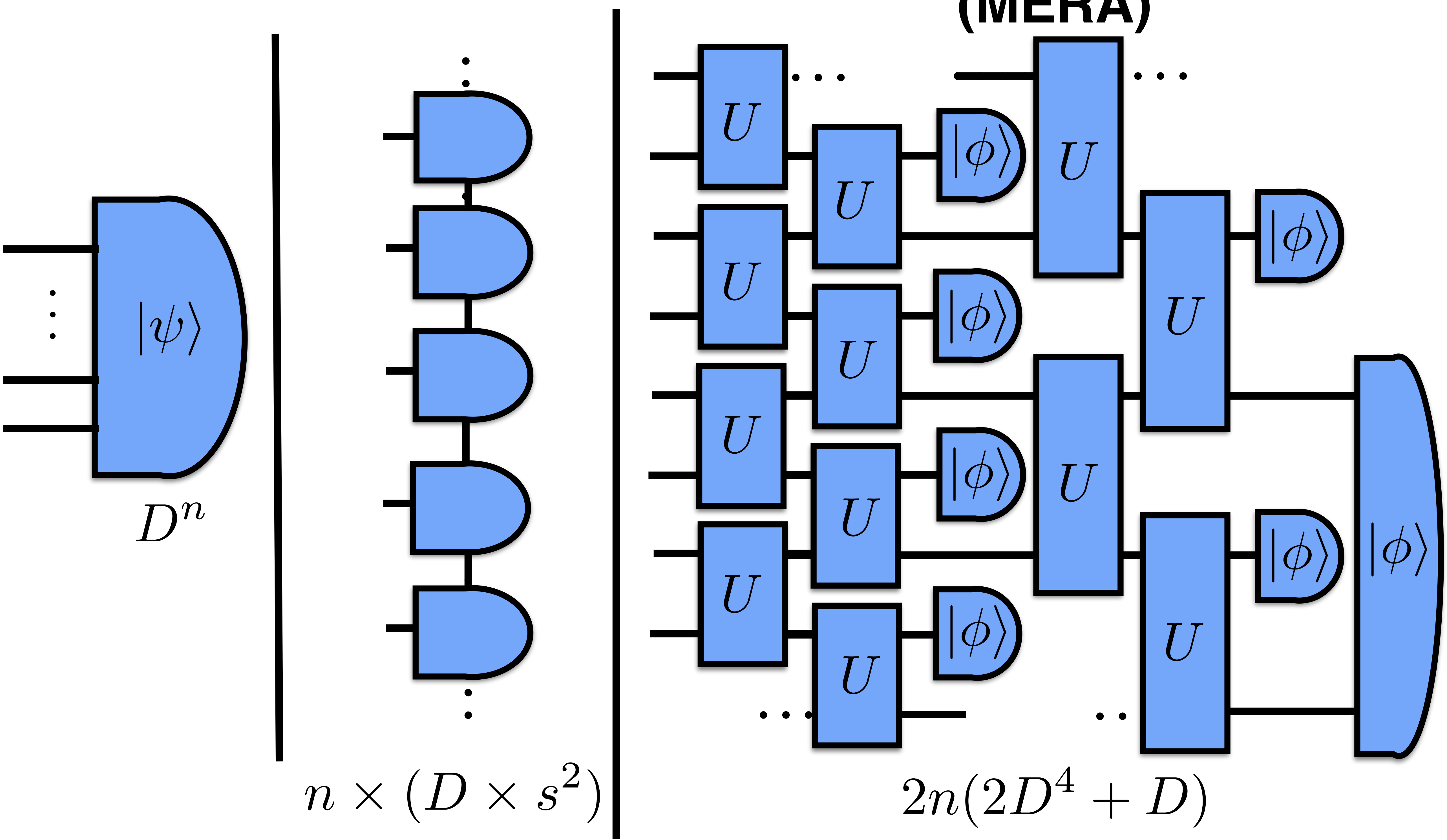
Entanglement area law (low)

Basis for tensor network ansatz

Tensor network ansatz(e)

Matrix Product States **(MPS)**

Multi-scale Entanglement Renormalization Ansatz **(MERA)**

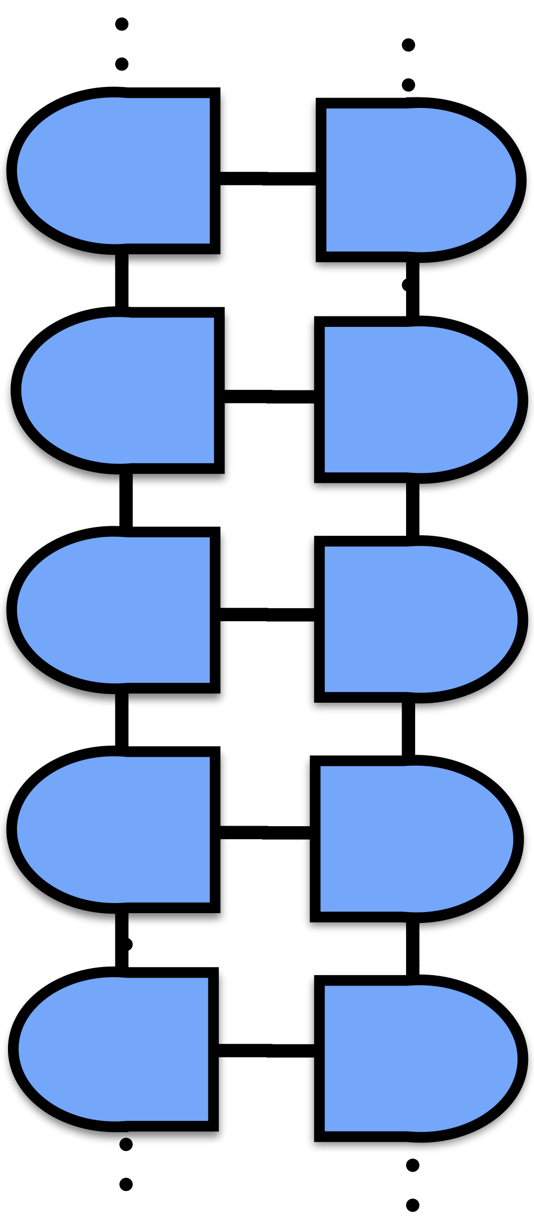


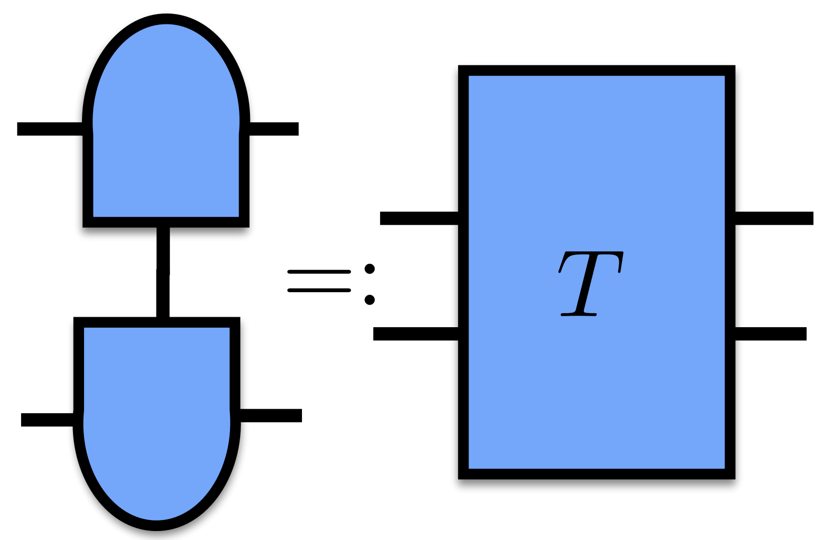
Normalization and expectation values

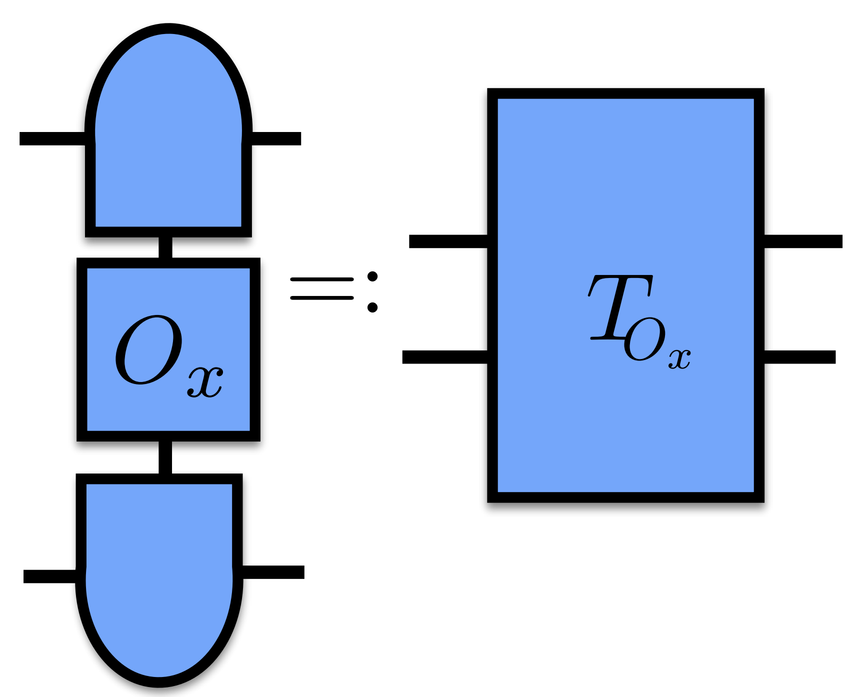
Example: Matrix Product States **(MPS)**

