Complexity for quantum field theory and bi-invariant Finsler manifolds

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- Background;
- Axiomatization of complexity;
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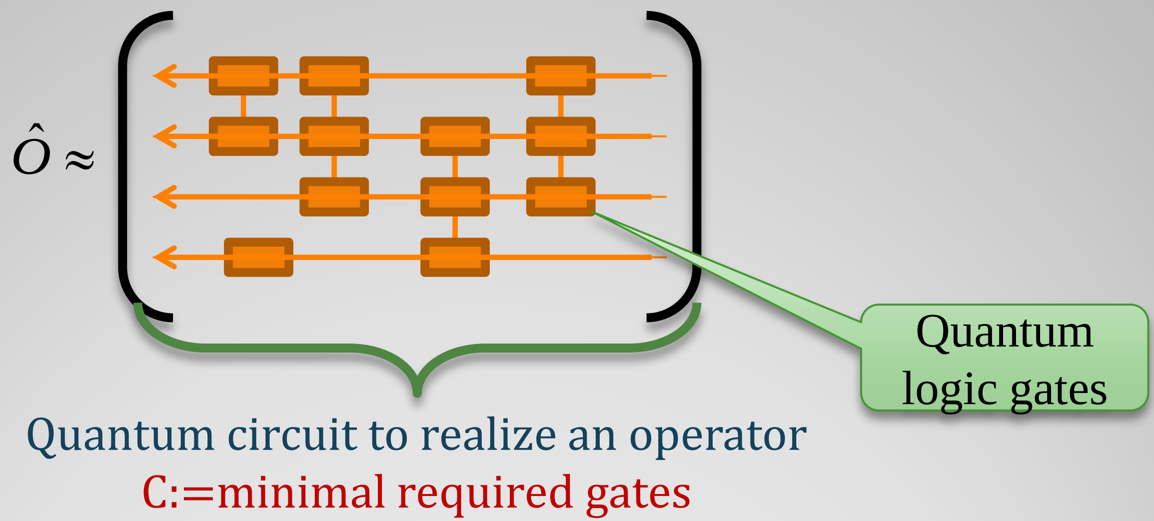
1. Background

Complexity in quantum circuits

- In quantum circuits, the complexity is usually defined in the finite (discreet) Hilbert space.
- Two states can be associated by an unitary operator $\hat{O}$ which could be simulated by a quantum circuit.



- Complexity for one state to the other state is defined as the minimal required gates when we realized this operator by quantum circuits.

- Daniel Harlow and Patrick Hayden first noted that the computational complexity may play role in black holes physics; JHEP06(2013)085

- In 2014, Susskind proposed an idea that complexity is essential to understand the properties of black holes horizon.
  Fortsch. Phys., 64:44-48, 2016.

- In 2014 and 2016, Susskind proposed two different conjectures to compute complexity by holography.

Phys. Rev., D90(12):126007, 2014

Phys. Rev. Lett., 116(19):191301, 2016

Complexity in holography

- In 2016-2017, there are more than 100 papers involved the complexity in holography and black holes!

- However, some foundations about complexity are still unclear!

- How do we give the complexity a well-defined mathematical foundation?
- What is the meaning of complexity in quantum field theory?
- What can it tell us by using this conception?
- …….

Complexity: the foundation is still absent!

2. Axiomatization of complexity

- The complexity for any operator should be nonnegative;

$$C(\hat{O}) \geq 0, \forall \hat{O} \in D$$

- As the identity can be realized without referring any gate, its complexity should be zero; if an operator has zero complexity, then it can be realized by a quantum circuit without any gate and so this operator must be identity.

$$C(\hat{O}) = 0 \Leftrightarrow \hat{O} = \hat{I}$$

- As the complexity should stand for the minimal required gates, following inequality must be true,

