Entanglement Entropy and Decoupling in the Universe

Noburo Shiba (YITP, Kyoto)

with Y. Nakai (Rutgers) and M. Yamada (Tufts)

Phys.Rev. D96 (2017) no.12, 123518 [arXiv:1709.02390]

Holography, Quantum Entanglement and Higher Spin Gravity II

@YITP,Kyoto 2018/3/15

Density matrix and Entropy

In quantum mechanics, a physical state is described by a vector which belongs to the Hilbert space of the total system.

However, in many situations, it is more convenient to consider quantum mechanics of a subsystem.

In a finite temperature system which contacts with heat bath, we ignore the part of heat bath and consider a physical state given by a statistical average (Mixed state).

Density matrix

A physical observable $\langle O \rangle = \mathrm{Tr}_{\mathcal{H}_{\mathrm{tot}}}[O \cdot \rho_{\mathrm{tot}}]$

The definition of (Renyi) entanglement entropy

We decompose the total Hilbert space into subsystems A and B.

$$H_{tot} = H_A \otimes H_B$$

We trace out the degrees of freedom of B and consider
the reduced density matrix of A.

$$\rho_A = Tr_B \rho_{tot}$$

EE is defined as von Neumann entropy.

$$S_A := -tr_A \rho_A \log \rho_A$$

The Renyi entropy is the generalization of EE and defined as

$$S_A^{(n)} := \frac{1}{1-n} \log Tr({\rho_A}^n)$$

Entanglement entropy

- ✓ **When the total system is a pure state,**

  The entanglement entropy represents the strength of quantum entanglement.

- ✓ **When the total system is a mixed state,**

  The entanglement entropy includes thermodynamic entropy.

Any application to particle phenomenology or cosmology ??

Decoupling in the Universe

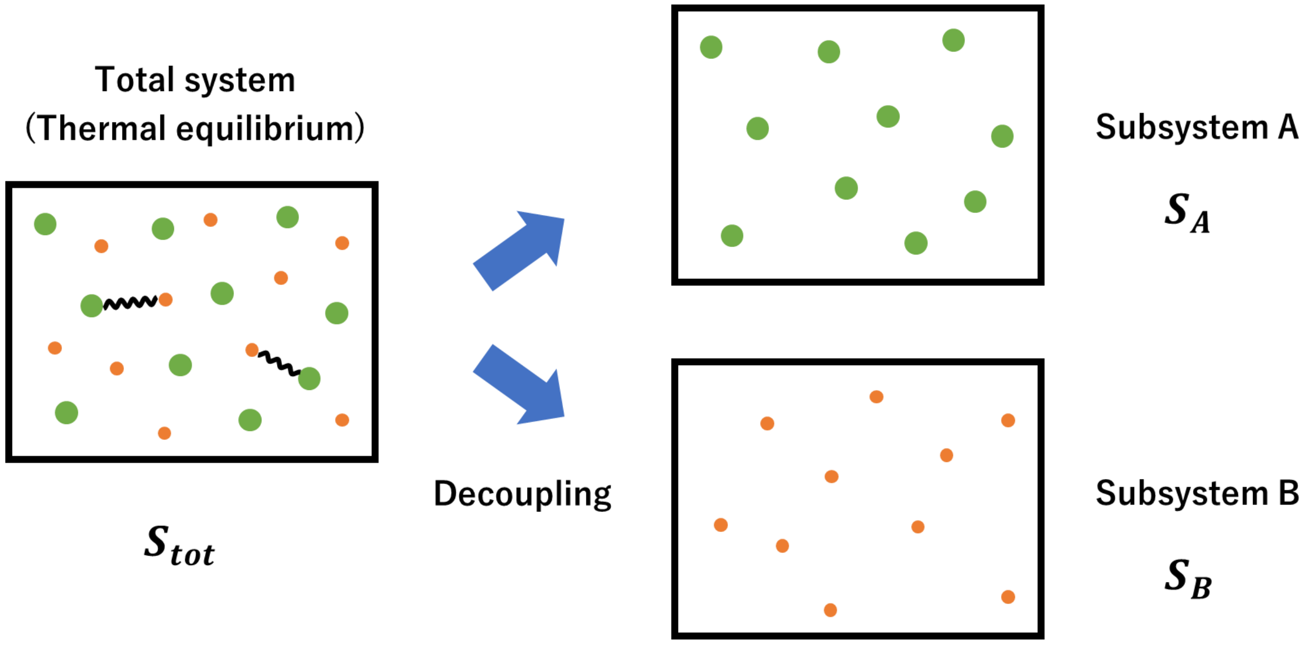
Particles A are no longer in thermal contact with particles B when the interaction rate is smaller than the Hubble expansion rate.

$$\Gamma_{AB} = n_B \langle \sigma_{AB} |v| \rangle < H \sim T^2 / M_{\rm Pl}$$

e.g. neutrino decoupling

Decoupled subsystems are treated separately.