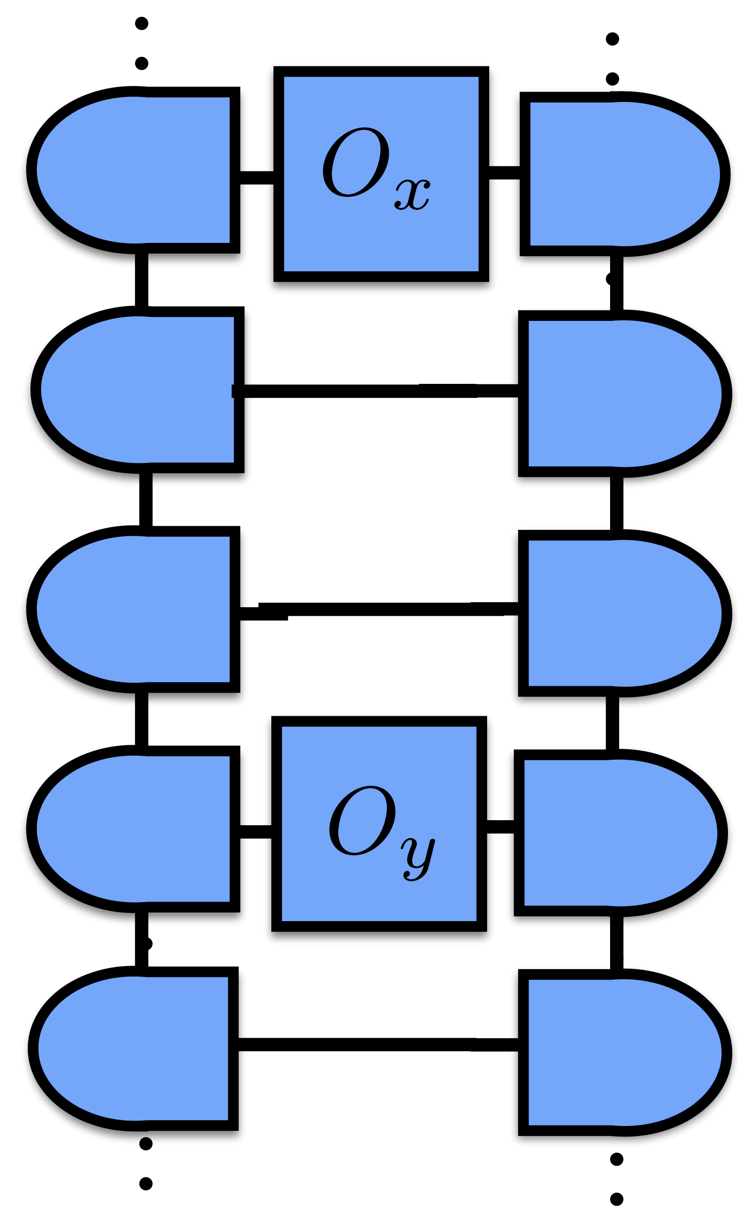
$$1 = \langle \psi | \psi \rangle = Tr[T^n]$$

$$\langle \psi | O_x O_y | \psi \rangle = Tr[T_{O_x} T^{d_1} T_{O_y} T^{d_2}]$$

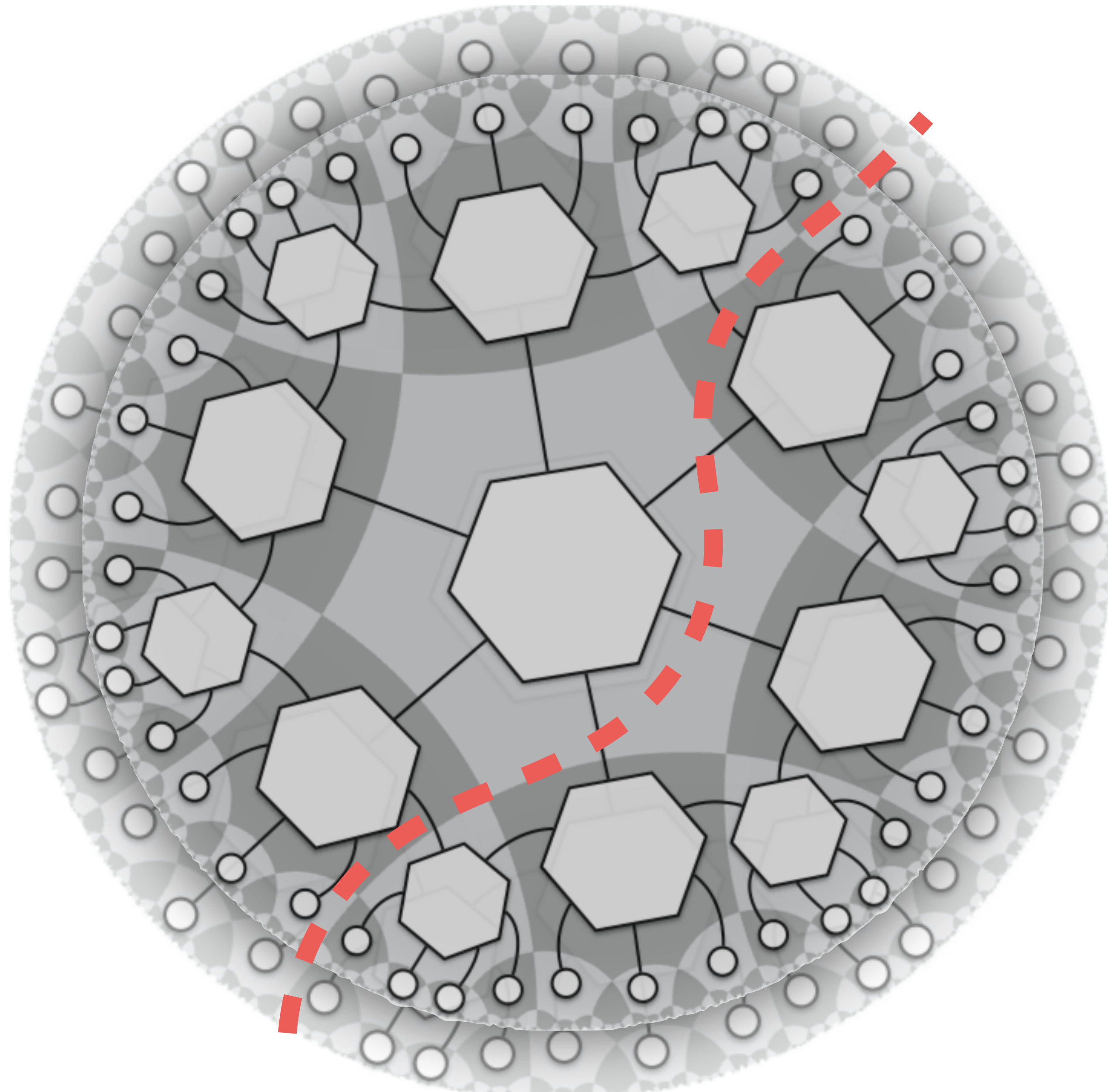






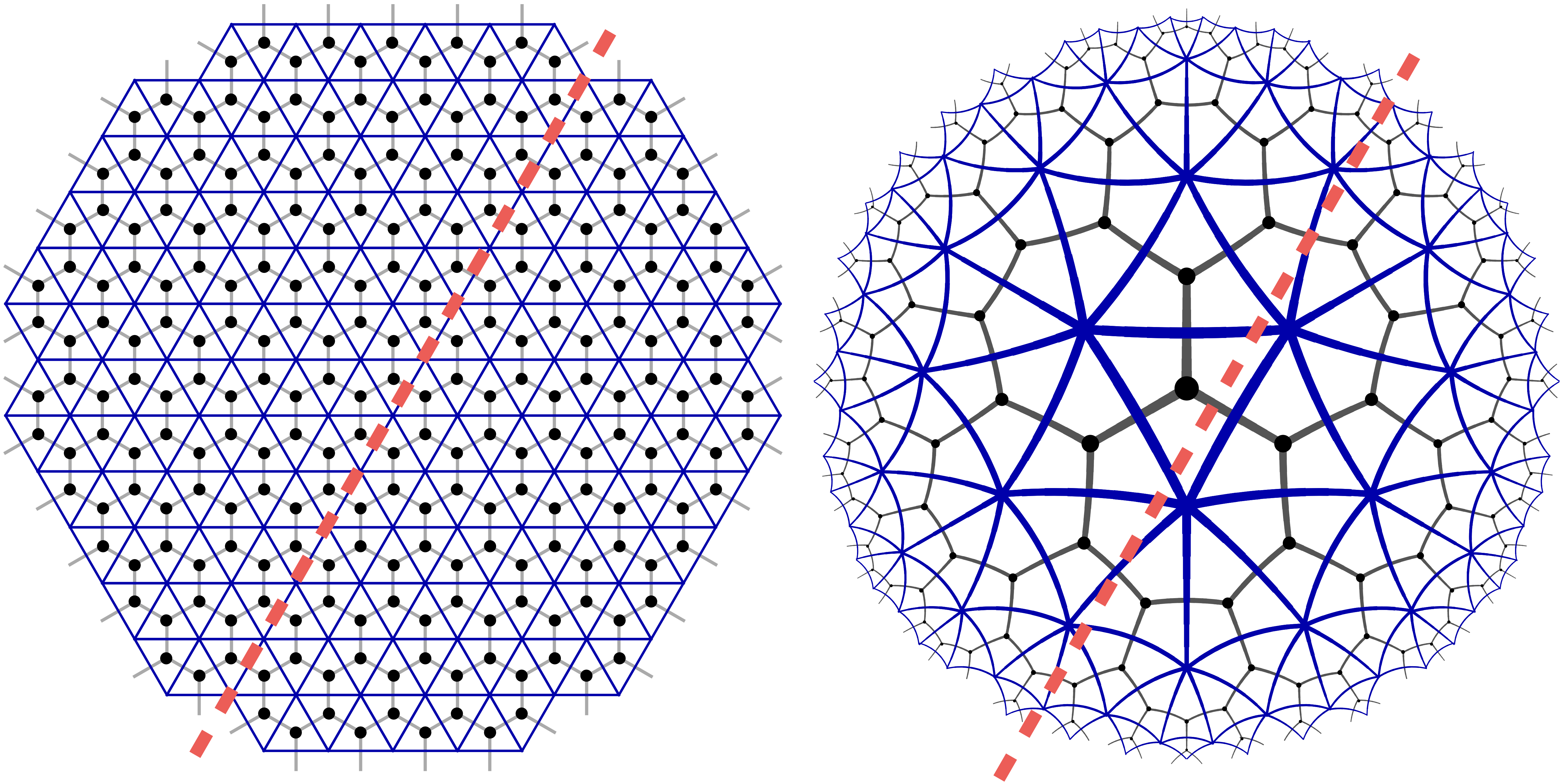


Beyond MPS and MERA



Computer memory required (norm & expectation values) is exponential in the “tree-width” of tensor network graph.

Curse of polynomial tree-width



Most tensor networks have polynomial tree-width.
Only 1D MPS (constant TW) and (1D) MERA-like (log n).

What tools can we use to run large tensor network experiments beyond H_2?

Exactly solvable TN

Gaussian fermion states

- Continuous parameter space.
- *Planar graph* connectivity (otherwise inefficient)
- Bond dimension d= 2
- Free-fermion spectrum
- Allows approximation Ising CFT lattice models.