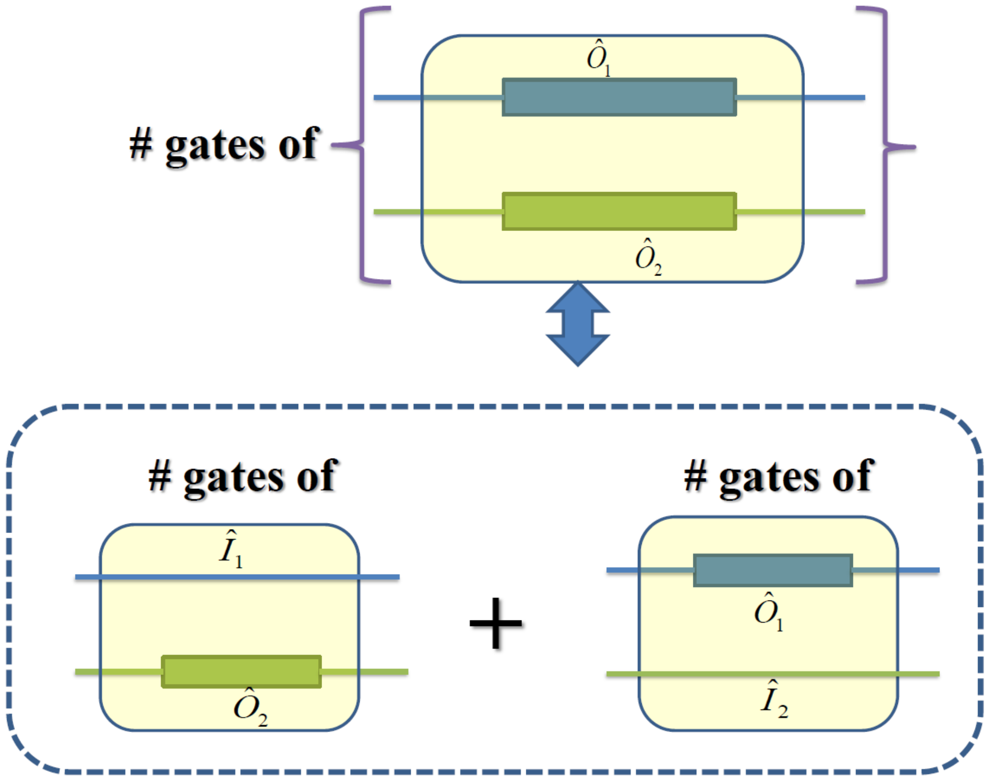
$$C(\hat{O}_1 \hat{O}_2) \leq C(\hat{O}_1) + C(\hat{O}_2), \forall \hat{O}_1, \hat{O}_2 \in D$$

Basic properties of complexity

Cartesian product

The fourth axiom :

$$C[(\hat{O}_1,\hat{O}_2)]$$
$$= C[(\hat{O}_1,\hat{I}_2)] + C[(\hat{I}_1,\hat{O}_2)]$$



- As any quantum circuit can only realize a unitary transformation, it is enough to only consider the complexity for operators in SU(n) groups.

- **Assumption 1**: for any generator H, the complexity of $\exp(H\varepsilon)$ depends on ε smoothly if $\varepsilon \geq 0$;

- **Assumption 2**: complexity has path-reversal symmetry.

Two assumptions for SU(n) grups

Complexity and Finsler geometry

- In fact, just by four axioms and two assumptions, we can prove following surprising result:

The complexity for SU(n) group can **only** be given by geodesic length of a **bi-invariant Finsler geometry** in SU(n) group with following Finsler structure:

$$F(c,\dot{c}) = \lambda \mathrm{Tr}\left(\sqrt{HH^\dagger}\right) = \lambda \mathrm{Tr}\left(\sqrt{\dot{c}\dot{c}^\dagger}\right)$$

In general it is not Riemannian

Here H is defined as $H = \dot{c}c^{-1}$ or $c^{-1}\dot{c}$

- In fact, it is enough for any quantum mechanics and quantum field theories.

- We define a *cost* for a curve/process by the curve length,

$$L[c] := \int_0^1 F(c, \dot{c}) \mathrm{d}s = \lambda \int_0^1 \mathrm{Tr}\left(\sqrt{\dot{c}\dot{c}^\dagger}\right) \mathrm{d}s$$

- Thus, finding the complexity of operator is becomes finding the minimal cost curve to connect identity and this operator.

$$C\left(\hat{O}\right) := \min\left\{ L[c] \middle| \forall c,\ \text{s.t.},\ c(0) = \hat{I}, c(1) = \hat{O} \right\}$$

This means we need to find the geodesics in Finsler manifolds.