Neutrino temperature drops independently from photon temperature.

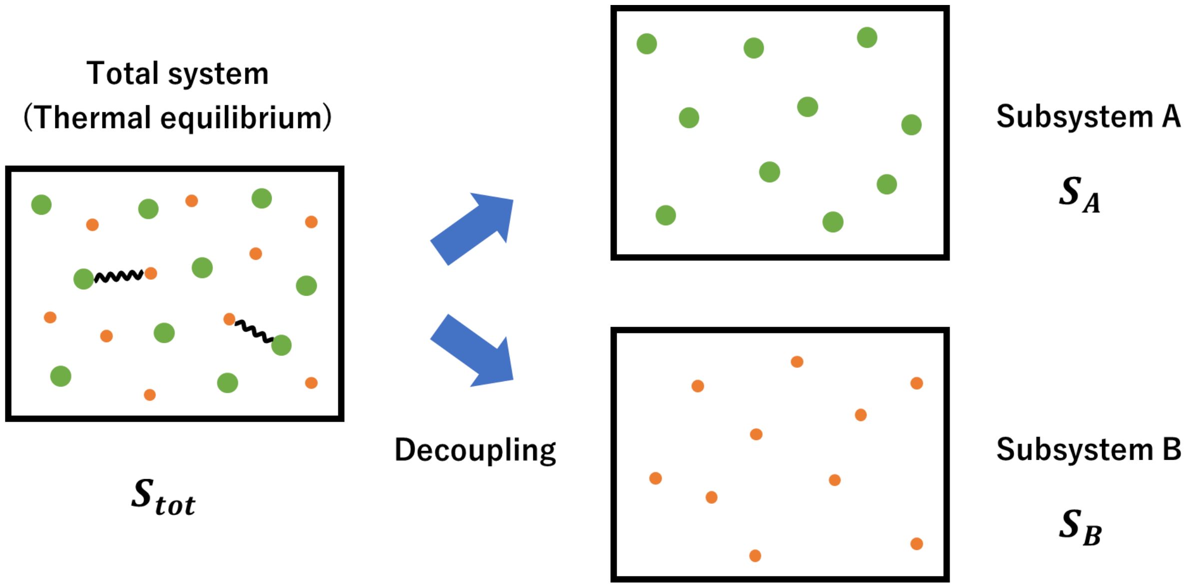


Thermodynamic entropy is expected to be conserved in each subsystem.

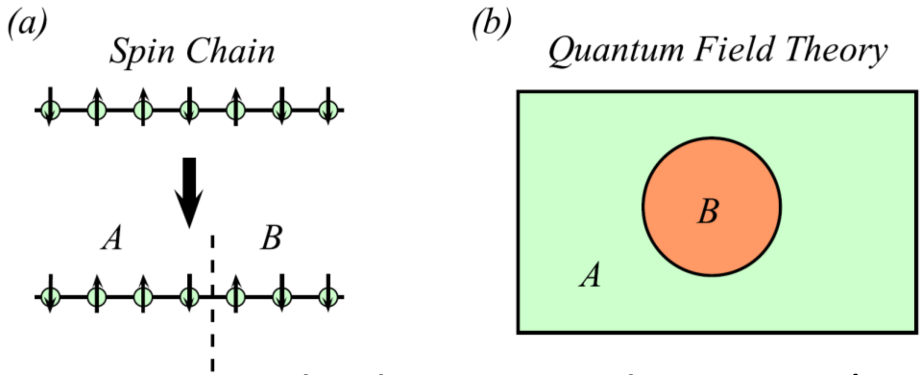
Decoupling in the Universe

However, **this paradigm assumes that thermal equilibrium is maintained in the subsystems during and after the decoupling.**

Can we refine the argument of decoupling?



Trace out the information of the other subsystem.



Nishioka, Ryu, Takayanagi (2009)

Thermodynamic entropy is generalized to entanglement entropy!

Entanglement entropy

Before decoupling, the total system is in thermal equilibrium and the density matrix is given by a grand canonical ensemble.

$$\rho_{\text{tot}} = \frac{e^{-\beta(H_{\text{tot}} - \mu_A \hat{N}_A - \mu_B \hat{N}_B)}}{Z_{\text{tot}}}, \qquad Z_{\text{tot}} = \text{Tr}_{\mathcal{H}_{\text{tot}}} e^{-\beta(H_{\text{tot}} - \mu_A \hat{N}_A - \mu_B \hat{N}_B)}$$

After decoupling, in terms of the subsystem A,
the system B can be seen as an environment and be traced out.