(nullifiers)

$$n_j = \sum_\ell \alpha_j^\ell m_\ell$$

$$n_j|\psi\rangle = 0$$

Stabilizer states

- Discrete parameter space.
- Arbitrary graph connectivity
- Flexible bond dimension $d = p$ prime
- Flat spectrum
- Exact QEC properties.

$$I \equiv \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, X \equiv \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, Y \equiv \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, Z \equiv \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$$\begin{array}{cccccccc} g_1 & = & X & Z & Z & X & I \\ g_2 & = & I & X & Z & Z & X \\ g_3 & = & X & I & X & Z & Z \\ g_4 & = & Z & X & I & X & Z \end{array}$$

$$g_j|\psi\rangle = |\psi\rangle$$

Matchgate tensors
(Fermionic Gaussian states)

(a.k.a. free-fermions)

Fermionic Gaussian states

O(n²) parameters

Majorana operators:

$$m_{2k-1} = f_k + f_k^\dagger$$

$$m_{2k} = i(f_k^\dagger - f_k)$$

Anti-commutation: $\{m_j, m_k\}_+ = 2\delta_{j,k}$

Covariance-matrix: $\Gamma_{j,k}(\rho) = \mathrm{Tr}\left(\rho \frac{i}{2}[m_j, m_k]\right)$

Coherent (pure) Gaussian states:

$$|\psi\rangle =: U_g : |\emptyset\rangle = \mathcal{Z}^{-1/2} \exp\left(\frac{1}{2}\sum_{j,k=1}^{n} A_{j,k} f_j^\dagger f_k^\dagger\right)|\emptyset\rangle$$

Quadratic Hamiltonian:

$$H_g = \frac{i}{4}\sum_{j,k=1}^{2n} g_{j,k} m_j m_k$$

Gaussian unitary: $U_g = e^{-iH_g}$

Parent Hamiltonian

Pure gaussian states can be specified as ground states of a non-degenerate quadratic Hamiltonian

$$H_g = \frac{i}{4} \sum_{j,k=1}^{2n} g_{j,k} m_j m_k$$

Upon block diagonalization of the anti-hermitian matrix

$$H_g = \frac{i}{2} \sum_{k=1}^{n} \varepsilon_k \tilde{m}_{2k-1} \tilde{m}_{2k}$$

$$= \frac{1}{2} \sum_{k=1}^{n} \varepsilon_k (\tilde{f}_k^\dagger \tilde{f}_k - \tilde{f}_k \tilde{f}_k^\dagger)$$

Nullifiers: $\tilde{f}_k |\psi\rangle = 0$

The new fermionic operators are sometimes called nullifiers. They are linear combinations of the original ones.

Fermionic Wick's theorem

For two pure fermionic gaussian states:

$$\langle\phi_1|m_1^{x_1}m_2^{x_2}\dots m_{2n}^{x_{2n}}|\phi_2\rangle = \langle\phi_1|\phi_2\rangle \mathrm{Pf}\left(i\Delta_{|\vec{x}}\right)$$

Where in terms of the covariance matrices,

$$\Delta = \left(-2 + i\Gamma_1 - i\Gamma_2\right)\left(\Gamma_1 + \Gamma_2\right)^{-1}$$

And Pf is the Pfaffian, which satisfies:

$$\mathrm{Pf}(M)^2 = \mathrm{det}(M)$$

$$\mathrm{Pf}\,(M) = \frac{1}{2^k k!}\sum_{\sigma\in S_n}(-1)^\sigma M_{\sigma(1),\sigma(2)}\cdots M_{\sigma(n-1),\sigma(n)}.$$

Matchgate tensors

Pure gaussian states correspond to match-gate tensors when considering occupational basis.

$$|\psi_g\rangle = \sum_{j\in\{0,1\}^n} \mathcal{T}(j)(f_1^\dagger)^{j_1}\dots(f_n^\dagger)^{j_n}|\emptyset\rangle$$

Efficient closed operations on match-gate tensors.

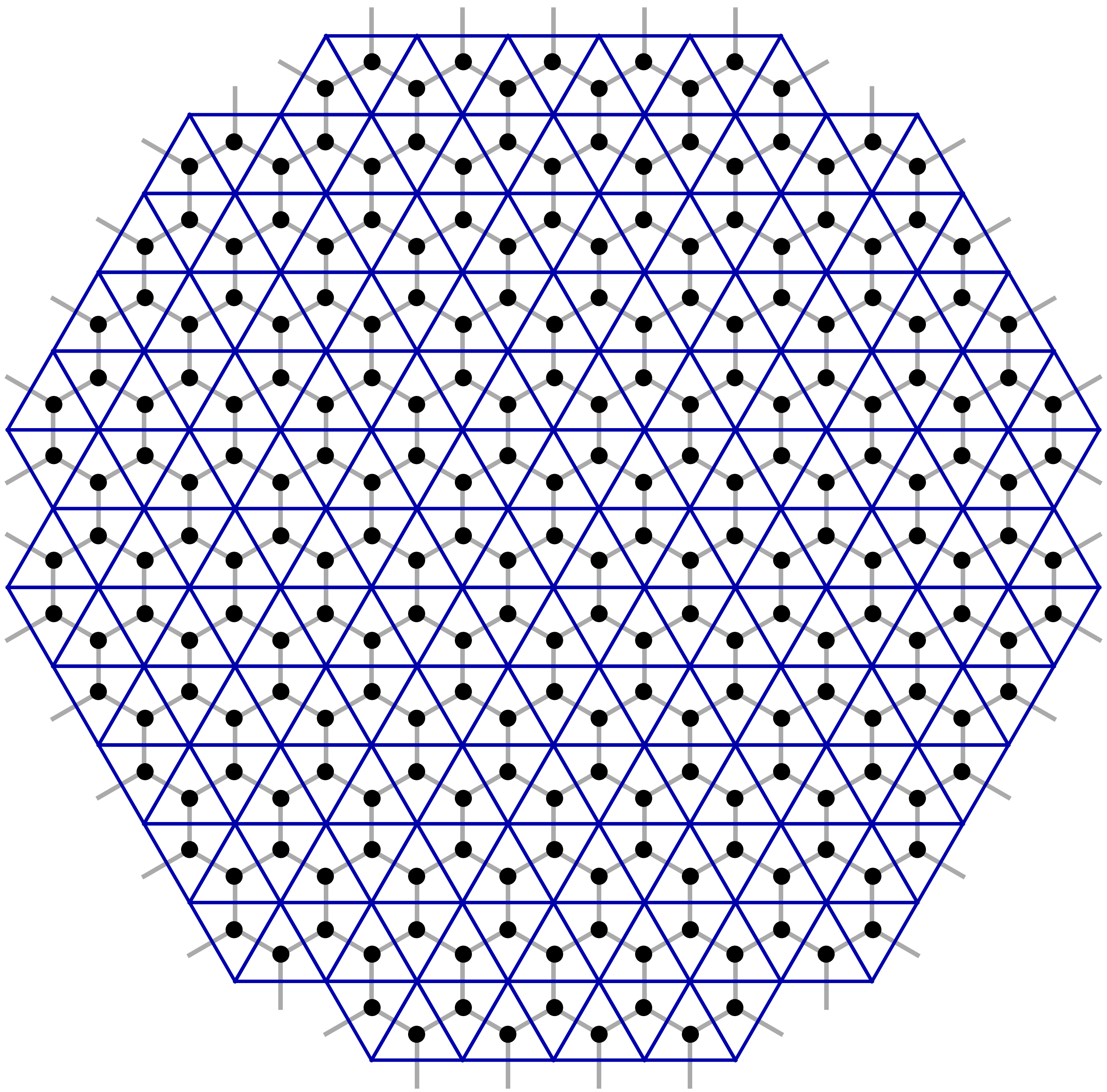
1. Cyclic reordering of labels.
2. Tensor product of two.
3. Contraction of neighboring labels.

Contract **Planar** match gate TNs efficiently

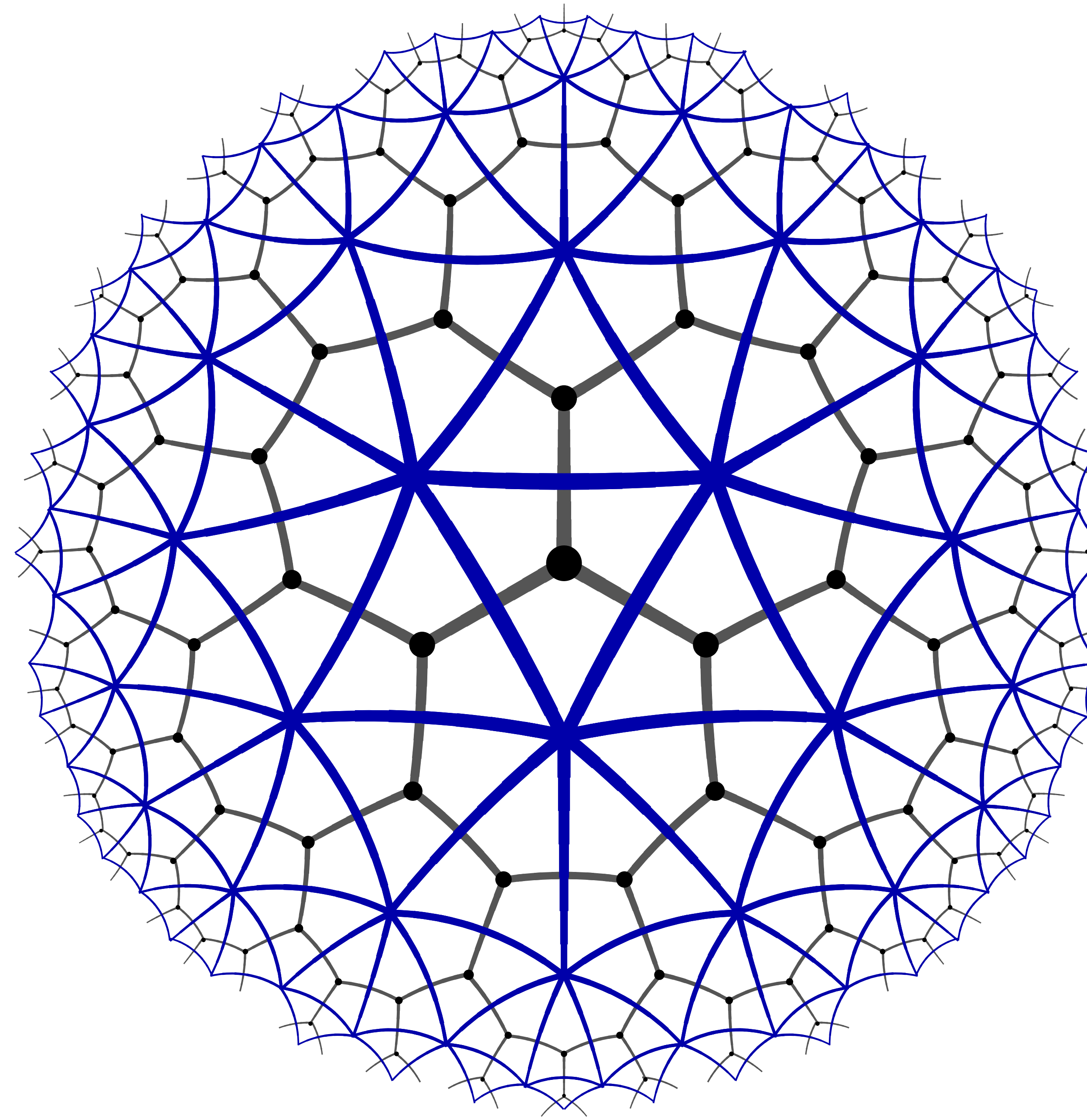
Sergey Bravyi, *Contraction of matchgate tensor networks on non-planar graphs*
arXiv:0801.2989 Contemporary Mathematics, Vol. 482, pp. 179-211 (2009)