Complexity is geodesic length

- Due to the bi-invariance, the geodesic can be easy obtained. In fact, we have following theorem: [Journal of Basic and Applied Sciences **5**, 607 (2011), Mathematical Sciences **7**, 37 (2013)]

> the curve c(s) is a geodesic if and only if there is a constant generator H such that c(s)=exp(Hs)

- With the condition $\hat{O} = c(1) = \exp(H)$, we can solve H formally. Let's define

$$\ln(\hat{O}) := \{H_i \mid \forall H \in \mathrm{su}(n), \text{s.t.}, \exp(H) = \hat{O}\}$$

- Then we see

$$C(O) = \min\left\{\mathrm{Tr}\left(\sqrt{HH^{\dagger}}\right) \middle| \forall H \in \ln(\hat{O})\right\}$$

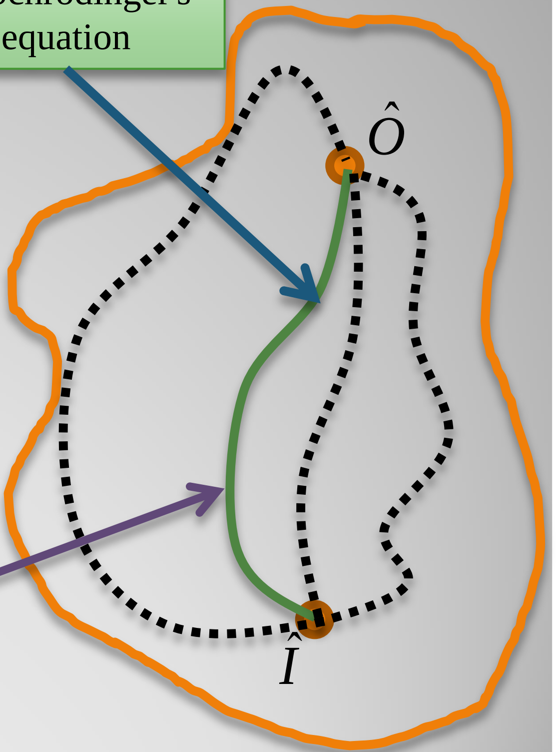
Geodesic and complexity

3. Some applications

- We can prove the Schroding's equation for isolated system is a consequence of *minimal complexity* principle:

For an isolated system, the time evolution operator will go along the curve such that the complexity in this process is locally minimal!

Curve generated by Schrödinger's equation



The process to realize *O* with minimal complexity

Applications

- The complexity between two states is given by density matrixes such that:

$$\mathcal{C}(\rho_1,\rho_2) = -\ln\left\{1 - \frac{1}{4}\left[\mathrm{Tr}\sqrt{(\rho_1-\rho_2)^2}\right]^2\right\}$$

- For two pure states $\rho_i = |\psi_i\rangle\langle\psi_1|$, we can prove this is just

$$\mathcal{C}(\rho_1,\rho_2) = -\ln\left|\langle\psi_1|\psi_2\rangle\right|^2$$

- In addition, if one of them is ground state and the other is field eigenstate, it can be expressed as Euclidean path integral:

$$\mathcal{C}(\rho_1,\rho_2) = -\ln\int_{\phi(x,0)=\phi_0(x)} D[\phi]\exp\left(-\frac{1}{\hbar}S_E[\phi]\right)$$

- For classical limit, its leading term reads

$$\mathcal{C}(\rho_1,\rho_2) \approx \frac{1}{\hbar}\min\left\{S_E[\phi]-S_E[0]\,|\,\forall\phi(x,\tau),\ \text{s.t.},\ \phi(x,0)=\phi_0(x)\right\}$$

Applications

- The complexity between a thermofield double state and vacuum state is given

$$\mathcal{C}(\rho_1, \rho_2) \propto VT^{d-1}$$

- The complexity between ground state and directly product state in coordinates representation in CFT is given by

$$\mathcal{C}(\rho_1, \rho_2) \propto V\Lambda^{d-1}$$

- Both these two results agree with CV and CA conjectures!!!

- For a system with classical chaos, the complexity in classical limit will growth linear for large time

$$\mathcal{C} = \lambda_L t, \;\; t \gg 1 \text{ and } \hbar \to 0$$

Applications

Summary

- Four axioms was abstracted to define complexity. They are suitable for discrete and continuous systems.
- With two additional simple assumptions for continuous cases, the complexity for any operator in SU(n) group is determined uniquely up to a factor;
- A *complexity principle* was proposed, which can replace Schrödinger's equation in isolated systems;
- When it is applied into the detailed physical cases and quantum states, all good results become correct!

For more details, Please read our paper arXiv: 1803.01797