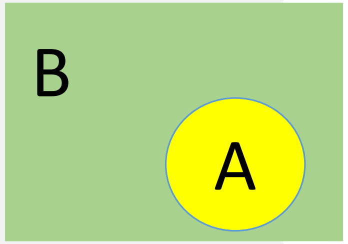
Entanglement entropy $S_A = -\text{Tr}_{\mathcal{H}_A}\left[\rho_A \log \rho_A\right]$

A unitary evolution : $\rho_A(t_0) \to \rho_A(t) = e^{-iH_A(t-t_0)} \rho_A(t_0)\, e^{iH_A(t-t_0)}$

The entanglement entropy is conserved due to the nature of the trace!

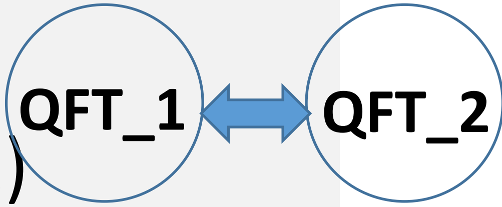
Research background

- ✓ **Entanglement entropy has been extensively discussed for the case of a spatial decomposition.**



- ✓ **Instead, in this work, we analyze EE between two QFTs which live in a common spacetime and interacting with each other.**

  other works
  e.g. A. Mollabashi, NS, T. Takayanagi, (2014),C. Xu, (2011)



- ✓ **A general formulation of perturbation theory is still lacking.**

We formulate perturbation theory and present Feynman rules.

We discuss its applications to cosmology.

Path integral formulation

We consider two interacting fields : $\phi_A(t,x) \quad \phi_B(t,x)$

The total system is in thermal equilibrium.

The usual finite-temperature field theory $\tau = it$

The partition function of the total system : $\beta = 1/T$

$$Z_{\text{tot}}^{(\beta)} = \int \mathcal{D}\phi_A \mathcal{D}\phi_B \exp\left(-\int_0^\beta d\tau \int d^d x \, \mathcal{L}(\phi_A, \phi_B) \right)$$

The fields are (anti-)periodic in imaginary time $\tau \in (0, \beta)$

Path integral formulation

Trace out the density matrix over the Hilbert space of B.

The reduced density matrix :

$$\langle\phi_A|\,\rho_A\,|\phi'_A\rangle$$

$$= \frac{1}{Z_{\rm tot}^{(\beta)}} \int \mathcal{D}\phi_A \mathcal{D}\phi_B|_{\phi_A(0)=\phi'_A,\phi_A(\beta)=\phi_A} \exp\left(-\int_0^\beta d\tau \int d^d x\, \mathcal{L}(\phi_A,\phi_B)\right)$$

NOT the form of a grand canonical ensemble in general.

The entanglement entropy is defined as the von Neumann entropy.

$$S_A = -\mathrm{Tr}_{\mathcal{H}_A}\left[\rho_A \log \rho_A\right]$$

Path integral formulation

It is not easy to evaluate the trace of $\rho_A \log \rho_A$

We first calculate the Renyi entropy and take the limit $n \to 1$

$$S_A^{(n)} = \frac{1}{1-n} \log \left(\mathrm{Tr} \rho_A^n \right) \qquad \lim_{n\to 1} S_A^{(n)} = S_A$$

$$\mathrm{Tr}\rho_A^n = \frac{\widetilde{Z}_{\mathrm{tot}}^{(n\beta)}}{(Z_{\mathrm{tot}}^{(\beta)})^n} \equiv \frac{1}{(Z_{\mathrm{tot}}^{(\beta)})^n} \int \mathcal{D}\phi_A \mathcal{D}\phi_B \, \exp\left(-\sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau \int d^d x \, \mathcal{L}^{(j)}(\phi_A, \phi_B^{(j)}) \right)$$

$\phi_A \qquad \tau \in (0, n\beta)$

$\mathcal{L}^{(j)}(\phi_A, \phi_B^{(j)}) = \mathcal{L}(\phi_A, \phi_B^{(j)})$

$\phi_B^{(j)} \; (j = 1, \cdots, n) \qquad \tau \in ((j-1)\beta, \, j\beta - \epsilon)$