Numerical experiments
Matchgate TNs

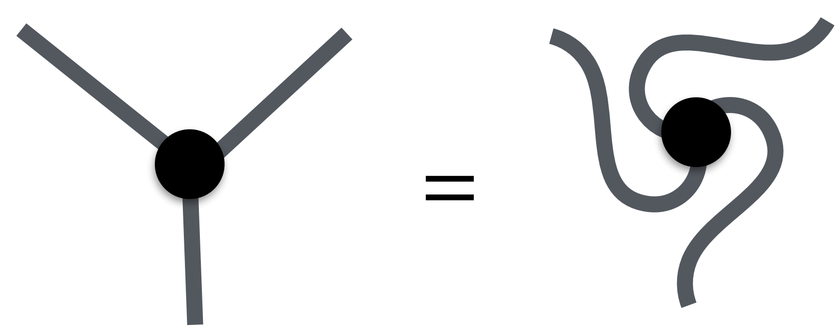
Flat geometry



H_2 geometry



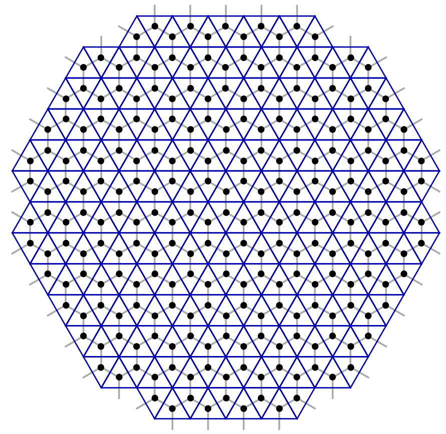
Single (tensor rank = 3)
cyclic invariant tensor.
Just one parameter left!

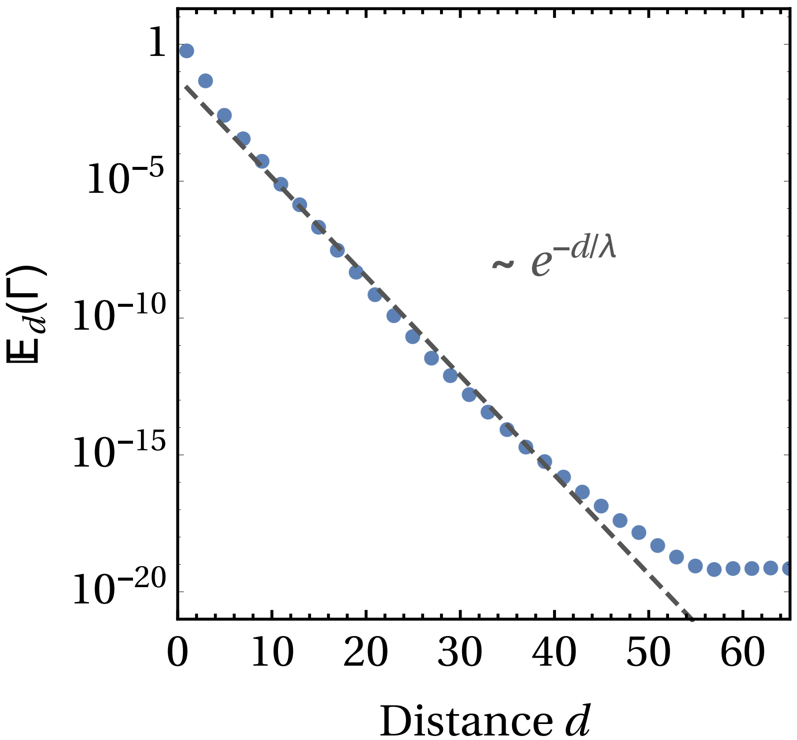


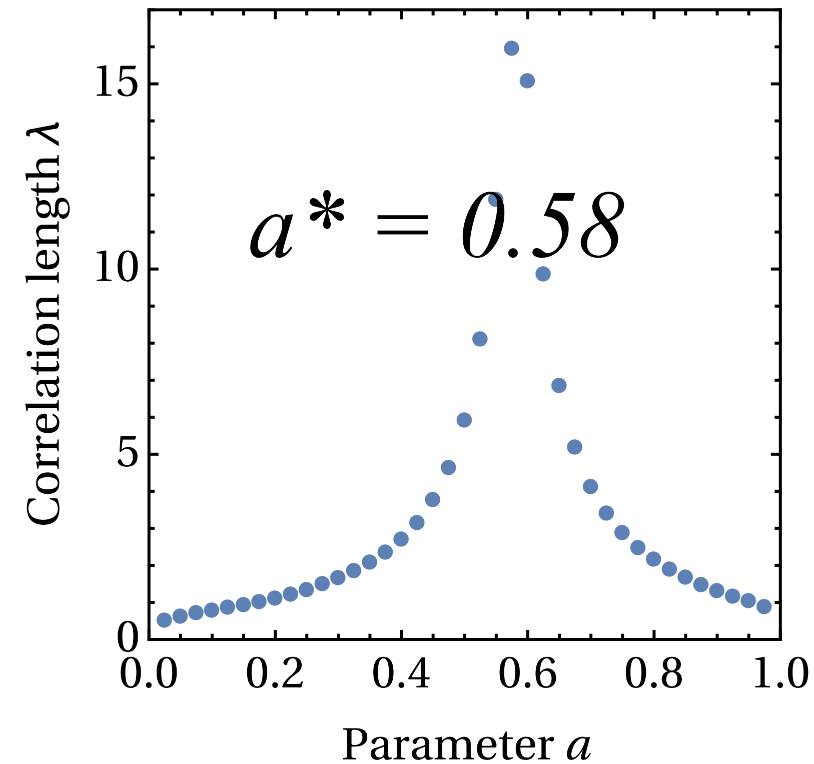
Covariance decay

$a = 0.25$

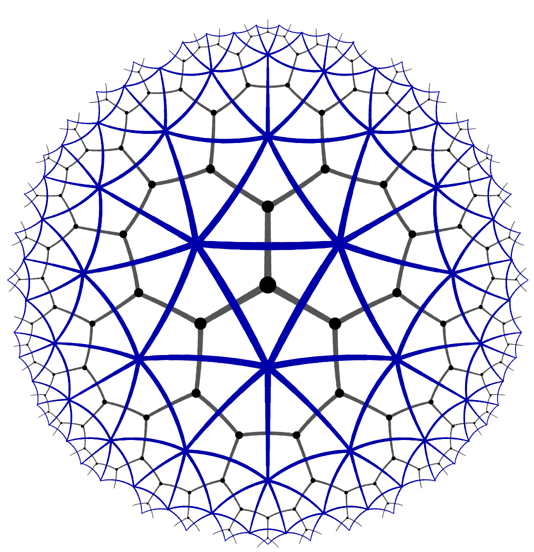
Correlation length

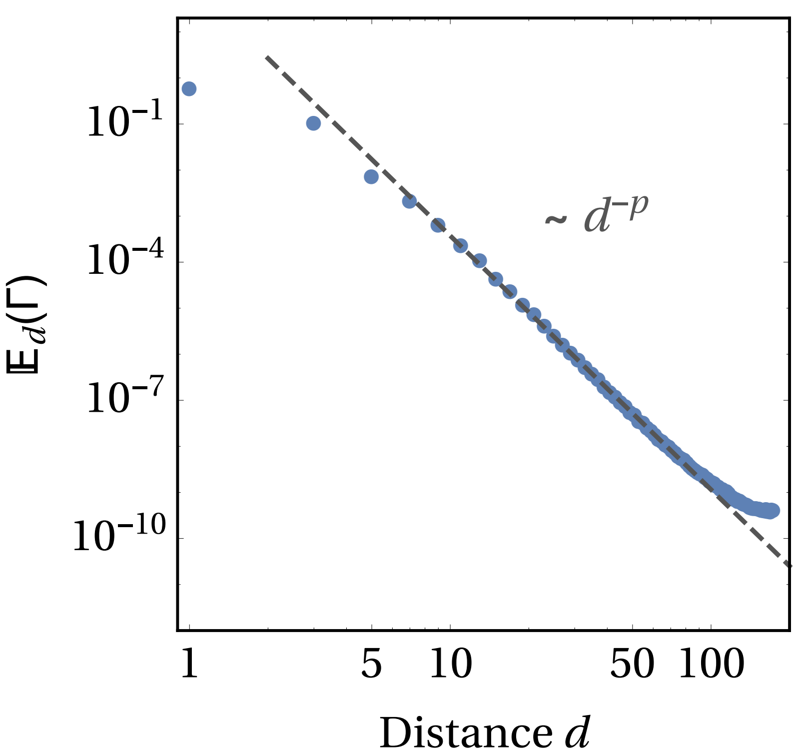


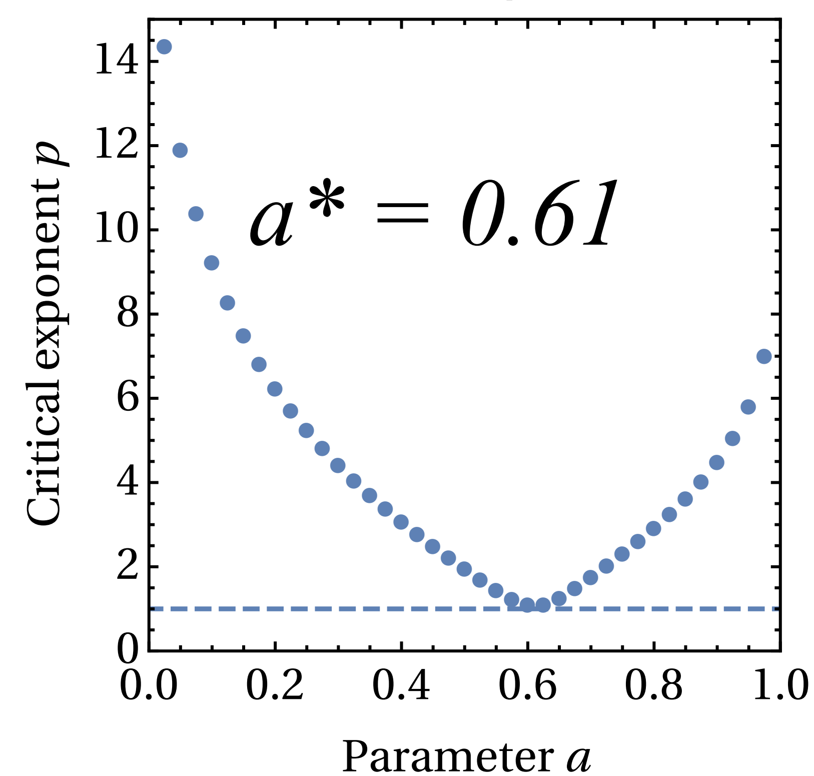




Critical exponent





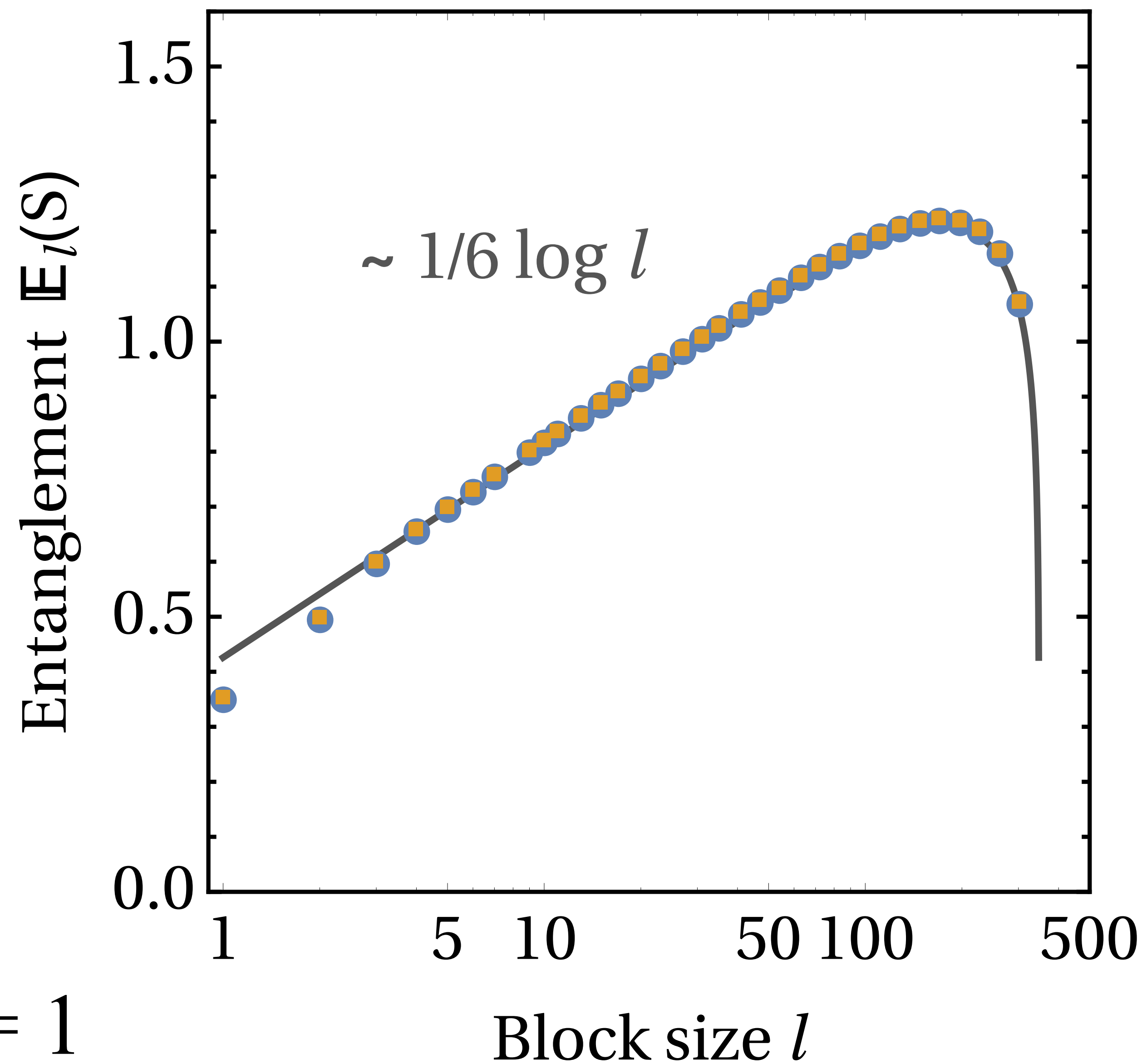


Ising CFT entanglement

Blue: flat tiling at $a = 0.58$

Orange: hyperbolic tiling at $a = 0.61$

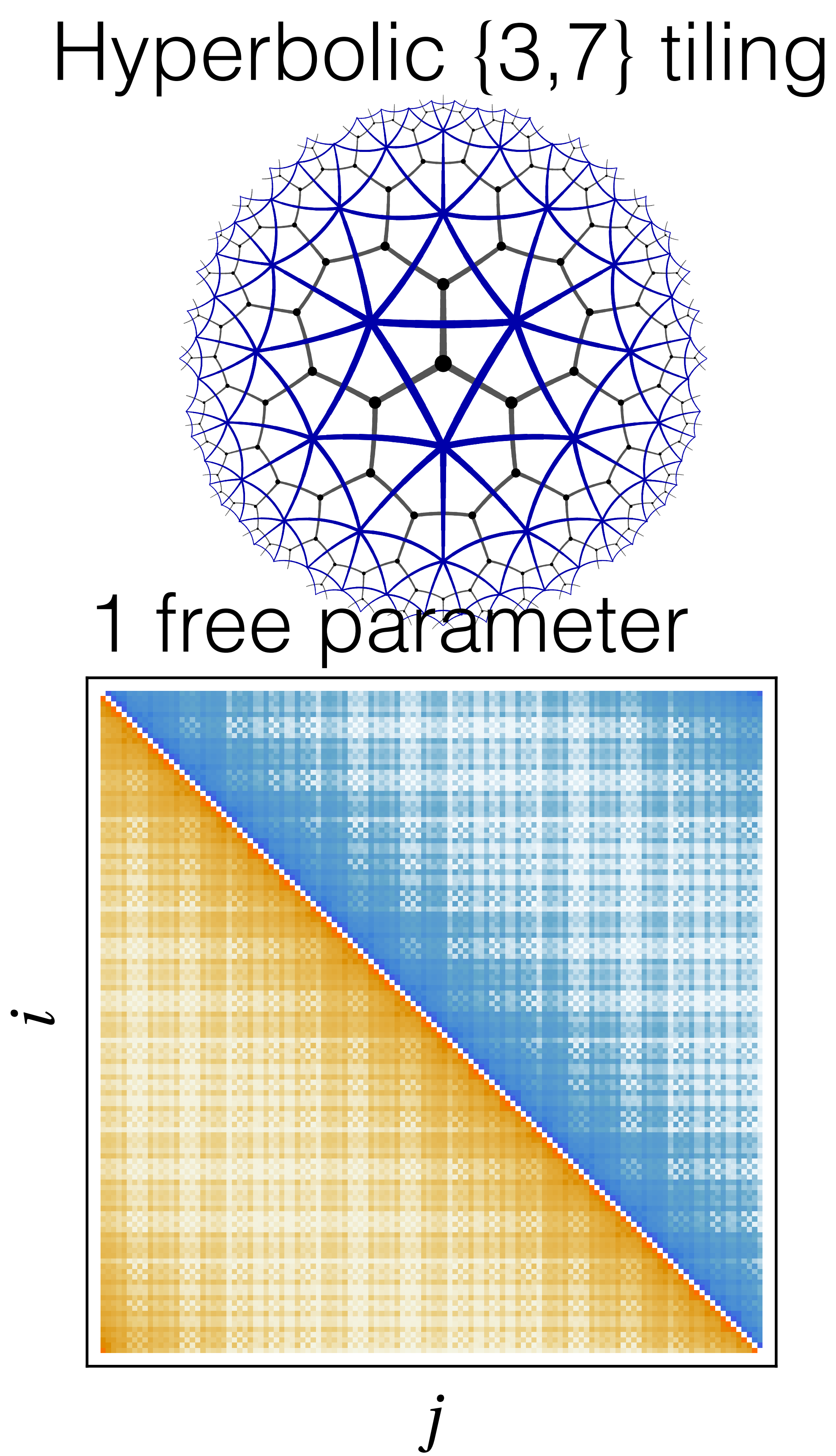
Solid: theory for $c = 1/2$, fit at $\varepsilon = 0.08$ (in lattice units)