Path integral formulation

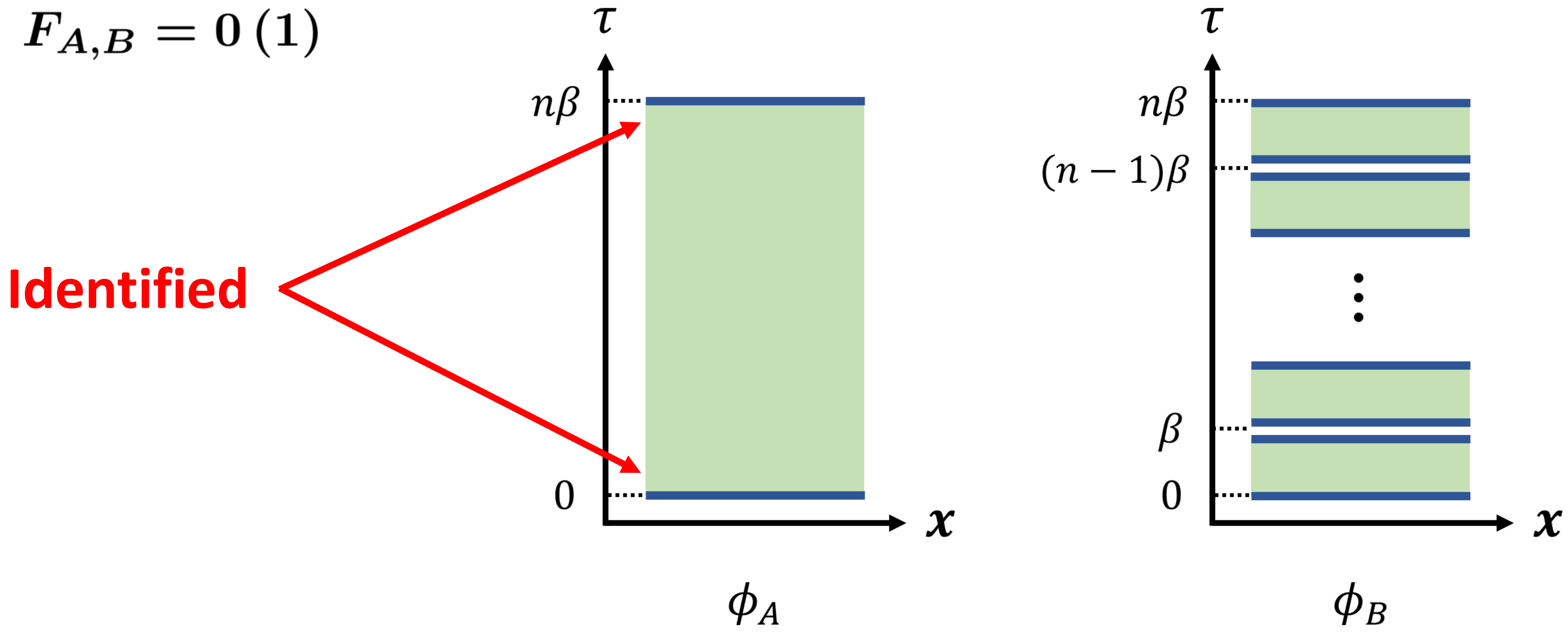
$$\phi_A(0) = (-1)^{F_A}\phi_A(n\beta),$$

ϕ_A has periodicity of $n\beta$.

$$\phi_B^{(j)}((j-1)\beta) = (-1)^{F_B}\phi_B^{(j)}(j\beta - \epsilon)$$

$\phi_B^{(j)}$ has periodicity of β.

$$F_{A,B} = 0\,(1)$$



Path integral formulation

Non-interacting part

Quantum correction

$$S_A^{(n)} = S_{A,0}^{(n)} + \underline{\frac{1}{1-n}\left(\log \frac{\widetilde{Z}_{\mathrm{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)}\,(Z_{B,0}^{(\beta)})^n} - n \log \frac{Z_{\mathrm{tot}}^{(\beta)}}{Z_{A,0}^{(\beta)}\,Z_{B,0}^{(\beta)}}\right)}$$

$$Z_{\alpha,0}^{(\beta)} = \int \mathcal{D}\phi_\alpha \,\exp\left(-\int_0^\beta d\tau \int d^d x\, \mathcal{L}_{\alpha,0}(\phi_\alpha)\right)$$

$(\alpha = A, B)$

$$S_{A,0}^{(n)} = \frac{1}{1-n} \log \frac{Z_{A,0}^{(n\beta)}}{(Z_{A,0}^{(\beta)})^n}$$

$n \to 1$

The thermodynamic entropy of a free field

Zeroth-order contributions

Neutral scalar field

$$\mathcal{L}_0(\phi_i) = \frac{1}{2}\left[\left(\frac{\partial \phi_i}{\partial \tau}\right)^2 + (\nabla \phi_i)^2 + M_i^2 \phi_i^2\right]$$

The result of the functional integration : See e.g. Kapusta, Gale (2011).

$$\log Z^{(\beta)}_{\phi_i,0} = V \int \frac{d^3p}{(2\pi)^3} \left\{-\frac{1}{2}\beta\omega - \log\left(1 - e^{-\beta\omega}\right)\right\} \qquad \omega = \sqrt{p^2 + M_i^2}$$

➡ $$S_{A,0} = V \int \frac{d^3p}{(2\pi)^3} \left\{\frac{\beta\omega}{e^{\beta\omega} - 1} - \log\left(1 - e^{-\beta\omega}\right)\right\}$$

Perturbative expansion

Consider a scalar-scalar system in d+1 dimensions.

$$\mathcal{L}_I = \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda}{4}\phi_A^2\phi_B^2$$

Preserve two independent parities : $\phi_A \to -\phi_A \qquad \phi_B \to -\phi_B$

Formulate perturbation theory and present Feynman rules !

$$S_A^{(n)} = S_{A,0}^{(n)} + \frac{1}{1-n}\left(\underbrace{\log \frac{\widetilde{Z}_{\text{tot}}^{(n\beta)}}{Z_{A,0}^{(n\beta)}\,(Z_{B,0}^{(\beta)})^n}}_{\text{(ii)}} - n\,\underbrace{\log \frac{Z_{\text{tot}}^{(\beta)}}{Z_{A,0}^{(\beta)}\;Z_{B,0}^{(\beta)}}}_{\text{(i)}}\right)$$

Perturbative expansion

The term (i) $$\mathcal{S}^{(\beta)} = \mathcal{S}_0^{(\beta)} + \mathcal{S}_I^{(\beta)}$$

Averaged over the unperturbed ensemble.

$$\log \frac{Z_{\text{tot}}^{(\beta)}}{Z_{A,0}^{(\beta)}\, Z_{B,0}^{(\beta)}} = \log\left(1 + \sum_{l=1}^{\infty} \frac{1}{l!} \frac{\int \mathcal{D}\phi_A \mathcal{D}\phi_B\, e^{-\mathcal{S}_0^{(\beta)}} (\mathcal{S}_I^{(\beta)})^l}{\int \mathcal{D}\phi_A \mathcal{D}\phi_B\, e^{-\mathcal{S}_0^{(\beta)}}}\right)$$

The usual finite temperature perturbation theory.

Propagator : $$D_\alpha^{(\beta)}(\tau, x) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} e^{i(\omega_m \tau + p\cdot x)} \widetilde{D}_\alpha^{(\beta)}(\omega_m, p)$$

$(\alpha = A, B)$

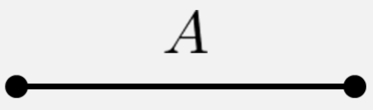
$$= \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d} \frac{e^{i(\omega_m \tau + p\cdot x)}}{\omega_m^2 + p^2 + M_\alpha^2}$$

$\omega_m = 2\pi m T \ \ (m \in \mathrm{Z})$

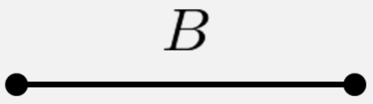
Feynman rules (Position space)

The term (i)

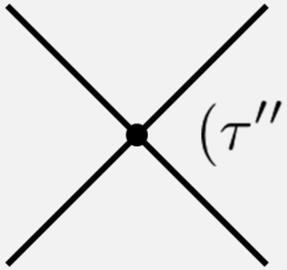
Kapusta, Gale (2011).

1. $(\tau, \boldsymbol{x})$ 

$(\tau', \boldsymbol{x}') \quad = D_A^{(\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

2. $(\tau, \boldsymbol{x})$ 

$(\tau', \boldsymbol{x}') \quad = D_B^{(\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

3. 

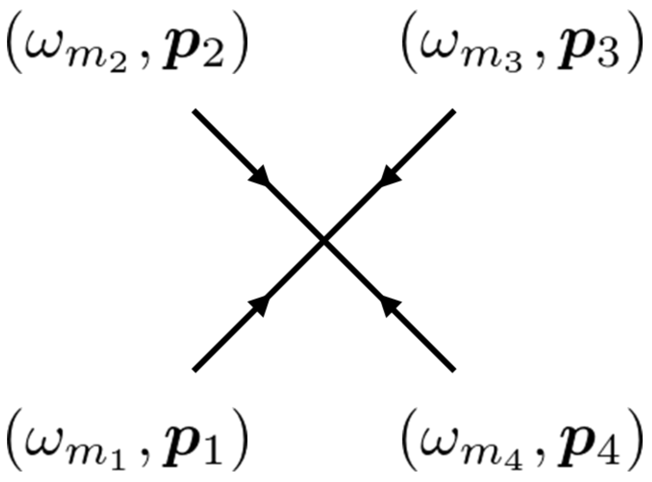
$= -\lambda \int_0^\beta d\tau'' \int d^d x''$

4. Divide by the symmetry factor.

Draw all topologically inequivalent diagrams at a given order.

Only connected diagrams contribute.

Feynman rules (Momentum space)



$$\longleftrightarrow \quad \int_0^\beta d\tau'' \int d^d x''\, e^{i(\omega_{m_1}\tau'' + \boldsymbol{p}_1\cdot\boldsymbol{x}'')} e^{i(\omega_{m_2}\tau'' + \boldsymbol{p}_2\cdot\boldsymbol{x}'')} \times e^{i(\omega_{m_3}\tau'' + \boldsymbol{p}_3\cdot\boldsymbol{x}'')} e^{i(\omega_{m_4}\tau'' + \boldsymbol{p}_4\cdot\boldsymbol{x}'')}$$

$$= (2\pi)^d \delta^{(d)}(\boldsymbol{p}_{\text{in}} - \boldsymbol{p}_{\text{out}})\, \beta \delta_{\omega_{\text{in}},\, \omega_{\text{out}}} \,.$$

Energy and momentum are conserved at each vertex.