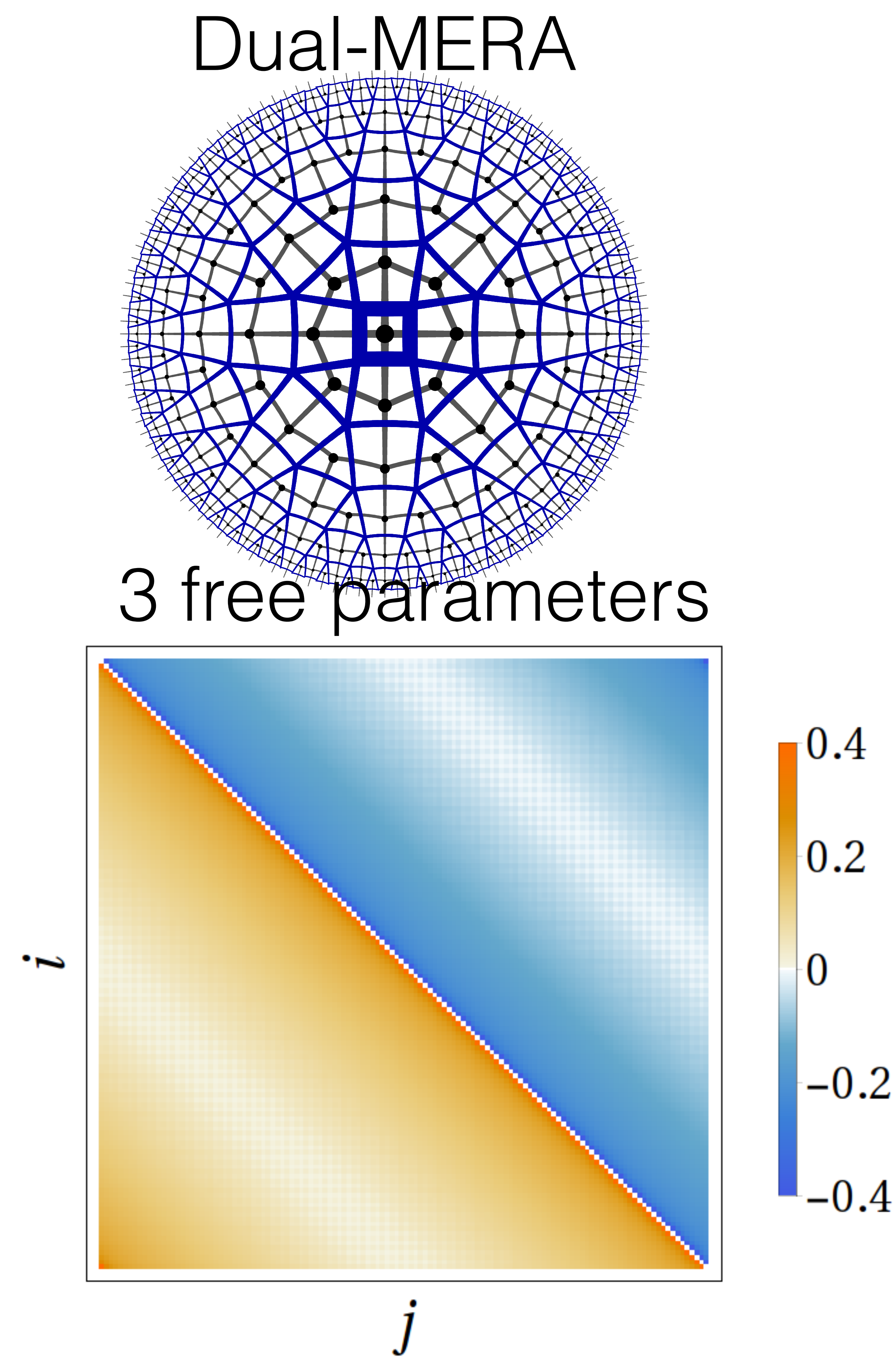


cutoff $= \varepsilon \neq$ lattice spacing $= 1$

$$S_\ell = \frac{c}{3} \ln \left(\frac{L}{\pi \epsilon} \sin \frac{\pi \ell}{L} \right) \simeq \frac{c}{3} \ln \frac{\ell}{\epsilon} + O\left((\ell / L)^2 \right)$$

Bulk lattice breaks TI

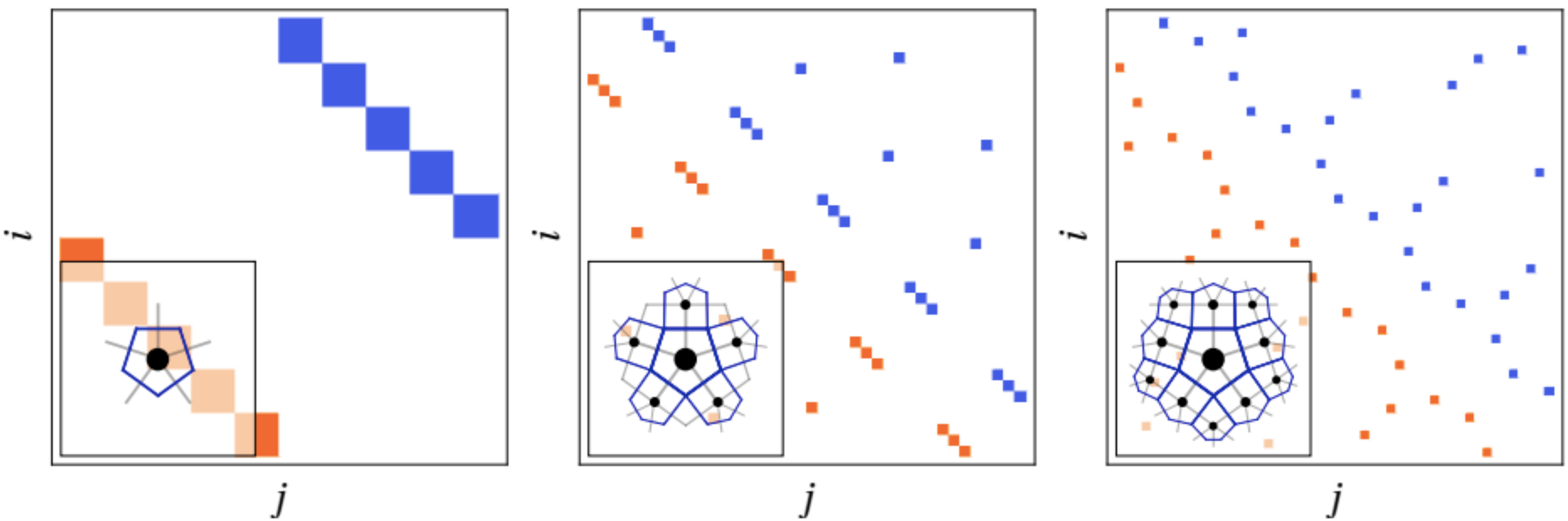




Holographic pentagon code as a majorana dimer model

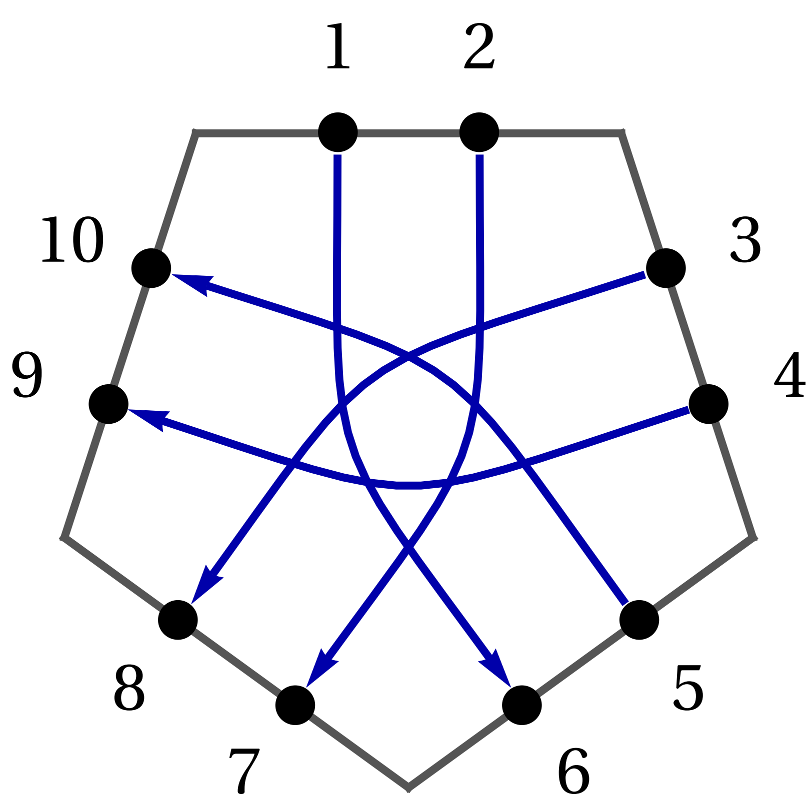
{0,1} input on 5-qubit code yields match gate tensors.

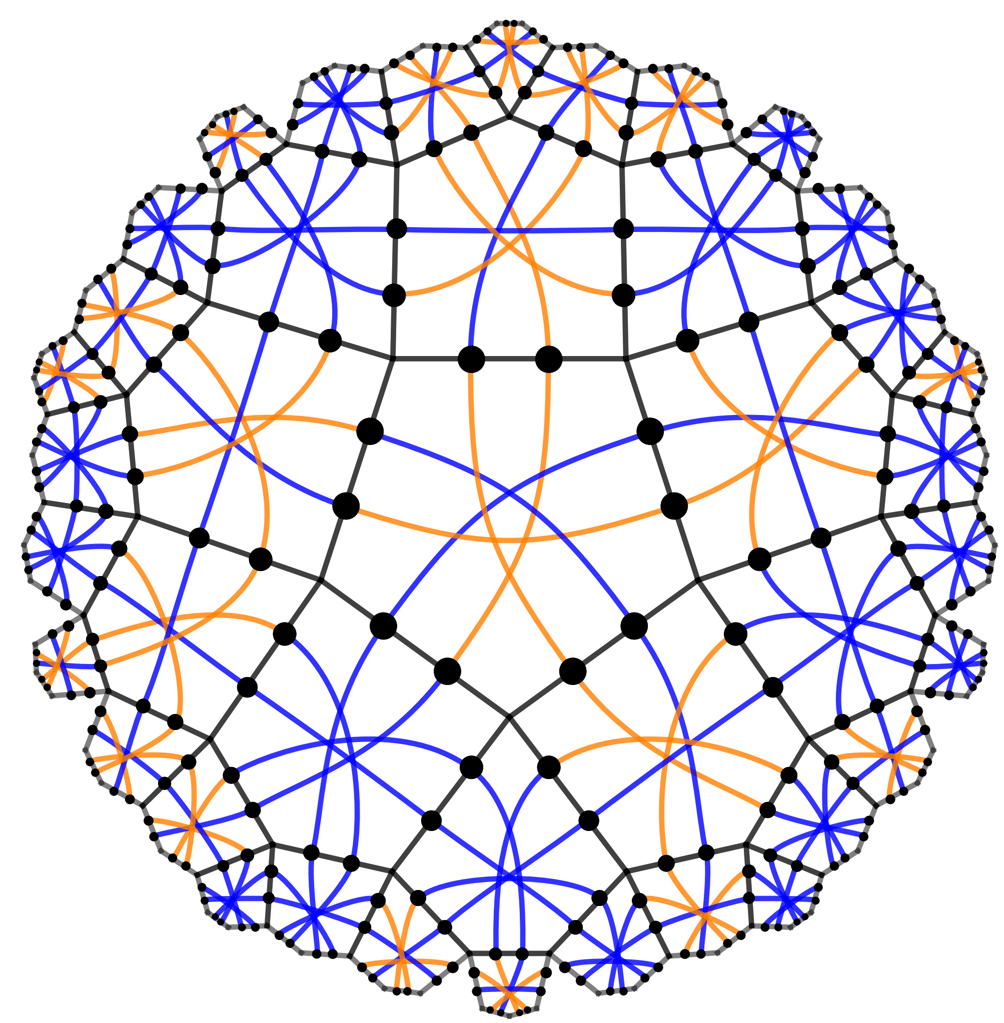
Covariance matrix



Geodesic tracing of +-1 majorana correlations

1 qubit = 2 majorana (Jordan-Wigner transformation)





Stabilizer tensor networks

Generalized Pauli group

$$X = \sum_{j=1}^{p} |j+1\rangle\langle j|$$

$$Z = \sum_{j=1}^{p} \omega^j |j\rangle\langle j|$$

$$Z^\dagger X^\dagger Z X = \omega$$

$$\omega = e^{2i\pi/p}$$

Order p unitary ops.

$$X^p = Z^p = 1 = XX^\dagger = ZZ^\dagger$$

The generalized Pauli group on n qupits is generated by the X and Z operators on all n qupits. Up to phases, it has p^{2n} elements p^{2n+1} with phases.

Stabilizer groups & states

A Pauli stabilizer group is a group, generated by commuting elements from the generalized Pauli group (and without $\omega 1$).

$$\mathcal{S} = \langle S_1, \ldots, S_m \rangle$$

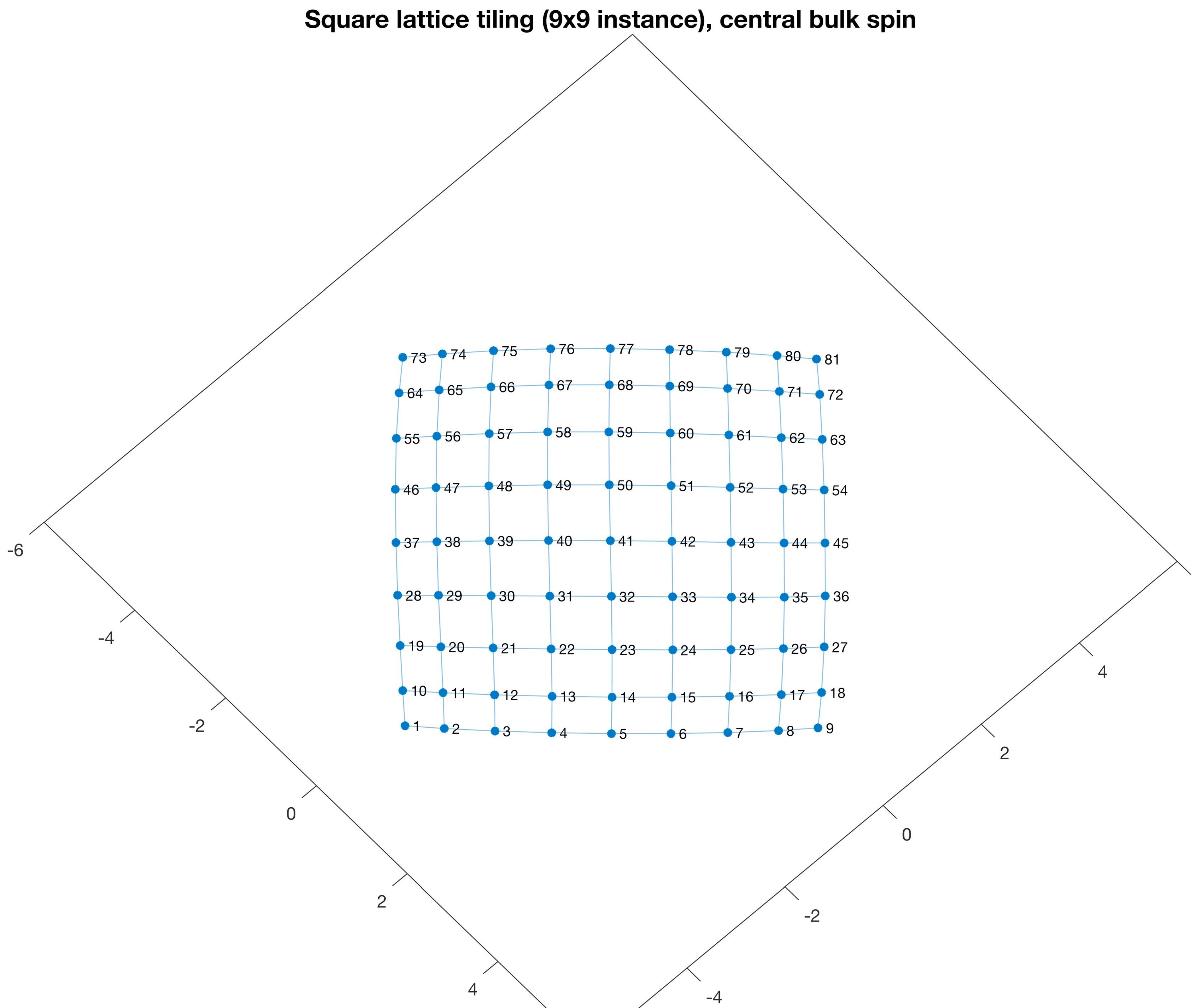
If the *m* generators are independent, the stabilizer group defines a p^{n-m} dimensional subspace, a unique state if $m=n$.

$$S_j |\psi\rangle = |\psi\rangle$$

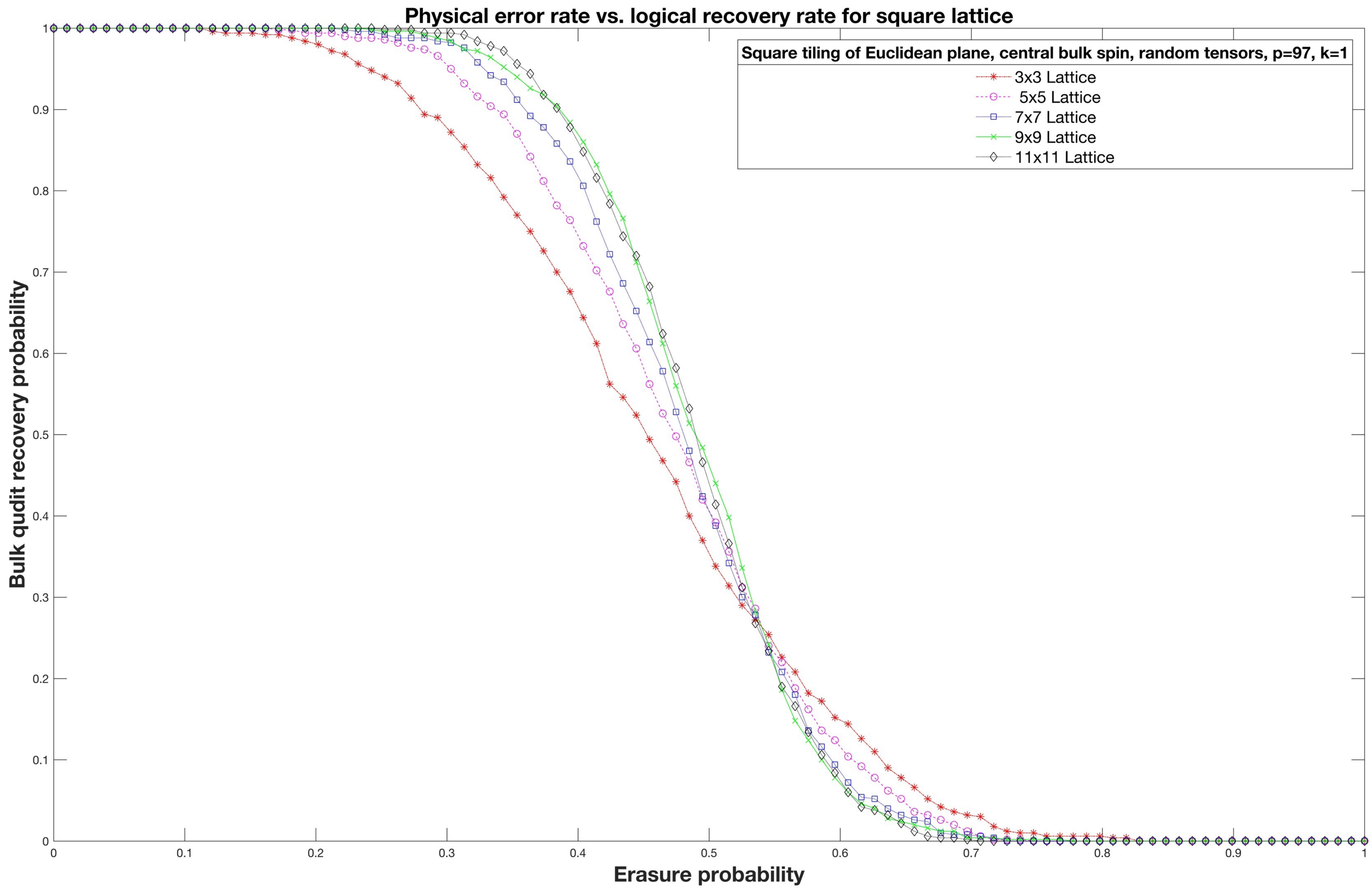
Tensors associated to stabilizer states are efficiently described and contracted. This is the content of the Gottesman-Knill theorem.

Aaronson, S., & Gottesman, D. (2004). Improved simulation of stabilizer circuits. arXiv:quant-ph/0406196 Physical Review A, 70(5), 52328.