1. **For each propagator of $\phi_{A,B}$, assign a factor $\frac{1}{\beta}\sum_m \int \frac{d^d p}{(2\pi)^d} \widetilde{D}^{(\beta)}_{A,B}(\omega_m, p)$.**
2. **Include a factor $-\lambda(2\pi)^d \delta^{(d)}(p_{\text{in}} - p_{\text{out}})\, \beta\delta_{\omega_{\text{in}},\, \omega_{\text{out}}}$ for each vertex.**
3. **Divide by the symmetry factor.**

Perturbative expansion

The term (ii)

$$\widetilde{\mathcal{S}}^{(n\beta)} = \widetilde{\mathcal{S}}_0^{(n\beta)} + \widetilde{\mathcal{S}}_I^{(n\beta)}$$

$$= \int_0^{n\beta} d\tau \int d^d x\, \mathcal{L}_0(\phi_A) + \sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau \int d^d x\, \mathcal{L}_0^{(j)}(\phi_B^{(j)})$$

$$+ \sum_{j=1}^{n} \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau \int d^d x\, \mathcal{L}_I^{(j)}(\phi_A, \phi_B^{(j)})$$

We can consider this action as the theory of (n+1) scalar fields.

$$\widetilde{\omega}_m = 2\pi m T/n \ (m \in \mathbf{Z})$$

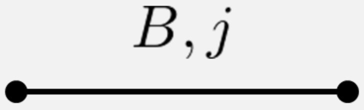
$$D_A^{(n\beta)}(\tau, x) = \frac{1}{n\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d}\, e^{i(\widetilde{\omega}_m \tau + p\cdot x)}\, \widetilde{D}_A^{(n\beta)}(\widetilde{\omega}_m, p) \qquad (0 \le \tau < n\beta)\,,$$

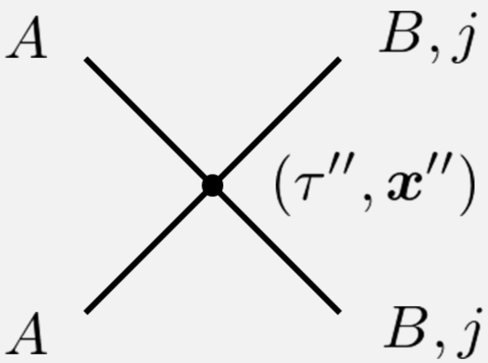
$$D_{B,j}^{(\beta)}(\tau, x) = \frac{1}{\beta} \sum_{m=-\infty}^{\infty} \int \frac{d^d p}{(2\pi)^d}\, e^{i(\omega_m \tau + p\cdot x)}\, \widetilde{D}_{B,j}^{(\beta)}(\omega_m, p) \qquad ((j-1)\beta \le \tau < j\beta)$$

Feynman rules (Position space)

The term (ii)

1. $(\tau, \boldsymbol{x})$  $(\tau', \boldsymbol{x}')$ $= D_A^{(n\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

2. $(\tau, \boldsymbol{x})$  $(\tau', \boldsymbol{x}')$ $= D_{B,j}^{(\beta)}(\tau - \tau', \boldsymbol{x} - \boldsymbol{x}')$

3. 

$= -\lambda \int_{(j-1)\beta}^{j\beta - \epsilon} d\tau'' \int d^d x''$

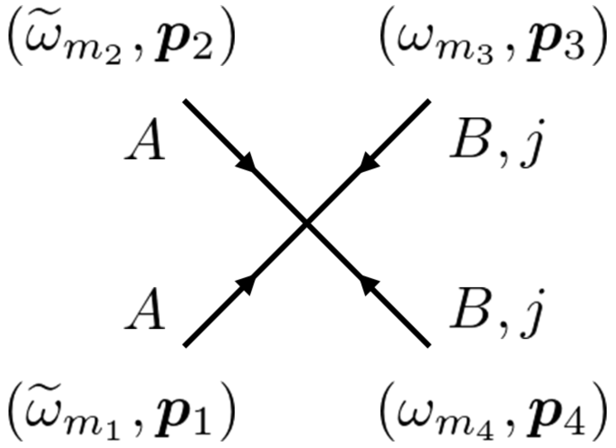
4. Divide by the symmetry factor.

There exists a line for each $\phi_B^{(j)}$.

Lines with different *j*s do not directly connect with each other.

Only connected diagrams contribute.