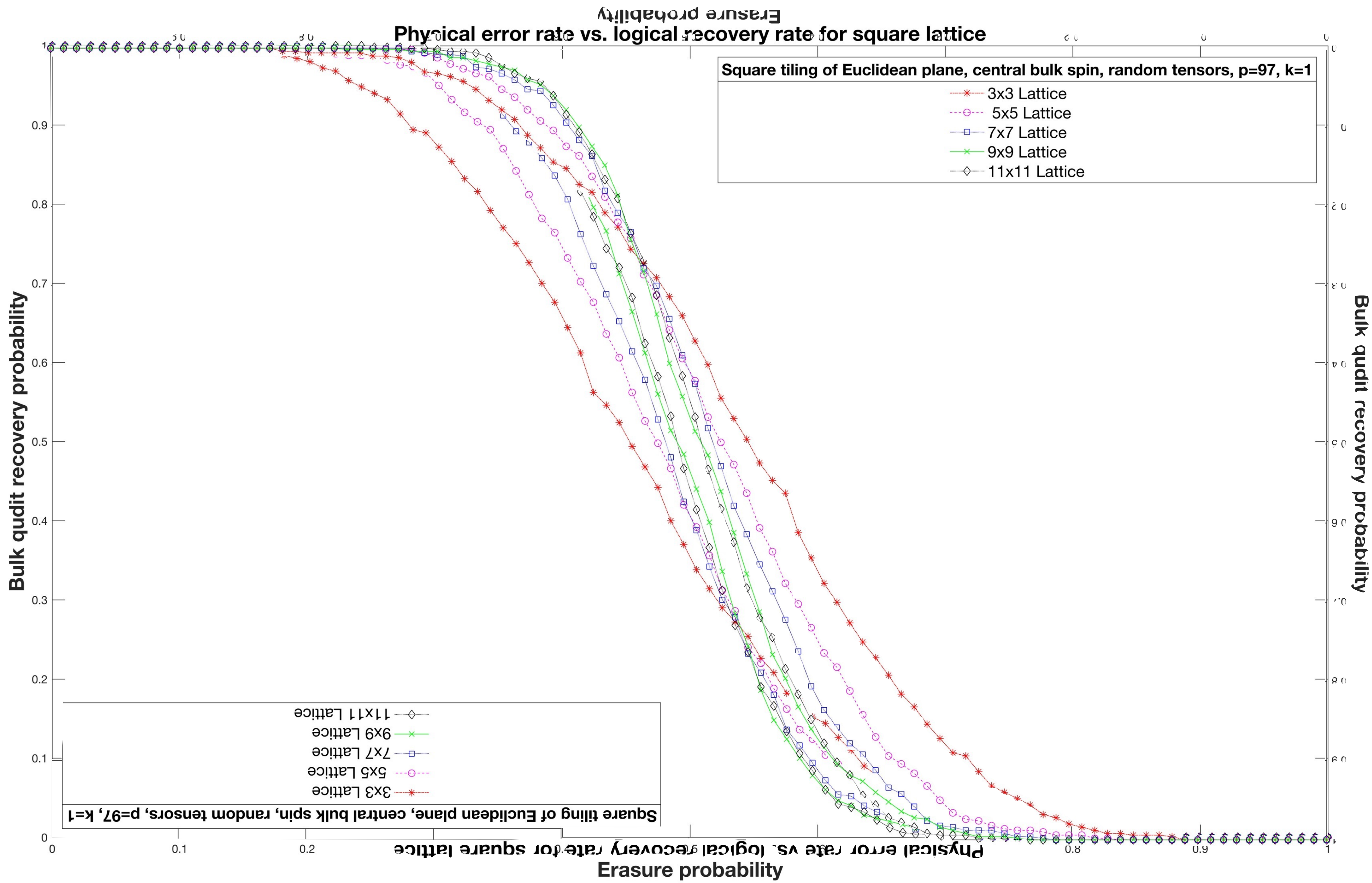
Checking entanglement wedge hypothesis with random stabilizer tensor networks



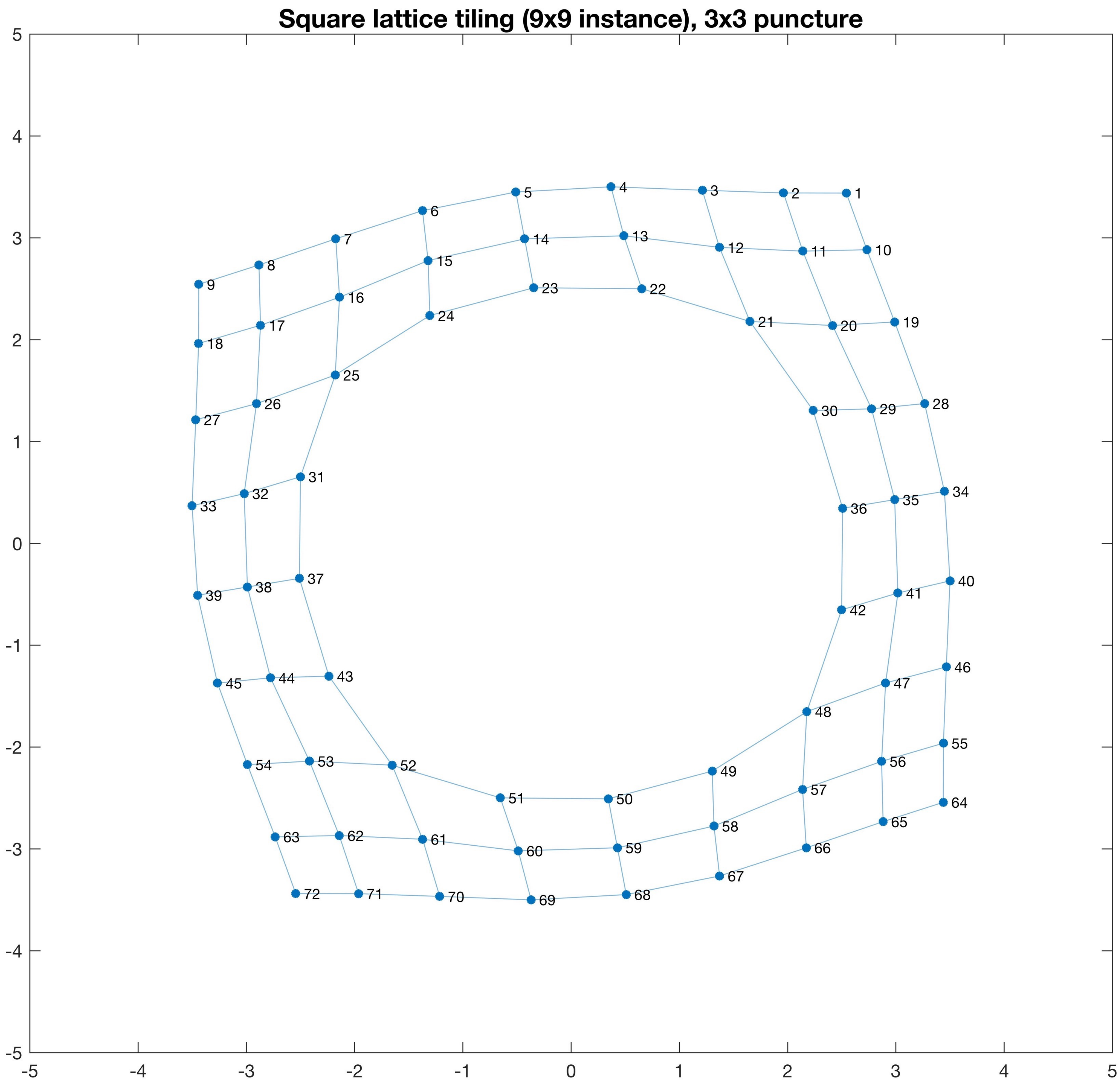
Checking entanglement wedge hypothesis with random stabilizer tensor networks



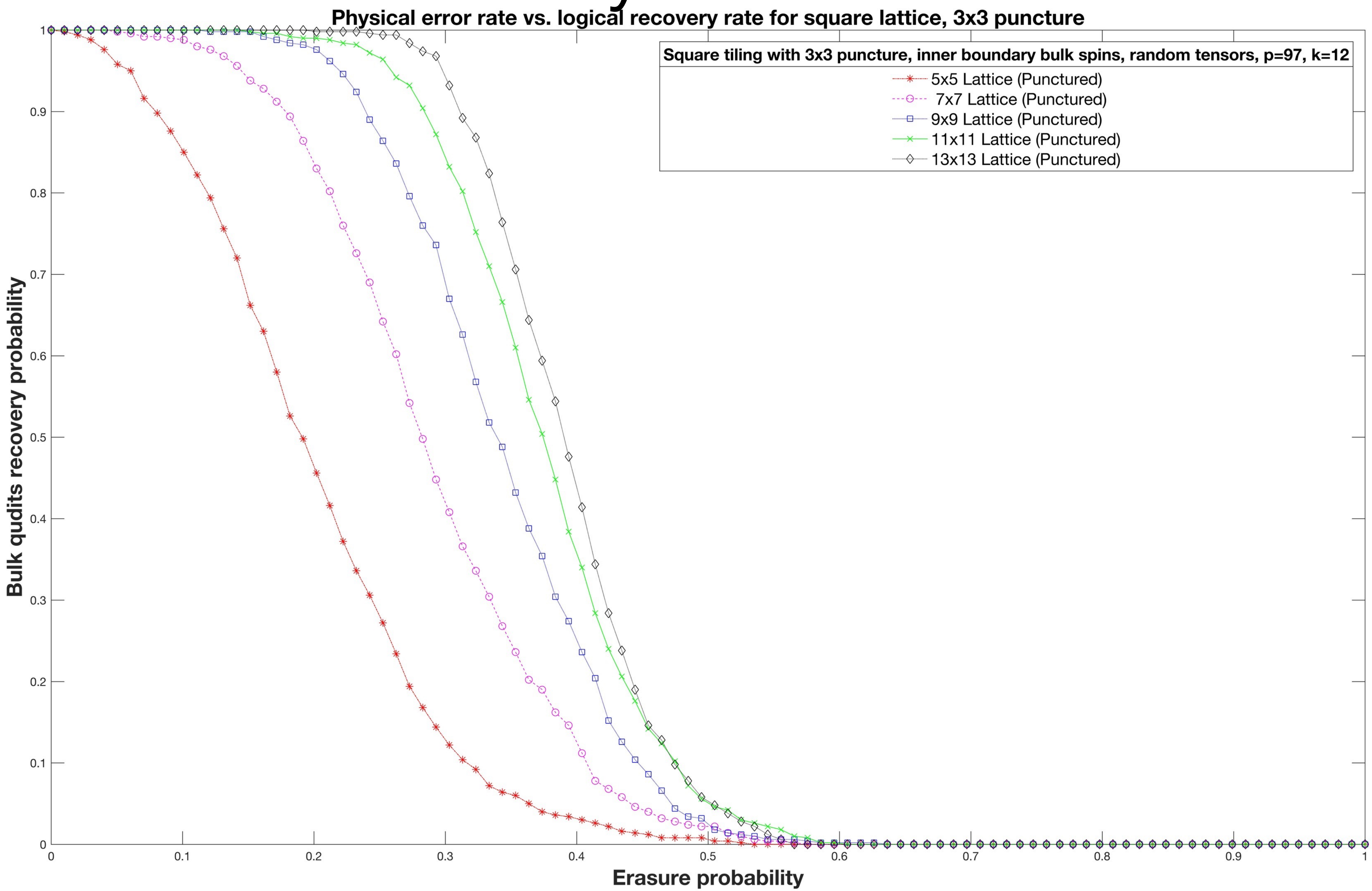
Checking entanglement wedge hypothesis with random stabilizer tensor networks



Punching holes



Recovery takes a hit for boundary close to hole



Conclusion

1. Tensor network structure does not immediately guarantee entanglement structure.
   - Fault free theory? Chaos needed ?
   - Fault bulk TI ?

2. Stabilizer TN approximate entanglement features suggested by discrete geometry in holography.
   - Approxiamate of the entanglement wedge hypothesis.
   - Produce synthetic entanglement structures and QECs.

3. Exactly solvable tensor networks provide a valuable tool for the study of tensor network holography.
   - What are your questions ?

Thank you and
Thank them for the hard work

Alexander Jahn

Fermionic
numeric

Marek Gluza

Matchgate
formalism

Arkin Tikku

Stabilizer TNs