Feynman rules (Momentum space)



$$\longleftrightarrow \int_{(j-1)\beta}^{j\beta-\epsilon} d\tau'' \int d^d x''\, e^{i(\widetilde{\omega}_{m_1}\tau''+\boldsymbol{p}_1\cdot\boldsymbol{x}'')} e^{i(\widetilde{\omega}_{m_2}\tau''+\boldsymbol{p}_2\cdot\boldsymbol{x}'')} \times e^{i(\omega_{m_3}\tau''+\boldsymbol{p}_3\cdot\boldsymbol{x}'')} e^{i(\omega_{m_4}\tau''+\boldsymbol{p}_4\cdot\boldsymbol{x}'')}$$

$$= (2\pi)^d \delta^{(d)}(\boldsymbol{p}_{\rm in}-\boldsymbol{p}_{\rm out})\, \beta\, e^{i(\omega_{\rm in}-\omega_{\rm out})(j-1)\beta} f_{\omega_{\rm in},\omega_{\rm out}} \,.$$

$$f_{\omega_{\rm in},\omega_{\rm out}} \equiv \frac{1}{\beta}\int_0^\beta d\tau''\, e^{i(\omega_{\rm in}-\omega_{\rm out})\tau''} = \begin{cases} 1 & \text{for } \omega_{\rm in} = \omega_{\rm out} \\ \dfrac{e^{i(\omega_{\rm in}-\omega_{\rm out})\beta}-1}{i(\omega_{\rm in}-\omega_{\rm out})\beta} & \text{for } \omega_{\rm in} \neq \omega_{\rm out} \end{cases}$$

$$\widetilde{\omega}_m = 2\pi m T/n \quad (m \in \mathbf{Z})$$

➡ **Energy is not necessarily conserved at a vertex.**

Feynman rules (Momentum space)

The term (ii)

1. For each propagator of ϕ_A, assign a factor $\frac{1}{n\beta}\sum_m \int \frac{d^d p}{(2\pi)^d}\, \widetilde{D}_A^{(n\beta)}(\widetilde{\omega}_m, p)$.

2. For each propagator of $\phi_B^{(j)}$, assign a factor $\frac{1}{\beta}\sum_m \int \frac{d^d p}{(2\pi)^d}\, \widetilde{D}_{B,j}^{(\beta)}(\omega_m, p)$.

3. Include a factor $-\lambda(2\pi)^d \delta^{(d)}(p_{\text{in}} - p_{\text{out}})\,\beta\, e^{i(\omega_{\text{in}}-\omega_{\text{out}})(j-1)\beta} f_{\omega_{\text{in}},\,\omega_{\text{out}}}$ for each vertex of $\frac{\lambda}{4}(\phi_A \phi_B^{(j)})^2$.

4. Include a factor $-\lambda_A(2\pi)^d \delta^{(d)}(p_{\text{in}} - p_{\text{out}})\, n\beta\delta_{\omega_{\text{in}},\,\omega_{\text{out}}}$ for each vertex of $\frac{\lambda_A}{4!}\phi_A^4$.

5. Include a factor $-\lambda_B(2\pi)^d \delta^{(d)}(p_{\text{in}} - p_{\text{out}})\, \beta\delta_{\omega_{\text{in}},\,\omega_{\text{out}}}$ for each vertex of $\frac{\lambda_B}{4!}\left(\phi_B^{(j)}\right)^4$.

6. Divide by the symmetry factor.

The coupled ϕ^4 theory

Calculate the leading order correction to the thermodynamic entropy.

$$\mathcal{L}(\phi_A, \phi_B) = \mathcal{L}_0 + \mathcal{L}_I + \mathcal{L}_{\text{counter}} \,,$$

$$\mathcal{L}_0(\phi_A, \phi_B) = \frac{1}{2}\left[(\partial_\mu \phi_A)^2 + M_A^2 \phi_A^2\right] + \frac{1}{2}\left[(\partial_\mu \phi_B)^2 + M_B^2 \phi_B^2\right],$$

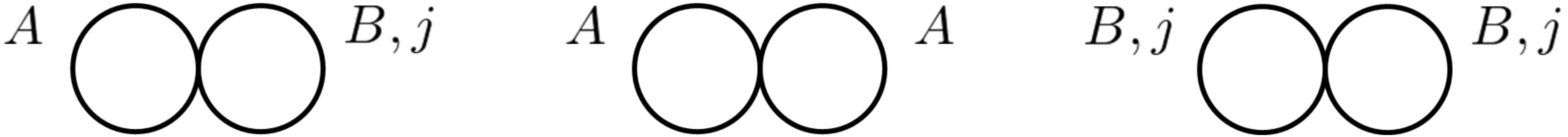
$$\mathcal{L}_I(\phi_A, \phi_B) = \frac{\lambda_A}{4!}\phi_A^4 + \frac{\lambda_B}{4!}\phi_B^4 + \frac{\lambda}{4}\phi_A^2\phi_B^2 \,, \qquad (\partial_\mu \phi)^2 \equiv (\partial_\tau \phi)^2 + (\nabla \phi)^2$$

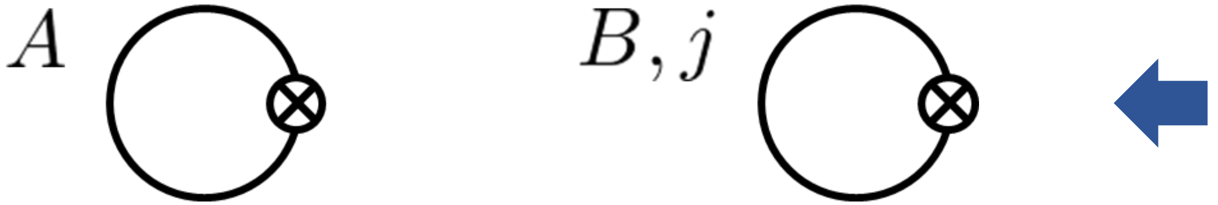
$$\mathcal{L}_{\text{counter}}(\phi_A, \phi_B) = \frac{1}{2}\left[\delta_{Z_A}(\partial_\mu \phi_A)^2 + \delta_{M_A}\phi_A^2\right] + \frac{1}{2}\left[\delta_{Z_B}(\partial_\mu \phi_B)^2 + \delta_{M_B}\phi_B^2\right]$$

$$+ \frac{\delta_{\lambda_A}}{4!}\phi_A^4 + \frac{\delta_{\lambda_B}}{4!}\phi_B^4 + \frac{\delta_\lambda}{4}\phi_A^2\phi_B^2$$

Divergence is renormalized by counter terms of the usual zero-temperature field theory.

The leading order diagrams





Counter terms

High temperature limit

$$S_A = VT^3 \left[\frac{2\pi^2}{45} - \frac{1}{12}\left(\frac{\lambda_A}{4!}\right) - \frac{1}{12}\left(\frac{\lambda}{4!}\right) + \cdots \right]$$

The correction terms are important when the couplings are sufficiently strong.

Instantaneous decoupling

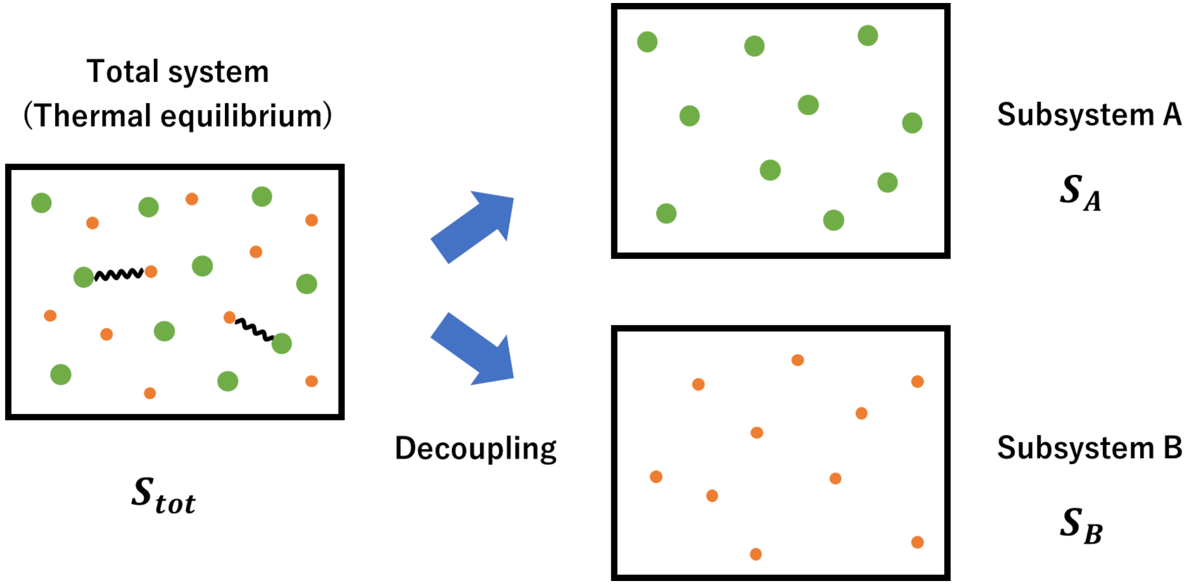
Decoupling may proceed much faster than the time scale of cosmic expansion and the time scale of interactions in a subsystem.

➡ **Thermal equilibrium cannot be maintained during the decoupling.**

Thermodynamic entropy is no longer a good fiducial quantity.

The form of the reduced density matrix is preserved before and after the decoupling.

Entanglement entropy is evaluated at the time just before the decoupling.



Summary

- ✓ **Thermodynamic entropy is generalized to entanglement entropy!**
- ✓ **Formulation of perturbation theory to derive the entanglement entropy of coupled quantum fields and Feynman rules.**
- ✓ **The correction may be important in circumstances of instantaneous decoupling.**